Rendiconti
Accademia Nazionale delle Scienze detta dei XL
Memorie di Matematica
110° (1992), Vol. XVI, fasc. 1, pagg. 1-19

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On the Growth of Coefficients of Entire Functions (**) (***)

SUMMARY. — The decomposition theorem proved in [1] for entire functions of Sato growth fails for those functions which have lower $(p, 1)$-type zero or $(p, 1)$-type infinity. Here we deal with this situation using the concept of proximate order introduced in [8].

Sulla crescita dei coefficienti delle funzioni intere

RIASSUNTO. — Il teorema di decomposizione dimostrato in [1] per le funzioni intere con crescita del tipo di Sato non si applica al caso in cui la funzione abbia uno zero di tipo $(p, 1)$ inferiore o un infinito di tipo $(p, 1)$. In questo lavoro ci si occupa appunto di questo caso, impiegando il concetto di ordine approssimato introdotto in [8].

0. - INTRODUCTION

Let $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ be a nonconstant entire function, where $\lambda_0 = 0$ and $\{\lambda_k\}_{k=1}^{\infty}$ is a strictly increasing seguence of positive integers and assume that $a_k \neq 0$ for $k = 1, 2, \ldots$. We set $M(r) \equiv M(r, f) = \max_{|z|=r} |f(z)|$; $\mu(r) \equiv \mu(r, f) = \max_{k \geq 1} \{|a_k| r^{\lambda_k}\}$ and $\nu \equiv \nu(r, f) = \max\{\lambda_k : \mu(r) = |a_k| r^{\lambda_k}\}$. Then $M(r)$, $\mu(r)$ and $\nu(r)$ are called respectively the maximum modulus, maximum term and rank of the maximum term, of $f(z)$ for $|z| = r$. The concepts of index-pair (p, q), $p \geq q \geq 1$, (p, q)-orders and (p, q)-types were introduced by Juneja *et al.* ([4], [5]).

The growth of an entire function $f(z)$ can be studied in terms of its (p, q)-orders and (p, q)-types. However, these parameters are inadequate for comparing the growth of those entire functions which are of the same (p, q)-orders but of infinite (p, q)-type. In order to refine this scale Nandan *et al.* [11] introduced the concept of proximate order for entire functions with index-pair (p, q) as follows.

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(**) This work was supported by The Swedish Institute, Stockholm.
(***) Memoria presentata il 18 Feb. 1991 da Giuseppe Scorza Dragoni, uno dei XL.

ISSN-0392-4106

DEFINITION 1: A positive function $\rho_{p,q}(r)$ defined on $[r_0, \infty)$, $r_0 > \exp^{[q-1]} 1$, is said to be a proximate order of an entire function with index-pair (p, q) if

(i) $\rho_{p,q}(r) \to \rho(p, q) \equiv \rho$ as $r \to \infty$, $b < \rho < \infty$,

(ii) $\Lambda_{[q]}(r) \rho'(r) \to 0$ as $r \to \infty$;

where $\rho'_{p,q}(r)$ denotes the derivative of $\rho_{p,q}(r)$, $b = 1$ if $p = q$, $b = 0$ if $p > q$ and, for convenience, $\Lambda_{[q]}(r) = \prod_{i=0}^{q} \log^{[i]} r$.

The generalized (p, q)-type T^* and generalized lower (p, q)-type t^* of $f(z)$ are defined as

$$(0.1) \qquad \lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} M(r)}{(\log^{[q-1]} r)^{\rho_{p,q}(r)}} = \begin{matrix} T^*(p, q) \equiv T^* \\ t^*(p, q) \equiv t^* \end{matrix}; \qquad 0 \leq t^* \leq T^* \leq \infty,$$

If the quantity T^* is different from zero and infinity then $\rho_{p,q}(r)$ is said to be the proximate order of a given entire function $f(z)$ with index-pair (p, q). Clearly, the proximate order and the corresponding generalized (p, q)-type of an entire function are not uniquely determined. For example, if the function $c / \log^{[q]} r$, $0 < c < \infty$ is added to the proximate order $\rho_{p,q}(r)$ then it can be seen that $\rho_{p,q}(r) + c / \log^{[q]} r$ is also a proximate order satisfying (i) and (ii) and consequently, the generalised (p, q)-type turns out to be $e^c T^*$. Following Levin [10], Nandan *et al.* [11] established that there exists a proximate order for every entire function with index-pair (p, q).

In a similar manner if t^* is different from zero and infinity then $\rho_{p,q}(r)$ is said to be the lower proximate order of the entire function $f(z)$. Kasana and Sahai [8] have proved the existence theorem for such functions.

DEFINITION 2: An entire function $f(z)$ is said to be of generalized (p, q)-growth $\{\rho, T^*\}$ with respect to a proximate order $\rho_{p,q}(r)$ if the (p, q)-order of $f(z)$ does not exceed ρ, if $f(z)$ has (p, q)-order ρ the generalized (p, q)-type does not exceed T^*.

The generalized (p, q)-growth number μ^* and generalized lower (p, q)-growth number δ^* of $f(z)$ are defined as

$$(0.2) \qquad \lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-2]} \nu(r)}{(\log^{[q-1]} r)^{\rho_{p,q}(r) - A}} = \begin{matrix} \mu^*(p, q) \equiv \mu^* \\ \delta^*(p, q) \equiv \delta^* \end{matrix},$$

where $A = 1$ if $(p, q) = (2, 2)$ and $A = 0$ if $(p, q) \neq (2, 2)$.

DEFINITION 3: An entire function $f(z)$ of (p, q)-order $\rho (b < \rho < \infty)$ is said to be of perfectly regular (p, q)-growth with respect to a proximate order $\rho_{p,q}(r)$ if $0 < t^* = T^* < \infty$.

DEFINITION 4: An entire function with index-pair (p, q) is said to be of minimal, normal or maximal (p, q)-type with respect to a proximate order according as T^* is zero, positive finite and infinite respectively.

In this paper we prove a decomposition theorem for entire functions which are not of perfectly regular (p, q)-growth in reference to $\rho_{p,q}(r)$. The second section deals with the result which describes the growth of a_n's for entire functions of (p, q)-order 0 or 1 and in other situations for entire functions of minimal generalized (p,q)-type. As a last result of this section we have obtained the necessary conditions for an entire function to be of perfectly regular (p,q)-growth. Finally, some inequalities connecting generalized (p,q)-type and generalized lower (p,q)-type of an entire function $f(z)$ with the ratio $|a_{n-1}/a_n|$ of the coefficients occurring in its power series expansion have been derived. Also, some results have been obtained involving coefficients and exponents of entire gap power series and generalized (p,q)-growth numbers.

It is known [11, Thm. 4] that $(\log^{[q-1]} r)^{\rho_{p,q}(r)-A}$ is a monotonically increasing function of r for $r > r_0$. Hence we can define the function $\phi(x)$ to be the unique solution of the equation

$$x = (\log^{[q-1]} r)^{\rho_{p,q}(r)-A} \Leftrightarrow \phi(x) = \log^{[q-1]} r, \tag{0.3}$$

where A has the same meaning as in (0.2).

Consequently, we have the following results [12]:

$$\lim_{x\to\infty} \frac{d[\log \phi(x)]}{d[\log x]} = \frac{1}{\rho - A}, \tag{0.4}$$

and for every η such that $0 < \eta < \infty$,

$$\lim_{x\to\infty} \frac{\phi(\eta x)}{\phi(x)} = \eta^{1/(\rho - A)}. \tag{0.5}$$

1. - Decomposition theorem

Basically, this theorem was obtained by Bajpai *et al.* [1] for entire functions of index $(p, 1)$ i.e., Sato growth [13]. Later on, in a subsequent paper [5], they themselves extended this to entire power series with index-pair (p,q). It has been observed that their decomposition theorem fails foe entire functions of infinite $(p, 1)$-type or infinite (p,q)-type and moreover, for entire functions of lower $(p, 1)$-type or lower (p,q)-type as zero. To deal with these situations we review their decomposition theorem and extend this also to entire gap power series.

Theorem 1: Let $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ be an entire function having (p,q)-order $\rho(b < \rho < \infty)$, generalized (p,q)-type T^* and generalized lower (p,q)-type t^*, and β be a number such that $0 < t^* < \beta < T^* < \infty$. Then

$$f(z) = g_\beta(z) + h_\beta(z),$$

where either the index-pair of $g_\beta(z)$ is less than (p,q) or $g_\beta(z)$ is of (p,q)-growth $\{\rho, \beta\}$ with respect to $\rho_{p,q}(r)$ of $f(z)$ and $h_\beta(z) = \sum_{k=0}^{\infty} b_k z^{\lambda_{m_k}}$, $(b_k \neq 0, \forall k)$ satisfies:

$$t^* \geq \beta \liminf_{k\to\infty} \frac{\log^{[p-2]}(\phi(\lambda_{m_k}))^{\rho}}{(\phi(\log^{[p-2]}\lambda_{m_{k+1}}))^{\rho}}, \tag{1.1}$$

where $\phi(t)$ is the function defined by (0.3).

PROOF: Let $g(z)=\sum_{k=0}^{\infty} c_k z^{\lambda_k}$, where

$$c_k=\begin{cases} a_k & \text{if } |a_k|\leq \exp\left\{-\lambda_k \exp^{[q-2]}\left(\dfrac{\phi(\log^{[p-2]}\lambda_k)}{(\beta/M)^{1/(\rho-A)}}\right)\right\}, \\ 0 & \text{otherwise.}\end{cases}$$

Clearly, $g(z)$ is an entire function. Assume that $g(z)$ has the index-pair (p', q') and (p', q')-order ρ'. In case $\{\lambda_{n_k}\}$ is the strictly increasing sequence of positive integers such that $c_{n_k}\neq 0$, $\forall k$, then it is known [4, Thm. 1] that

$$\rho' = P(L'(p', q')),$$

where

$$L'(p', q')=\limsup_{k\to\infty}\frac{\log^{[p-1]}\lambda_{n_k}}{\log^{[q-1]}|c_{n_k}|^{-1/\lambda_{n_k}}}.$$

Hence the index-pair (p', q') of $g(z)$ is of generalized (p, q)-growth $\{\rho, \beta\}$. Define

$$b_k(z)=f(z)-g_k(z)=\sum_{k=0}^{\infty} b_k z^{\lambda_k}.$$

Then,

$$\log|b_k| > -\lambda_{m_k}\exp^{[q-2]}\left(\frac{\phi(\log^{[p-2]}\lambda_{m_k})}{(\beta/M)^{1/(\rho-A)}}\right). \tag{1.2}$$

Let r_k be the unique root of the equation

$$\exp^{[q-2]}\left(\frac{\phi(\log^{[p-2]}\lambda_{m_k})}{(\beta/M)^{1/(\rho-A)}}\right)+ \tag{1.3}$$

$$+\frac{\phi(\lambda_{m_k})^{\rho-A}}{\rho-A}\,\frac{E_{[q-2]}(\phi(\log^{[p-2]}\lambda_{m_k})/(\beta/M)^{1/(\rho-A)})}{\Delta_{[p-2]}(\lambda_{m_k})}=\log r_k,$$

where $E_{[q]}(r)=\prod_{i=0}^{q}\exp^{[i]}r$ and

$$M=M(p,q)=\begin{cases}(\rho-1)^{\rho-1}/\rho^{\rho} & \text{if } (p,q)=(2,2),\\ 1/\rho e & \text{if } (p,q)=(2,1),\\ 1 & \text{otherwise.}\end{cases}$$

Further, assume that $r_k \leq r \leq r_{k+1}$. In view of (1.2) and (1.3), we have

$$\text{(1.4)} \quad \log M(r) \geq \log |b_k| + \lambda_{m_k} \log r \geq -\lambda_{m_k} \exp^{[q-2]}\left(\frac{\phi(\log^{[p-2]}\lambda_{m_k})}{(\beta/M)^{1/(\rho-A)}}\right) +$$

$$+ \lambda_{m_k} \log r_k = \frac{\lambda_{m_k}(\phi(\lambda_{m_k}))^{\rho-A}}{(\rho-A)\Delta_{[p-2]}(\lambda_{m_k})} E_{[q-2]}\left(\frac{\phi(\log^{[p-2]}\lambda_{m_k})}{(\beta/M)^{1/(\rho-A)}}\right).$$

Using the equation (1.3) for $(p, q) = (2, 1)$, we observe that $r_k \simeq \phi(\lambda_{m_k})/(\beta\rho)^{1/\rho}$ as $k \to \infty$ (since $(\phi(\lambda_{m_k}))^{\rho}/\lambda_{m_k} \to 1$ as $k \to \infty$) and hence (1.4) gives

$$\frac{\log M(r)}{r^{\rho_{2,1}(r)}} \geq \frac{(\phi(\lambda_{m_k}))^{\rho}}{\rho(r_{k+1})^{\rho_{2,1}(r)}} \simeq \frac{(\phi(\lambda_{m_k}))^{\rho}(\beta\rho)^{\rho_{2,1}(r)/\rho}}{\rho(\phi(\lambda_{m_{k+1}}))^{\rho_{2,1}(r)}}.$$

On passing to limitss, we get

$$t^* \geq \beta\left(\frac{\phi(\lambda_{m_k})}{\phi(\lambda_{m_{k+1}})}\right)^{\rho}.$$

Again, for $(p, q) = (2, 2)$, we observe from (1.3) that

$$\log r_k = \frac{(\rho-1)\phi(\lambda_{m_k})}{\beta^{1/(\rho-1)}\rho^{\rho/(\rho-1)}} + \frac{(\phi(\lambda_{m_k}))^{\rho}}{\beta^{1/(\rho-1)}\rho^{\rho/(\rho-1)}\lambda_{m_k}} =$$

$$= \frac{(\rho-1)\phi(\lambda_{m_k})}{\beta^{1/(\rho-1)}\rho^{\rho/(\rho-1)}}\left(1 + \frac{(\phi(\lambda_{m_k}))^{\rho-1}}{(\rho-1)\lambda_{m_k}}\right) \simeq \frac{(\rho-1)\phi(\lambda_{m_k})}{\beta^{1/(\rho-1)}\rho^{\rho/(\rho-1)}}\left(1 + \frac{1}{\rho-1}\right) = \frac{\phi(\lambda_{m_k})}{(\beta\rho)^{1/(\rho-1)}}.$$

Using this in (1.4) and further application of (0.5) yields

$$\frac{\log M(r)}{(\log r)^{\rho_{2,2}(r)}} \geq \frac{(\phi(\lambda_{m_k}))^{\rho}}{\beta^{1/(\rho-1)}\rho^{\rho/(\rho-1)}(\log r_{k+1})^{\rho_{2,2}(r)}} \simeq \frac{\beta^{(\rho_{2,2}(r)-1)/(\rho-1)}\rho^{(\rho_{2,2}(r)-\rho)/(\rho-1)}(\phi(\lambda_{m_k}))^{\rho}}{(\phi(\lambda_{m_{k+1}}))^{\rho_{2,2}(r)}}.$$

On taking limit, we get (1.1) for the case when $(p, q) = (2, 2)$.

Finally, consider the case $p \geq 3$. From (1.3),

$$\log^{[q-1]} r_k \simeq \frac{\phi(\log^{[p-2]}\lambda_{m_k})}{\beta^{1/\rho}} \quad \text{as } k \to \infty,$$

and from (1.4),

$$\log\log M(r) > O(1) + \rho \log \phi(\lambda_{m_k}) + \log \frac{\phi(\log^{[p-2]}\lambda_{m_k})}{\beta^{1/\rho}} +$$

$$+ \sum_{i=0}^{q-3} \exp^{[i]}\left(\frac{\phi(\log^{[p-2]}\lambda_{m_k})}{\beta^{1/\rho}}\right) - \sum_{i=2}^{p-1} \log^{[i]}\lambda_{m_k},$$

or

$$\log^{[p-1]} M(r) > (1 + o(1)) \log^{[p-2]} (\phi(\lambda_{m_k}))^{\rho}.$$

Hence

$$\frac{\log^{[p-1]} M(r)}{(\log^{[q-1]})^{\rho_{k+1}(r)}} > (1 + o(1)) \frac{\beta^{\rho_{k+1}(r)/\rho} \log^{[p-2]} (\phi(\lambda_{m_k}))^{\rho}}{(\phi(\log^{[p-2]} \lambda_{m_{k+1}}))^{\rho_{k+1}(r)}}.$$

On passing to limits (consider (0.1) for t^*), we get (1.1) and hence the theorem.

2. - In this section we establish the growth of coefficients for those entire functions which have zero or one as (p,q)-order and otherwise minimal generalized (p,q)-type. The next result of this section describes the necessary conditions for an entire function to be of perfectly regular (p,q)-growth with respect to a proximate order.

THEOREM 2: Let $f(z)$ be an entire function with index-pair (p,q) such that $\rho = b$, then for every $\delta > 0$,

$$\lim_{n\to\infty} \sup \frac{(\log^{[p-2]} \lambda_n)^{\delta}}{\log^{[q-1]} |a_n|^{-1/\lambda_n}} = 0.$$

Further, if $\rho > b$ and $f(z)$ is of minimal generalized (p,q)-type, then

$$\lim_{n\to\infty} \sup \frac{\phi(\log^{[p-2]} \lambda_n)}{\log^{[q-1]} |a_n|^{-1/\lambda_n}} = 0,$$

where b has the same meaning as in Definition 1.

PROOF: Since $\rho = b$, it follows from the definition of (p,q)-order that for given $\varepsilon > 0$ and $r > r_0$,

$$\log M(r,f) < \exp^{[p-2]} (\log^{[q-1]} r)^{b+\varepsilon}.$$

Using Cauchy's inequality we get

$$\text{(2.1)} \qquad \log |a_n| < \exp^{[p-2]} (\log^{[q-1]} r)^{b+\varepsilon} - \lambda_n \log r.$$

Choose the value of r satisfying

$$\text{(2.2)} \qquad r = \exp^{[q-1]} \left(\log^{[p-2]} \frac{\lambda_n}{b+\varepsilon} \right)^{1/(b+\varepsilon)}.$$

For $(p,q) = (2.1)$, (2.2) implies $r = (\lambda_n/\varepsilon)^{1/\varepsilon}$ and using this value in (2.1), we get

$$|a_n| < \left(\frac{e\varepsilon}{\lambda_n} \right)^{\lambda_n/\varepsilon}.$$

or

$$|a_n|^{1/\lambda_n} < \left(\frac{e\varrho}{\lambda_n}\right)^{1/\varrho},$$

which on taking limits gives

$$\text{(2.3)} \qquad \lim_{n\to\infty} \sup \lambda_n^{1/\varrho} |a_n|^{1/\lambda_n} < \infty .$$

In case of $(p, q) = (2, 2)$ we observe that $\log r = (\lambda_n/(1 + \varepsilon))^{1/(1+\varrho)}$ satisfies (2.2) and (2.1) is reduced to

$$\log |a_n| < \frac{\lambda_n}{1+\varepsilon} - \lambda_n \left(\frac{\lambda_n}{1+\varepsilon}\right)^{1/(1+\varrho)},$$

or

$$\log |a_n|^{-1/\lambda_n} > (1 + o(1)) \left(\frac{\lambda_n}{1+\varepsilon}\right)^{1/(1+\varrho)}.$$

Thus,

$$\text{(2.4)} \qquad \lim_{n\to\infty} \sup \frac{\lambda_n^{1/(1+\varrho)}}{\log |a_n|^{-1/\lambda_n}} \leq 1 .$$

Finally, for $(p, q) \neq (2, 1)$ and $(p, q) \neq (2, 2)$, (2.1) and (2.2) give

$$\log^{[q-1]} |a_n|^{-1/\lambda_n} > (1 + o(1)) \left(\log^{[p-2]} \frac{\lambda_n}{\varepsilon}\right)^{1/\varrho}, \quad p > q ,$$

or

$$\log^{[q-1]} |a_n|^{-1/\lambda_n} > (1 + o(1)) \left(\log^{[p-2]} \frac{\lambda_n}{1+\varepsilon}\right)^{1/(1+\varrho)}, \quad p = q .$$

This means that for all $p \geq q \geq 3$,

$$\text{(2.5)} \qquad \lim_{n\to\infty} \sup \frac{(\log^{[p-2]} \lambda_n)^{1/(\varrho+\varepsilon)}}{\log^{[q-1]} |a_n|^{-1/\lambda_n}} \leq 1 .$$

Clearly, (2.3), (2.4) and (2.5) combine to give

$$\text{(2.6)} \qquad \lim_{n\to\infty} \sup \frac{(\log^{[p-2]} \lambda_n)^{\varrho}}{\log^{[q-1]} |a_n|^{-1/\lambda_n}} < , \qquad \text{for every } \varrho > 0 .$$

If the lim sup in (2.6) is finite and positive for some $\varrho > 0$ then for every $\alpha > 0$, we have

$$\text{(2.7)} \qquad \lim_{n\to\infty} \sup \frac{(\log^{[p-2]} \lambda_n)^{\varrho+\alpha}}{\log^{[q-1]} |a_n|^{-1/\lambda_n}} = \infty .$$

(2.7) is a contradiction to what we obtained in (2.6) and thus the first part is proved.

In case when $\rho > b$, Kasana [5] has proved that generalized (p, q)-type of an entire function is given by

$$(2.8) \qquad \frac{T^*}{M} = \lim_{n \to \infty} \sup \left[\frac{\phi(\log^{[p-2]} \lambda_n)}{\log^{[q-1]} |a_n|^{-1/\lambda_n}} \right]^{\rho - A}.$$

If we put $T^* = 0$ in (2.8), the second result is immediate.

Finally, we study the subsequence $\{\lambda_{n_k}\}$ of λ_n such that, for $f = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$, one has

$$(2.9) \qquad |a_{n_{k-1}}(f)| > |a_{n_k}(f)| \quad \text{and} \quad a_n(f) = a_{n_{k-1}}(f) \quad \text{for} \quad \lambda_{n_{k-1}} \leq n < \lambda_{n_k}.$$

The next theorem shows how this sequence influences the growth of an entire function in reference to its generalized (p, q)-type and generalized lower (p, q)-type. This also describes the condition for f to be an entire function of perfectly regular (p, q)-growth with respect to a proximate order.

Theorem 3: Let $f(z)$ be an entire function having (p, q)-order $\rho (b < \rho < \infty)$, generalized (p, q)-type T^* and generalized lower (p, q)-type t^*. Let $\{\lambda_{n_k}\}$ be the sequence defined by (2.9). Then

$$t^* \leq T^* \lim_{k \to \infty} \inf \left[\frac{\phi(\log^{[p-2]} \lambda_{n_{k-1}})}{\phi(\log^{[p-2]} \lambda_{n_k})} \right]^{\rho}, \qquad p \geq 3.$$

Further, if $\{\lambda_{n_k}\}$ be the sequence of principal indices satisfying $\lambda_{n_{k-1}} = \lambda_{n_k}$ as $k \to \to \infty$, then

$$t^* \leq T^* \lim_{k \to \infty} \inf \left[\frac{\phi(\lambda_{n_{k-1}})}{\phi(\lambda_{n_k})} \right]^{\rho - A}.$$

Proof: Let us define a function $u(z)$ such that

$$u(z) = \sum_{n=1}^{\infty} (a_{n-1}(f) - a_n(f)) z^{\lambda_n} = \sum_{k=1}^{\infty} \alpha_k(f) z^{\lambda_{n_k}},$$

where

$$\alpha_k(f) = a_{n_{k-1}}(f) - a_{n_k}(f).$$

In view of the definition (2.9) it can be proved that $u(z)$ and $f(z)$ have the same (p, q)-order and generalized (p, q)-types such that

$$T^*(f) = T^*(u) \quad \text{and} \quad t^*(f) = t^*(u).$$

Thus, using (2.8) it can be shown that

$$\frac{T^*(f)}{M} = \lim_{m\to\infty} \sup \left[\frac{\phi(\log^{[p-2]}\lambda_{m_k})}{\log^{[q-1]}\alpha_{\lambda_{m_k}}^{-1/\lambda_{m_k}}(f)} \right]^{\rho-A}$$

Considering the above formula and Theorem 2 of Kasana *et al.* [8], we observe that for $p \geq 3$,

$$t^*(f) = \max_{\{k_m\}} \left\{ \lim_{m\to\infty} \inf \left(\frac{(\log^{[p-2]}\lambda_{m_{k-1}})}{\log^{[q-1]}\alpha_{\lambda_{m_k}}^{-1/\lambda_{m_k}}} \right)^{\rho} \right\} \leq \max_{\{k_m\}} \left\{ \lim_{m\to\infty} \sup \left(\frac{\phi(\log^{[p-2]}\lambda_{m_k})}{\log^{[q-1]}\alpha_{\lambda_{m_k}}^{-1/\lambda_{m_k}}(f)} \right)^{\rho} \right\}\cdot$$

$$\cdot \max_{\{k_m\}} \left\{ \lim_{m\to\infty} \inf \left(\frac{\phi(\log^{[p-2]}\lambda_{m_{k-1}})}{\phi(\log^{[p-2]}\lambda_{m_k})} \right)^{\rho} \right\} \leq T^*(f) \left\{ \lim_{k\to\infty} \inf \left(\frac{\phi(\log^{[p-2]}\lambda_{m_{k-1}})}{\phi(\log^{[p-2]}\lambda_{m_k})} \right)^{\rho} \right\}.$$

Similarly, for the case $p = 2$ and $q = 1$, let $\{\lambda_{m_k}\}$ be the sequence of principal indices such that $\lambda_{m_{k-1}} \simeq \lambda_{m_k}$ as $k \to \infty$, we have

$$t^*(f) \leq T^*(f) \lim_{k\to\infty} \inf \left(\frac{\phi(\lambda_{m_{k-1}})}{\phi(\lambda_{m_k})} \right)^{\rho-A}.$$

Corollary 1: If $f(z)$ is an entire function of perfectly regular (p, q)-growth with respect to $\rho_{p,q}(r)$, then

$$\log^{[p-2]}\lambda_{m_{k-1}} \simeq \log^{[p-2]}\lambda_{m_k} \quad \text{as } k \to \infty.$$

Corollary 2: If $f(z)$ is an entire function having (p, q)-order $\rho(b < \rho < \infty)$, (p, q)-type T and lower (p, q)-type t such that $(0 \leq t \leq T < \infty)$, then

$$t \leq T \lim_{k\to\infty} \inf \frac{\log^{[p-2]}\lambda_{m_{k-1}}}{\log^{[p-2]}\lambda_{m_k}}, \quad p \geq 3.$$

This inequality also holds for $p = 2$ if $\{\lambda_{m_k}\}$ is the sequence of principal indices satisfying $\lambda_{m_{k-1}} \simeq \lambda_{m_k}$ as $k \to \infty$.

3. - This section contains various inequalities, some of which are extensions of results in [3], [16] and [1].

Theorem 4: Let $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ be an entire function having (p, q)-order $\rho(b < \rho < \infty)$, generalized (p, q)-type T^* and generalized lower (p, q)-type t^*. Then

(3.1) $$YR^* \leq t^* \leq T^* \leq XQ^*,$$

where

$$R^* \equiv R^*(p,q) = \liminf_{k\to\infty} \left[\frac{\phi(\log^{[p-2]} \lambda_{k-1})}{\log^{[q-1]} \left| \frac{a_{k-1}}{a_k} \right|^{1/(\lambda_k - \lambda_{k-1})}} \right]^{p-A},$$

$$Q^* \equiv Q^*(p,q) = \limsup_{k\to\infty} \left[\frac{\phi(\log^{[p-2]} \lambda_k)}{\log^{[q-1]} \left| \frac{a_{k-1}}{a_k} \right|^{1/(\lambda_k - \lambda_{k-1})}} \right]^{p-A},$$

$X = 1/\rho$ if $p = 2$, $X = 1$ if $p \geq 3$, and

$$Y \equiv Y(p,q) = \begin{cases} \dfrac{e^{\alpha-1}}{\rho} & \text{if } (p,q) = (2,1), \\ \dfrac{\alpha^{1/(\rho-1)}}{\rho}\left(\dfrac{\rho-1}{\rho-\alpha}\right)^{\rho-1} & \text{if } (p,q) = (2,2), \\ 1 & \text{otherwise}, \end{cases}$$

such that

$$\alpha = \liminf_{k\to\infty} \frac{\lambda_{k-1}}{\lambda_k}.$$

Proof: From definition, $R^* \geq 0$. If $R^* = 0$, first part of the inequality (3.1) is tivial. Hence, let $R^* > 0$. In this case, for given $\varepsilon > 0$ and $k > k_0$, we have

$$\frac{\phi(\log^{[p-2]} \lambda_{k-1})}{\log^{[q-1]} \left| \frac{a_{k-1}}{a_k} \right|^{1/(\lambda_k - \lambda_{k-1})}} > (R^* - \varepsilon)^{1/(p-A)},$$

or

$$\log \left| \frac{a_{k-1}}{a_k} \right| < (\lambda_k - \lambda_{k-1}) \exp^{[q-2]} \left(\frac{\phi(\log^{[p-2]} \lambda_{k-1})}{(R^* - \varepsilon)^{1/(p-A)}} \right).$$

Putting $k = n_0, n_0 + 1, \ldots, n$ in above and adding the inequalities thus obtained, we get

$$\log \left| \frac{a_{n_0-1}}{a_n} \right| < \sum_{k=n_0}^{n} (\lambda_k - \lambda_{k-1}) \exp^{[q-2]} \left(\frac{\phi(\log^{[p-2]} \lambda_{k-1})}{(R^* - \varepsilon)^{1/(p-A)}} \right) <$$

$$< \lambda_n \psi(\lambda_{n-1}) - \lambda_{n_0-1} \psi(\lambda_{n_0-1}) - \sum_{k=n_0+1}^{n} (\psi(\lambda_{k-1}) - \psi(\lambda_{k-2})) \lambda_{k-1},$$

where

$$\log^{[\rho-2]}\psi(x) = \frac{\phi(\log^{[\rho-2]}x)}{(R^*-\varepsilon)^{1/(\rho-A)}}.$$

Hence

(3.2) $$\log\left|\frac{a_{n_0-1}}{a_n}\right| < \lambda_n\psi(\lambda_{n-1}) - \lambda_{n_0-1}\psi(\lambda_{n_0-1}) - \int_{\lambda_{n_0-1}}^{\lambda_{n-1}} xd[\psi(x)].$$

Considering (3.2) for $(p, q) = (2, 1)$,

$$\log\left|\frac{a_n}{a_{n_0-1}}\right| > \lambda_{n_0-1}\psi(\lambda_{n_0-1}) - \lambda_n\psi(\lambda_{n-1}) + \int_{\lambda_{n_0-1}}^{\lambda_{n-1}} xd\left[\log\frac{\phi(x)}{(R^*-\varepsilon)^{1/\rho}}\right],$$

or

$$\log|a_n| > O(1) - \lambda_n\psi(\lambda_{n-1}) + \int_{\lambda_{n_0-1}}^{\lambda_{n-1}} \frac{x\phi'(x)}{\phi(x)}dt.$$

Using the property (1.2), we get

$$\log|a_n|^{-1/\lambda_n} < o(1) + \psi(\lambda_{n-1}) - \frac{1}{\rho+\varepsilon}\frac{\lambda_{n-1}}{\lambda_n} \leq o(1) + \log\frac{\phi(\lambda_{n-1})}{(R^*-\varepsilon)^{1/\rho}} - \frac{\alpha}{\rho+\varepsilon}.$$

Hence

$$e^{\alpha/(\rho+\varepsilon)}(R^*-\varepsilon) < (1+o(1))\left(\frac{\phi(\lambda_{n-1})}{|a_n|^{-1/\lambda_n}}\right)^{\rho}.$$

Passing to limits and using [9, Thm. 1], we get

(3.3) $$e^{\rho-1}R^* \leqslant \rho t^*.$$

For the case $(p, q) = (2, 2)$, inequality (3.2) yields

$$\log\left|\frac{a_n}{a_{n_0-1}}\right| > O(1) - \lambda_n\psi(\lambda_{n-1}) + \int_{\lambda_{n_0-1}}^{\lambda_{n-1}} xd[\psi(x)] =$$

$$O(1) - \lambda_n\psi(\lambda_{n-1}) + \lambda_{n-1}\psi(\lambda_{n-1}) - \int_{\lambda_{n_0-1}}^{\lambda_{n-1}} \psi(x)\,dx =$$

$$= O(1) - \lambda_n\psi(\lambda_{n-1}) + \lambda_{n-1}\psi(\lambda_{n-1}) - \frac{\rho-1}{\rho(R^*-\varepsilon)^{1/(\rho-1)}}\left[\{\phi(x)\}^{\rho}\right]_{\lambda_{n_0-1}}^{\lambda_{n-1}},$$

or

$$\log |a_n|^{-1/\lambda_n} < \left(1 - \frac{\lambda_{n-1}}{\lambda_n}\right) \frac{\phi(\lambda_{n-1})}{(R^* - \varepsilon)^{1/(\rho-1)}} + (1 + o(1)) \frac{\rho - 1}{\rho \lambda_n} \frac{\{\phi(\lambda_{n-1})\}^{\rho}}{(R^* - \varepsilon)^{1/(\rho-1)}} =$$

$$= \left(1 - \alpha + \alpha \frac{\rho - 1}{\rho}\right) \frac{\phi(\lambda_{n-1})}{(R^* - \varepsilon)^{1/(\rho-1)}}.$$

Thus

$$R^* - \varepsilon < (1 + o(1)) \left(\frac{\rho - \alpha}{\rho}\right)^{\rho-1} \left(\frac{\phi(\lambda_{n-1})}{\log |a_n|^{-1/\lambda_n}}\right)^{\rho-1}.$$

Passing to limits as $n \to \infty$ (again, in view of [9, Thm. 1]; we get

$$\frac{\alpha^{1/(\rho-1)}}{\rho} \left(\frac{\rho - 1}{\rho - \alpha}\right)^{\rho-1} R^* \leq t^*. \tag{3.4}$$

Finally, let us consider the case when $(p, q) \neq (2, 1)$ and $(p, q) \neq (2, 2)$. In this situation (3.2) is reduced to

$$\log \left| \frac{a_{n_0-1}}{a_n} \right| < O(1) + \lambda_n \phi(\lambda_{n-1}) - \int_{\lambda_{n_0-1}}^{\lambda_{n-1}} t d[\phi(t)] <$$

$$< O(1) + \lambda_n \phi(\lambda_{n-1}) - \lambda_{n-1} \phi(\lambda_{n-1}) < O(1) + \lambda_n \phi(\lambda_{n-1}) - \lambda_{n_0-1} \phi(\lambda_{n-1}),$$

or

$$\log |a_n|^{-1/\lambda_n} < (1 + o(1)) \phi(\lambda_{n-1}) = (1 + o(1)) \exp^{[p-2]} \left(\frac{\phi(\log^{[p-2]} \lambda_{n-1})}{(R^* - \varepsilon)^{1/\rho}}\right).$$

Proceeding to limits as $n \to \infty$, we get

$$R^* \leq t^*.$$

This inequality together with (3.3) and (3.4) give $YR^* \leq t^*$ for all index-pairs (p, q).

In order to prove the third part of the inequality (3.1) we assume that $Q^* < \infty$. Then, for given $\varepsilon > 0$ and $k > k_0$, we have

$$\frac{\phi(\log^{[p-2]} \lambda_k)}{\log^{[q-1]} \left| \frac{a_{k-1}}{a_k} \right|^{1/(\lambda_k - \lambda_{k-1})}} < (Q^* + \varepsilon)^{1/(\rho - A)},$$

or

$$\log \left| \frac{a_{k-1}}{a_k} \right| > (\lambda_k - \lambda_{k-1}) \exp^{[q-2]} \left(\frac{\phi(\log^{[p-2]} \lambda_k)}{(Q^* + \varepsilon)^{1/(p-A)}} \right).$$

Putting $k = n_0, n_0 + 1, \ldots, n$ in above and adding the inequalities thus obtained we have

$$\log \left| \frac{a_{n_0-1}}{a_n} \right| > \sum_{k=n_0}^{n} (\lambda_k - \lambda_{k-1}) \xi(\lambda_k) >$$

$$> \lambda_n \xi(\lambda_n) - \lambda_{n_0-1} \xi(\lambda_{n_0}) - \sum_{k=n_0+1}^{n} (\xi(\lambda_k) - \xi(\lambda_{k-1})) \lambda_{k-1},$$

where

$$\log^{[q-2]} \xi(x) = \frac{\phi(\log^{[p-2]} x)}{(Q^* + \varepsilon)^{1/(p-A)}}.$$

Hence we have

$$\text{(3.5)} \qquad \log \left| \frac{a_{n_0-1}}{a_n} \right| > \lambda_n \xi(\lambda_n) - \lambda_{n_0-1} \xi(\lambda_{n_0}) - \int_{\lambda_{n_0}}^{\lambda_n} x d[\xi(x)].$$

Consider (3.5) for $(p, q) = (2, 1)$. Then

$$\log \left| \frac{a_{n_0-1}}{a_n} \right| > O(1) + \lambda_n \xi(\lambda_n) - \int_{\lambda_{n_0}}^{\lambda_n} x d \left[\frac{\phi(x)}{(Q^* + \varepsilon)^{1/\rho}} \right] >$$

$$> O(1) + \lambda_n \log \left(\frac{\phi(\lambda_n)}{(Q^* + \varepsilon)^{1/\rho}} \right) - \frac{1}{\rho - \varepsilon} (\lambda_n - \lambda_{n_0+1}),$$

or

$$\log |a_n|^{-1/\lambda_n} > o(1) + \log \left(\frac{\phi(\lambda_n)}{(Q^* + \varepsilon)^{1/\rho}} \right) - \frac{1}{\rho - \varepsilon},$$

which implies

$$|a_n|^{-1/\lambda_n} e^{1/(\rho - \varepsilon)} > (1 + o(1)) \frac{\phi(\lambda_n)}{(Q^* + \varepsilon)^{1/\rho}}.$$

Proceeding to limits as $n \to \infty$, we have

$$Q^* \geqslant \frac{1}{e} \lim_{n\to\infty} \sup \left(\frac{\psi(\lambda_n)}{|a_n|^{-1/n}} \right)^{\rho}. \tag{3.6}$$

Next, for $(p, q) = (2,2)$, we observe that

$$\log \left| \frac{a_{m_n-1}}{a_n} \right| > \lambda_n \xi(\lambda_n) - \lambda_{m_n} \xi(\lambda_{m_n-1}) - \int_{\lambda_{m_n}}^{\lambda_n} x d[\xi(x)] > O(1) + \int_{\lambda_{m_n}}^{\lambda_n} \xi(x)\, dx.$$

Hence

$$\log |a_n|^{-1} > O(1) + \frac{1}{(Q^* + \varepsilon)^{1/(\rho-1)}} \int_{\lambda_{m_n}}^{\lambda_n} \psi(x)\, dx,$$

or

$$\log |a_n|^{-1} > O(1) + \frac{\rho - 1}{\rho (Q^* + \varepsilon)^{1/(\rho-1)}} [\{\psi(x)\}^{\rho}]_{\lambda_{m_n}}^{\lambda_n},$$

or

$$\log |a_n|^{-1/\lambda_n} > o(1) + \frac{(\rho - 1)\psi(\lambda_n)}{\rho (Q^* + \varepsilon)^{1/(\rho-1)}} \left(\frac{(\psi(\lambda_n))^{\rho-1}}{\lambda_n} - \frac{(\psi(\lambda_{m_n}))^{\rho}}{\lambda_n \psi(\lambda_n)} \right),$$

which further implies that

$$\log |a_n|^{-1/\lambda_n} > o(1) + \frac{(\rho - 1)\psi(\lambda_n)}{\rho (Q^* + \varepsilon)^{1/(\rho-1)}}$$

i.e.,

$$Q^* + \varepsilon > (1 + o(1)) \left(\frac{\rho - 1}{\rho} \right)^{\rho-1} \left(\frac{\psi(\lambda_n)}{\log |a_n|^{-1/\lambda_n}} \right)^{\rho-1}.$$

Taking limits, we have

$$Q^* \geqslant \left(\frac{\rho - 1}{\rho} \right)^{\rho-1} \lim_{n\to\infty} \sup \left(\frac{\psi(\lambda_n)}{\log |a_n|^{-1/\lambda_n}} \right)^{\rho-1}. \tag{3.7}$$

Finally, if the index-pair (p, q) of the function $f(z)$ is such that $3 \leqslant p < \infty$, then by (3.5)

$$\log\left|\frac{a_{n_k-1}}{a_n}\right| > \lambda_n \xi(\lambda_n) - \lambda_{n_k-1}\xi(\lambda_{n_k}) - \int_{\lambda_{n_k}}^{\lambda_n} x d[\xi(x)] >$$

$$> O(1) + \lambda_n \xi(\lambda_n) + \int_{\lambda_{n_k}}^{\sqrt{\lambda_n}} x d[\xi(x)] + \int_{\sqrt{\lambda_n}}^{\lambda_n} x d[\xi(x)] > O(1) + \lambda_n \xi(\lambda_n) + o(\lambda_n),$$

so that

$$\log |a_n|^{-1/\lambda_n} > o(1) + \xi(\lambda_n),$$

or

$$\log^{[p-1]} |a_n|^{-1/\lambda_n} > \frac{\phi(\log^{[p-2]}\lambda_n)}{(Q^* + \varepsilon)^{1/\rho}}.$$

Passing to limits as $n \to \infty$, we get $Q^* \geq T^*$ for $p \geq 3$. Since this inequality is seen to be true by (3.6) and (3.7) when the index-pair of the function is $(2,1)$ or $(2,2)$, we have $XQ^* \geq T^*$ and this completes the proof.

Corollary 3: Let $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ be an entire function having (p,q)-order $\rho(b < \rho < \infty)$, generalized (p,q)-type T^* and generalized lower (p,q)-type t^* such that

(i) $\log^{[p-2]}\lambda_{k-1} \simeq \log^{[p-2]}\lambda_k$ as $k \to \infty$,

(ii) $R^* = Q^*$.

Then $f(z)$ is of perfectly regular (p,q)-growth with respect to $\bar{\rho}_{p,q}(r)$, and $T^* = t^* = XR^*$.

Theorem 5: Let $f(z) = \sum_{k=0}^{\infty}$ be an entire function of (p,q)-order $\rho(b < \rho < \infty)$ and generalized (p,q)-type T^* and suppose that $|a_k/a_{k+1}|^{1/(\lambda_{k+1}-\lambda_k)}$ forms a nondecreasing function of k for $k > k_0$, then

$$T^* \geq MQ^*. \tag{3.8}$$

Proof: Let

$$\theta(k) = \left|\frac{a_k}{a_{k+1}}\right|^{1/(\lambda_{k+1}-\lambda_k)}.$$

Then

$$\log\left|\frac{a_k}{a_{k+1}}\right| = (\lambda_{k+1} - \lambda_k)\log\theta(k).$$

Adding these equations for $k = n_0, n_0 + 1, \dots, n - 1$, we get, since $\theta(k)$ is nondecreasing,

$$\log \left| \frac{a_{n_0}}{a_n} \right| = \sum_{k=n_0}^{n-1} (\lambda_{k-1} - \lambda_k) \log \theta(k) <$$

$$< \log \theta(n-1) \sum_{k=n_0}^{n-1} (\lambda_{k+1} - \lambda_k) = (\lambda_n - \lambda_{n_0}) \log \left| \frac{a_{n-1}}{a_n} \right|^{1/(\lambda_n - \lambda_{n-1})},$$

or

$$\log |a_n|^{-1/\lambda_n} < (1 + o(1)) \log \left| \frac{a_{n-1}}{a_n} \right|^{1/(\lambda_n - \lambda_{n-1})}.$$

Hence

$$\left[\frac{\phi(\log^{[p-2]} \lambda_n)}{\log^{[q-1]} |a_n|^{-1/\lambda_n}} \right]^{\rho - A} > \left[\frac{\phi(\log^{[p-2]} \lambda_n)}{\log^{[q-1]} \left| \frac{a_{n-1}}{a_n} \right|^{-1/(\lambda_n - \lambda_{n-1})}} \right]^{\rho - A}.$$

Passing to limits we get the desired result (2.8) on using [5, Thm. 1].

Remark 1: If we assume $\rho_{p,q}(r) = \rho$ for all $r > r_0$ and define $\phi(x) = x^{1/(\rho - A)}$ then Theorems 4 and 5 include the results of Juneja [3], Srivastava and Singh [16] for $(p,q) = (2,1)$ and Bajpai et al. [1], for $(p,q) = (p,1)$.

Theorem 6: Let $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ be an entire function with index-pair (p,q) and μ^* and δ^* be (p,q)-growth number and lower (p,q)-growth number, respectively of $f(z)$ with respect to a proximate order $\rho_{p,q}(r)$. Then

$$\delta^* \leq \mu^* \liminf_{k \to \infty} \frac{\log^{[p-2]} \lambda_k}{\log^{[p-2]} \lambda_{k+1}}. \tag{3.9}$$

Further, if $\theta(k) = |a_k / a_{k+1}|^{1/(\lambda_{k+1} - \lambda_k)}$ forms a strictly increasing sequence for $k > k_0$, then

$$\mu^* = Q^* \quad \text{and} \quad \delta^* = R^*, \tag{3.10}$$

where Q^* and R^* are defined in Thm. 1.

PROOF: Let r_t be the value of r at which $\nu(r)$ jumps from a value less than or equal to $\lambda_{\nu(t)}$ to a value greater than or equal to $\lambda_{\nu(t)+1}$. Then

$$\xi^* \leqslant \liminf_{t\to\infty} \frac{\log^{[p-2]}\nu(r_t-0)}{(\log^{[q-1]} r_t)^{\rho(r_t)-A}} \leqslant$$

$$\leqslant \limsup_{t\to\infty} \frac{\log^{[p-2]}\nu(r_t+0)}{(\log^{[q-1]} r_t)^{\rho(r_t)-A}} \liminf_{t\to\infty} \frac{\log^{[p-2]}\nu(r_t-0)}{\log^{[p-2]}\nu(r_t+0)} \leqslant \mu^* \liminf_{k\to\infty} \frac{\log^{[p-2]}\lambda_k}{\log^{[p-2]}\lambda_{k+1}}.$$

This proves (3.9). For proving (3.10) we have, from (0.2), for given $\varepsilon > 0$ and sufficiently large values of k,

$$\log^{[p-2]}\nu(r) < (\mu^* + \varepsilon)(\log^{[q-1]} r)^{\rho_{p,q}(r)-A}. \tag{3.11}$$

Since $\{\theta(k)\}$ forms a srtrictly increasing sequence of k, the k-th term will be the maximum term for $|z| = r$, if and only if

$$\nu(r) = \lambda_k \quad \text{and} \quad \mu(r) = |a_k|\, r^{\lambda_k}, \qquad \text{for } \theta(k-1) \leqslant r < \theta(k).$$

Thus, in view of (3.11), we have

$$\log^{[p-2]}\lambda_k < (\mu^* + \varepsilon)(\log^{[q-1]} r)^{\rho-A},$$

or

$$\phi\left(\frac{\log^{[p-2]}\lambda_k}{\mu^* + \varepsilon}\right) < \phi\left((\log^{[q-1]} r)^{\rho_{p,q}(r)-A}\right).$$

Using (1.1) and the property (1.3), we have

$$\frac{\phi(\log^{[p-2]}\lambda_k)}{(\mu^* + \varepsilon)^{1/(\rho-A)}} < \log^{[q-1]} r.$$

Hence

$$\frac{\phi(\log^{[p-2]}\lambda_k)}{\log^{[q-1]} r} < (\mu^* + \varepsilon)^{1/(\rho-A)},$$

which on taking limits gives

$$\limsup_{k\to\infty} \left[\frac{\phi(\log^{[p-2]}\lambda_k)}{\log^{[q-1]} \left|\frac{a_k}{a_{k-1}}\right|^{1/(\lambda_{k+1}-\lambda_k)}}\right]^{\rho-A} \leqslant \mu^*,$$

and hence

$$Q^* \leqslant \mu^*. \tag{3.12}$$

Further, from (0.2) we have

$$\log^{[p-2]} \nu(r) > (\mu^* - \varepsilon)(\log^{[q-1]} r)^{\rho_{n+1}(r) - A}$$

for a sequence of values of $r = r_1, r_2, \ldots, r_k \to \infty$. Thus (3.11), for k's corresponding to these values of r_k's yields

$$\log^{[p-2]} \lambda_k > (\mu^* - \varepsilon)(\log^{[q-1]} r_k)^{\rho(r_k) - A}$$

or

$$\left(\frac{\phi(\log^{[p-2]} \lambda_k)}{\log^{[q-1]} \left| \frac{a_{k-1}}{a_k} \right|^{1/(\lambda_k - \lambda_{k-1})}} \right)^{\rho(r_k) - A} > \mu^* - \varepsilon .$$

Since $\rho(r_n) \to \rho$ as $n \to \infty$, on taking limits and combining the result with (3.12), we get

$$Q^* = \mu^* .$$

The case $\delta^* = R^*$ can be handled in a similar fashion.

Remark 2: For our studies in this paper we have preferred (p,q)-growth to (α,β)-growth which was introduced by Seremata [14] and later, on extensively discussed by Balašov [2] and Shah [15].

Let L^0 denote the class of functions h satisfying the following conditions (H, i) and (H, ii):

(H, i) $h(x)$ is defined on $[a, \infty)$ and is positive strictly increasing, differentiable and tends to ∞ as $x \to \infty$.

(H, ii) $$\lim_{x \to \infty} \frac{h((1 + 1/\psi(x))x)}{h(x)} = 1,$$

for every function $\psi(x)$ such that $\psi(x) \to \infty$ as $x \to \infty$.

Let Λ denote the class of functions h satisfying conditions (H, i) and (H, iii);

(H, iii) $$\lim_{x \to \infty} \frac{h(cx)}{h(x)} = 1,$$

for every $c > 0$.

Let $f(z)$ be any entire function and suppose that $\alpha(x) \in \Lambda$, $\beta(x) \in L^0$. Write

$$\frac{\rho(\alpha, \beta, f)}{\lambda(\alpha, \beta, f)} = \lim_{r \to \infty} \frac{\sup}{\inf} \frac{\alpha(\log M(r, f))}{\beta(\log r)} .$$

Then $\rho(\alpha, \beta, f)$ is called the generalized order of f and $\lambda(\alpha, \beta, f)$ the generalized lower order of f.

It has been observed that for $\alpha = \beta$ the results of these authors are not valid (cf.

Kapoor and Nautiyal [6]). Hence to study entire functions of slow growth the functions α and β are defined in a different way for which independent discussion is required and interestingly, (p, q)-scale covers all cases simultaneously.

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