On the Growth of Coefficients of Entire Functions (**)(***)

SUMMARY. — The decomposition theorem proved in [1] for entire functions of Sato growth fails for those functions which have lower $(p, 1)$-type zero or $(p, 1)$-type infinity. Here we deal with this situation using the concept of proximate order introduced in [8].

Sulla crescita dei coefficienti delle funzioni intere

RIASSUNTO. — Il teorema di decomposizione dimostrato in [1] per le funzioni intere con crescita del tipo di Sato non si applica al caso in cui la funzione abbia uno zero di tipo $(p, 1)$ inferiore o un infinito di tipo $(p, 1)$. In questo lavoro ci si occupa appunto di questo caso, impiegando il concetto di ordine approssimato introdotto in [8].

0. - INTRODUCTION

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be a nonconstant entire function, where $\lambda_0 = 0$ and $\{\lambda_k\}_{k=1}^{\infty}$ is a strictly increasing sequence of positive integers and assume that $a_k \neq 0$ for $k = 1, 2, \ldots$. We set $M(r) = \mu(r, f) + \mu(r, f) = \max_{k=1}^{\infty} \{a_k \cdot r^k\}$ and $\nu(r, f) = \max_{k=1}^{\infty} \{\lambda_k : \mu(r) = |a_k| \cdot r^k\}$. Then $M(r), \mu(r)$ and $\nu(r)$ are called respectively the maximum modulus, maximum term and rank of the maximum term, of $f(z)$ for $|z| = r$. The concepts of index-pair $(p, q)$, $p \geq q \geq 1$, $(p, q)$-orders and $(p, q)$-types were introduced by Juneja et al. ([4],[5]).

The growth of an entire function $f(z)$ can be studied in terms of its $(p, q)$-orders and $(p, q)$-types. However, these parameters are inadequate for comparing the growth of those entire functions which are of the same $(p, q)$-orders but of infinite $(p, q)$-type. In order to refine this scale Nandan et al. [11] introduced the concept of proximate order for entire functions with index-pair $(p, q)$ as follows.

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DEFINITION 1: A positive function \( \varphi_{p,q}(r) \) defined on \([r_0, \infty), r_0 > \exp^{q-1} 1 \), is said to be a proximate order of an entire function with index-pair \((p, q)\) if

(i) \( \varphi_{p,q}(r) \to \rho(p, q) \equiv \rho \) as \( r \to \infty \), \( b < \rho < \infty \),
(ii) \( \Delta_{p,q}(r) \rho'(r) \to 0 \) as \( r \to \infty \);

where \( \varphi'_{p,q}(r) \) denotes the derivative of \( \varphi_{p,q}(r) \), \( b = 1 \) if \( p = q \), \( b = 0 \) if \( p > q \) and, for convenience, \( \Delta_{p,q}(r) = \prod_{i=0}^{q} \log^{i+1} r \).

The generalized \((p, q)\)-type \( T^* \) and generalized lower \((p, q)\)-type \( \tau^* \) of \( f(z) \) are defined as

\[
\lim_{r \to \infty} \sup_{\tau < \infty} \frac{\log^{\tau} M(r)}{(\log^{\tau+1} r)^{\frac{p}{p+1}}} = T^*(p, q) = T^*, \quad 0 \leq \tau^* \leq T^* \leq \infty.
\]

If the quantity \( T^* \) is different from zero and infinity then \( \varphi_{p,q}(r) \) is said to be the proximate order of a given entire function \( f(z) \) with index-pair \((p, q)\). Clearly, the proximate order and the corresponding generalized \((p, q)\)-type of an entire function are not uniquely determined. For example, if the function \( c/\log^{\frac{p}{p+1}} r \), \( 0 < c < \infty \) is added to the proximate order \( \varphi_{p,q}(r) \) then it can be seen that \( \varphi_{p,q}(r) + c/\log^{\frac{p}{p+1}} r \) is also a proximate order satisfying (i) and (ii) and consequently, the generalised \((p, q)\)-type turns out to be \( e^{\tau^*} T^* \). Following Levin[10], Nandan et al. [11] established that there exists a proximate order for every entire function with index-pair \((p, q)\).

In a similar manner if \( \tau^* \) is different from zero and infinity then \( \varphi_{p,q}(r) \) is said to be the lower proximate order of the entire function \( f(z) \). Kasana and Sahai[8] have proved the existence theorem for such functions.

DEFINITION 2: An entire function \( f(z) \) is said to be of generalized \((p, q)\)-growth \( \{ \rho, T^* \} \) with respect to a proximate order \( \varphi_{p,q}(r) \) if the \((p, q)\)-order of \( f(z) \) does not exceed \( \rho \), if \( f(z) \) has \((p, q)\)-order \( \rho \) the generalized \((p, q)\)-type does not exceed \( T^* \).

The generalized \((p, q)\)-growth number \( \mu^* \) and generalized lower \((p, q)\)-growth number \( \delta^* \) of \( f(z) \) are defined as

\[
\lim_{r \to \infty} \sup_{\tau < \infty} \frac{\log^{\tau+1} \nu(r)}{(\log^{\tau+2} r)^{\frac{p}{p+1}}} = \mu^*(p, q) \equiv \mu^*,
\]

where \( A = 1 \) if \((p, q) = (2, 2)\) and \( A = 0 \) if \((p, q) \neq (2, 2)\).

DEFINITION 3: An entire function \( f(z) \) of \((p, q)\)-order \( \rho(b < \rho < \infty) \) is said to be of perfectly regular \((p, q)\)-growth with respect to a proximate order \( \varphi_{p,q}(r) \) if \( 0 < \tau^* = T^* < \infty \).

DEFINITION 4: An entire function with index-pair \((p, q)\) is said to be of minimal, normal or maximal \((p, q)\)-type with respect to a proximate order according as \( T^* \) is zero, positive finite and infinite respectively.

In this paper we prove a decomposition theorem for entire functions which are not of perfectly regular \((p, q)\)-growth in reference to \( \varphi_{p,q}(r) \). The second section deals with the result which describes the growth of \( u^*_i \)'s for entire functions of \((p, q)\)-order \( 0 \) or 1.
and in other situations for entire functions of minimal generalized \((p, q)\)-type. As a last result of this section we have obtained the necessary conditions for an entire function to be of perfectly regular \((p, q)\)-growth. Finally, some inequalities connecting generalized \((p, q)\)-type and generalized lower \((p, q)\)-type of an entire function \(f(z)\) with the ratio \(|a_{n-1}/a_n|\) of the coefficients occurring in its power series expansion have been derived. Also, some results have been obtained involving coefficients and exponents of entire gap power series and generalized \((p, q)\)-growth numbers.

It is known [11, Thm. 4] that \((\log [r^{1 - \gamma} r])^{\gamma, \gamma - A}\) is a monotonically increasing function of \(r\) for \(r > r_0\). Hence we can define the function \(\psi(x)\) to be the unique solution of the equation

\[
(0.3) \quad \phi = (\log [r^{1 - \gamma} r])^{\gamma, \gamma - A} \implies \phi(x) = \log [r^{1 - \gamma} r],
\]

where \(A\) has the same meaning as in (0.2).

Consequently, we have the following results [12]:

\[
(0.4) \quad \lim_{x \to \infty} \frac{d [\log \phi(x)]}{d [\log x]} = \frac{1}{\rho - A},
\]

and for every \(\eta\) such that \(0 < \eta < \infty\),

\[
(0.5) \quad \lim_{x \to \infty} \frac{\phi(\eta x)}{\phi(x)} = \eta^{1/(\rho - A)},
\]

1. Decomposition theorem

Basically, this theorem was obtained by Bai, et al. [11] for entire functions of index \((p, 1)\) i.e., Sato growth [13]. Later on, in a subsequent paper [5], they themselves extended this to entire power series with index-pair \((p, q)\). It has been observed that their decomposition theorem fails for entire functions of infinite \((p, 1)\)-type or infinite \((p, q)\)-type and moreover, for entire functions of lower \((p, 1)\)-type or lower \((p, q)\)-type as zero. To deal with these situations we review their decomposition theorem and extend this also to entire gap power series.

**Theorem 1:** Let \(f(z) = \sum_{\beta = 0}^\infty a_{\beta} z^{\beta}\) be an entire function having \((p, q)\)-order \(\rho (\beta < \rho < \infty)\), generalized \((p, q)\)-type \(T^*\) and generalized lower \((p, q)\)-type \(t^*\), and \(\beta\) be a number such that \(0 < t^* < \beta < T^* < \infty\). Then

\[
f(z) = g_{\beta}(z) + b_{\beta}(z),
\]

where either the index-pair of \(g_{\beta}(z)\) is less than \((p, q)\) or \(g_{\beta}(z)\) is of \((p, q)\)-growth \(\{\rho, \beta\}\) with respect to \(\rho_{p, q}(r)\) of \(f(z)\) and \(b_{\beta}(z) = \sum_{k = 0}^\infty b_{\beta} z^{\beta k}\), \(b_{\beta} \neq 0, \forall k\) satisfies:

\[
(1.1) \quad t^* \geq \beta \lim_{m \to \infty} \frac{\log [\rho - 21](\phi(\lambda m))]^*}{(\phi(\log [\rho - 21] \lambda m_{*}))^*},
\]

where \(\phi(\lambda)\) is the function defined by (0.3).
Proof: Let \( g(z) = \sum_{k=0}^{\infty} c_k z^k \), where

\[
c_k = \begin{cases} 
  a_k & \text{if } |a_k| \leq \exp \left( -\lambda_k \exp^{\beta - 2}\left( \frac{g(\log^{p - 2} \lambda_k)}{\beta M^{1/(q - A)}} \right) \right), \\
  0 & \text{otherwise.}
\end{cases}
\]

Clearly, \( g(z) \) is an entire function. Assume that \( g(z) \) has the index-pair \( (p', q') \) and \((p', q')\)-order \( \rho' \). In case \( \{\lambda_k\} \) is the strictly increasing sequence of positive integers such that \( c_k \neq 0, \forall k \), then it is known \([4, \text{Thm. 1}]\) that

\[
\rho' = P(L'(p', q')),
\]

where

\[
L'(p', q') = \lim_{k \to \infty} \sup \frac{\log^{q - 1} \lambda_k}{\log^{q - 1} \left| c_m \right|^{1/\lambda_m}}.
\]

Hence the index-pair \( (p', q') \) of \( g(z) \) is of generalized \((p, q)\)-growth \( \{\rho, \beta\} \). Define

\[
b_k(z) = f(z) - g_k(z) = \sum_{k=0}^{\infty} b_k z^{\lambda_k}.
\]

Then,

\[
\log |b_2| > -\lambda_m \exp^{\beta - 2}\left( \frac{g(\log^{p - 2} \lambda_m)}{\beta M^{1/(q - A)}} \right).
\]

Let \( r_k \) be the unique root of the equation

(1.3) \[ \exp^{\beta - 2}\left( \frac{g(\log^{p - 2} \lambda_m)}{\beta M^{1/(q - A)}} \right) + \]

\[
\frac{\Delta_{p - 2} (\log^{p - 2} \lambda_m)}{\rho - A} \frac{E_{p - 2}(\log^{p - 2} \lambda_m)}{\Delta_{p - 2} (\lambda_m)} = \frac{\log r_k,}{\log r_k,}
\]

where \( E_{i}(r) = \prod_{i=0}^{\infty} \exp^{l_j} r \) and

\[
M = M(p, q) = \begin{cases} 
  (p - 1)^{-1}/p & \text{if } (p, q) = (2, 2), \\
  1/p & \text{if } (p, q) = (2, 1), \\
  1 & \text{otherwise.}
\end{cases}
\]
Further, assume that \( r_n \leq r \leq r_{n+1} \). In view of (1.2) and (1.3), we have

\[
(1.4) \quad \log M(r) \geq \log |b_k| + \lambda_m \log r \geq -\lambda_m \exp^{(r-2)} \left( \frac{\phi(\log^{p-2}(\lambda_m))}{(\beta/M)^{1/p}} \right) + \\
+ \lambda_m \log r_k = \frac{\lambda_m (\phi(\lambda_m))^{p-A}}{(\beta/M)^{1/p} (\lambda_m)^{p-A} E_{p-2} (\phi(\log^{p-2}(\lambda_m)) (\beta/M)^{1/p} (\lambda_m)^{p-A} E_{p-2})}.
\]

Using the equation (1.3) for \((p, q) = (2, 1)\), we observe that \( r_k = \phi(\lambda_m)/(\beta p)^{1/p} \) as \( k \to \infty \) (since \( \phi(\lambda_m) / \lambda_m \to 1 \) as \( k \to \infty \)) and hence (1.4) gives

\[
\log M(r) \geq \left( \frac{(\phi(\lambda_m))^{p}}{\beta (r_k + 1)^{1/(r_k+1)}} \right) = \left( \frac{(\phi(\lambda_m))^{p}}{\beta (r_k + 1)^{1/(r_k+1)}} \right) \cdot \left( \frac{(\phi(\lambda_{m+1}))^{p}}{\beta (r_k + 1)^{1/(r_k+1)}} \right).
\]

On passing to limits, we get

\[
\\^* \geq \beta \left( \frac{(\phi(\lambda_m))}{(\phi(\lambda_{m+1}))} \right).
\]

Again, for \((p, q) = (2, 2)\), we observe from (1.3) that

\[
\log r_k = \frac{(\phi(\lambda_m))^{p}}{\beta^{1/(p-1)}} + \frac{(\phi(\lambda_m))^{p}}{\beta^{1/(p-1)}} = \\
= \frac{(\phi(\lambda_m))^{p}}{\beta^{1/(p-1)}} + \frac{(\phi(\lambda_m))^{p}}{\beta^{1/(p-1)}} = \frac{(\phi(\lambda_m))^{p}}{\beta^{1/(p-1)}} (1 + \frac{1}{\beta - 1}) = \frac{(\phi(\lambda_m))^{p}}{\beta^{1/(p-1)}}.
\]

Using this in (1.4) and further application of (0.5) yields

\[
\log M(r) \geq \frac{(\phi(\lambda_m))^{p}}{\beta^{1/(p-1)}} = \frac{(\phi(\lambda_m))^{p}}{\beta^{1/(p-1)}} (\phi(\lambda_{m+1}))^{p/(p+1)}.
\]

On taking limit, we get (1.1) for the case when \((p, q) = (2, 2)\).

Finally, consider the case \( p \geq 3 \). From (1.3),

\[
\log^{p-1} r_k = \frac{\phi(\log^{p-2}(\lambda_m))}{\beta^{1/p}} \quad \text{as } k \to \infty,
\]

and from (1.4),

\[
\log M(r) > O(1) + \rho \log \phi(\lambda_m) + \log \frac{\phi(\log^{p-2}(\lambda_m))}{\beta^{1/p}} + \\
+ \sum_{i=0}^{p-1} \exp^{(i)} \left( \frac{\phi(\log^{p-2}(\lambda_m))}{\beta^{1/p}} \right) - \sum_{i=2}^{p-1} \log^{(i)} \lambda_m,
\]
or
\[
\log^{(p-1)} M(r) > (1 + o(1)) \log^{p-2} (\phi(\lambda_{n_0}))^p .
\]

Hence
\[
\frac{\log^{(p-1)} M(r)}{\log^{(q-1)} \lambda_n} > (1 + o(1)) \frac{\log^{p-2} (\phi(\lambda_{n_0}))^p}{\phi(\log^{p-2} \lambda_{n_{m+1}}) \lambda_{n_{m+1}}} .
\]

On passing to limits (consider (0.1) for \( t^* \)), we get (1.1) and hence the theorem.

2. In this section we establish the growth of coefficients for those entire functions which have zero or one as \( (p, q) \)-order and otherwise minimal generalized \( (p, q) \)-type. The next result of this section describes the necessary conditions for an entire function to be of perfectly regular \( (p, q) \)-growth with respect to a proximate order.

**Theorem 2:** Let \( f(z) \) be an entire function with index-pair \( (p, q) \) such that \( p = b \), then for every \( \epsilon > 0 \),
\[
\lim_{n \to \infty} \sup \frac{(\log^{p-2} \lambda_n)^\epsilon}{\log^{(q-1)} |a_n|^{-1/\lambda_n}} = 0 .
\]

Further, if \( p > b \) and \( f(z) \) is of minimal generalized \( (p, q) \)-type, then
\[
\lim_{n \to \infty} \sup \frac{\phi(\log^{p-2} \lambda_n)}{\log^{(q-1)} |a_n|^{-1/\lambda_n}} = 0 ,
\]
where \( b \) has the same meaning as in Definition 1.

**Proof:** Since \( p = b \), it follows from the definition of \( (p, q) \)-order that for given \( \epsilon > 0 \) and \( r > r_0 \),
\[
\log M(r, f) < \exp^{(q-2)} (\log^{(q-1)} r)^b + \epsilon .
\]

Using Cauchy's inequality we get
\[
\log |a_n| < \exp^{(q-2)} (\log^{(q-1)} r)^b + \epsilon - \log r .
\]

Choose the value of \( r \) satisfying
\[
r = \exp^{(q-1)} \left( \frac{\lambda_n}{b + \epsilon} \right)^{1/(b + \epsilon)} .
\]

For \( (p, q) = (2, 1) \), (2.2) implies \( r = (\lambda_n/\epsilon)^{1/\epsilon} \) and using this value in (2.1), we get
\[
|a_n| < \left( \frac{\epsilon x}{\lambda_n} \right)^{1/\epsilon} ,
\]
or
\[ |a_n|^{1/\lambda_n} < \left( \frac{e \varepsilon}{\lambda_n} \right)^{1/\lambda_n}, \]

which on taking limits gives
\[ \lim_{n \to \infty} \sup \lambda_n^{1/\lambda_n} |a_n|^{1/\lambda_n} < \infty. \]  

In case of \((p, q) = (2, 2)\) we observe that \(\log r = (\lambda_n/(1 + \varepsilon))^{1/(1 + \varepsilon)}\) satisfies (2.2) and (2.1) is reduced to
\[ \log |a_n| < \frac{\lambda_n}{1 + \varepsilon} - \lambda_n \left( \frac{\lambda_n}{1 + \varepsilon} \right)^{1/(1 + \varepsilon)}, \]
or
\[ \log |a_n|^{-1/\lambda_n} > (1 + o(1)) \left( \frac{\lambda_n}{1 + \varepsilon} \right)^{1/(1 + \varepsilon)}. \]

Thus,
\[ \lim_{n \to \infty} \sup \frac{\lambda_n^{1/(1 + \varepsilon)}}{\log |a_n|^{-1/\lambda_n}} \leq 1. \]

Finally, for \((p, q) \neq (2, 1)\) and \((p, q) \neq (2, 2)\), (2.1) and (2.2) give
\[ \log^{p - 1} |a_n|^{-1/\lambda_n} > (1 + o(1)) \left( \log^{p - 1} \frac{\lambda_n}{\varepsilon} \right)^{1/\lambda_n}, \quad p > q, \]
or
\[ \log^{p - 1} |a_n|^{-1/\lambda_n} > (1 + o(1)) \left( \log^{p - 1} \frac{\lambda_n}{1 + \varepsilon} \right)^{1/(1 + \varepsilon)}, \quad p = q. \]

This means that for all \(p > q \geq 3,\)
\[ \lim_{n \to \infty} \sup \frac{(\log^{p - 1} \lambda_n)^{1/(1 + \varepsilon)}}{\log^{q - 1} |a_n|^{-1/\lambda_n}} \leq 1. \]

Clearly, (2.3), (2.4) and (2.5) combine to give
\[ \lim_{n \to \infty} \sup \frac{(\log^{p - 1} \lambda_n)^{1/\varepsilon}}{\log^{q - 1} |a_n|^{-1/\lambda_n}} < \infty, \quad \text{for every } \varepsilon > 0. \]

If the \(\limsup\) in (2.6) is finite and positive for some \(\varepsilon > 0\) then for every \(\alpha > 0,\) we have
\[ \lim_{n \to \infty} \sup \frac{(\log^{p - 1} \lambda_n)^{1/\varepsilon + \alpha}}{\log^{q - 1} |a_n|^{-1/\lambda_n}} = \infty. \]
(2.7) is a contradiction to what we obtained in (2.6) and thus the first part is proved.

In case when $p > b$, Kasana [5] has proved that generalized $(p, q)$-type of an entire function is given by

\[
T^* = \lim_{s \to -} \sup_{z \in M} \left( \frac{\phi(\log^{p-2} \lambda_k)}{\log^{p-1} |a_n|^{-1/\omega_n}} \right)^{s-A}.
\]

If we put $T^* = 0$ in (2.8), the second result is immediate.

Finally, we study the subsequence $\{n_k\}$ of $\lambda_n$ such that, for $f = \sum_{n=0}^\infty a_n z^{\lambda_n}$, one has

\[
|a_{n_k-1}(f)| > |a_{n_k}(f)| \quad \text{and} \quad a_n(f) = a_{n_k-1}(f) \quad \text{for} \quad \lambda_{n_k-1} \leq n < \lambda_{n_k}.
\]

The next theorem shows how this sequence influences the growth of an entire function in reference to its generalized $(p, q)$-type and generalized lower $(p, q)$-type. This also describes the condition for $f$ to be an entire function of perfectly regular $(p, q)$-growth with respect to a proximate order.

\textbf{Theorem 3:} Let $f(z)$ be an entire function having $(p, q)$-order $\rho(b < p < \infty)$, generalized $(p, q)$-type $T^*$ and generalized lower $(p, q)$-type $t^*$. Let $\{\lambda_{n_k}\}$ be the sequence defined by (2.9). Then

\[
t^* \leq T^* \lim_{k \to -} \inf \left( \frac{\phi(\log^{p-2} \lambda_{n_k-1})}{\phi(\log^{p-2} \lambda_{n_k})} \right)^{s-A}, \quad p \geq 3.
\]

Further, if $\{\lambda_{n_k}\}$ be the sequence of principal indices satisfying $\lambda_{n_k-1} = \lambda_{n_k}$ as $k \to -\infty$, then

\[
t^* \leq T^* \lim_{k \to -} \inf \left( \frac{\phi(\lambda_{n_k-1})}{\phi(\lambda_{n_k})} \right)^{s-A}.
\]

\textbf{Proof:} Let us define a function $u(z)$ such that

\[
u(z) = \sum_{n=1}^\infty (a_{n-1}(f) - a_n(f)) z^{\lambda_n} = \sum_{n=1}^\infty a_q(f) z^{\lambda_n},
\]

where

\[
a_q(f) = a_{n_q-1}(f) - a_{n_q}(f).
\]

In view of the definition (2.9) it can be proved that $u(z)$ and $f(z)$ have the same $(p, q)$-order and generalized $(p, q)$-types such that

\[T^*(f) = T^*(u) \quad \text{and} \quad t^*(f) = t^*(u).
\]
Thus, using (2.8) it can be shown that

\[
\frac{T^* (f)}{M} = \lim_{m \to \infty} \sup \left[ \frac{\phi(\log^{p-2} \lambda_{m,1})}{\log^{p-1} \alpha_{m}^{-1/2m}} (f) \right]^{a_A}.
\]

Considering the above formula and Theorem 2 of Kasana et al. [8], we observe that for \( p \geq 3 \),

\[
t^* (f) = \max_{(\lambda_m)} \left\{ \lim_{m \to \infty} \min_{\left( \lambda_{m,1} \right)} \left( \frac{\log^{p-2} \lambda_{m,1}}{\log^{q-1} \alpha_{m}^{-1/2m}} \right)^{a_A} \right\} \leq \max_{(\lambda_m)} \left\{ \lim_{m \to \infty} \sup \left( \frac{\phi(\log^{p-2} \lambda_{m,1})}{\log^{q-1} \alpha_{m}^{-1/2m}} (f) \right)^{a_A} \right\}.
\]

Similarly, for the case \( p = 2 \) and \( q = 1 \), let \( \{\lambda_m\} \) be the sequence of principal indices such that \( \lambda_{m,1} = \lambda_m \) as \( k \to \infty \), we have

\[
t^* (f) \leq T^* (f) \lim_{k \to \infty} \inf \left( \frac{\phi(\lambda_{m,1})}{\phi(\lambda_m)} \right)^{a_A}.
\]

**Corollary 1**: If \( f(z) \) is an entire function of perfectly regular \((p,q)\)-growth with respect to \( \rho_{p,q} (r) \), then

\[
\log^{p-2} \lambda_{m,1} = \log^{p-2} \lambda_m \quad \text{as} \quad k \to \infty.
\]

**Corollary 2**: If \( f(z) \) is an entire function having \((p,q)\)-order \( \rho(b < \rho < \infty) \), \((p,q)\)-type \( T \) and lower \((p,q)\)-type \( t \) such that \( 0 \leq t \leq T < \infty \), then

\[
t \leq T \lim_{k \to \infty} \inf \frac{\log^{p-2} \lambda_{m,1}}{\log^{p-2} \lambda_m}, \quad p \geq 3.
\]

This inequality also holds for \( p = 2 \) if \( \{\lambda_m\} \) is the sequence of principal indices satisfying \( \lambda_{m,1} = \lambda_m \) as \( k \to \infty \).

3. This section contains various inequalities, some of which are extensions of results in [3, 16] and [1].

**Theorem 4**: Let \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) be an entire function having \((p,q)\)-order \( \rho(b < \rho < \infty) \), generalized \((p,q)\)-type \( T^* \) and generalized lower \((p,q)\)-type \( t^* \). Then

\[
YR^* \leq t^* \leq T^* \leq XQ^* ,
\]
where
\[ R^* = R^*(p, q) = \lim_{k \to \infty} \inf \left[ \frac{\phi(\log^{(p-2)}(\lambda_{k-1}))}{\log^{(p-1)}(d_{k-1}/d_k)^{1/(\lambda_k - \lambda_{k-1})}} \right]^{1-A}, \]
\[ Q^* = Q^*(p, q) = \lim_{k \to \infty} \sup \left[ \frac{\phi(\log^{(p-2)}(\lambda_k))}{\log^{(p-1)}(d_{k-1}/d_k)^{1/(\lambda_k - \lambda_{k-1})}} \right]^{1-A}. \]

\[ X = 1/p \text{ if } p = 2, \ X = 1 \text{ if } p \geq 3, \text{ and} \]

\[ Y = Y(p, q) = \begin{cases} \frac{\epsilon^{(p-1)}}{\rho} & \text{if } (p, q) = (2, 1), \\ \frac{1}{\rho} \left( \frac{p-1}{\rho - \alpha} \right)^{1/(p-1)} & \text{if } (p, q) = (2, 2), \\ 1 & \text{otherwise}, \end{cases} \]

such that
\[ \alpha = \lim_{k \to \infty} \inf \frac{\lambda_k-1}{\lambda_k}. \]

Proof: From definition, \( R^* \geq 0 \). If \( R^* = 0 \), first part of the inequality (3.1) is trivial. Hence, let \( R^* > 0 \). In this case, for given \( \epsilon > 0 \) and \( k > k_0 \), we have
\[ \frac{\phi(\log^{(p-2)}(\lambda_{k-1}))}{\log^{(p-1)}(d_{k-1}/d_k)^{1/(\lambda_k - \lambda_{k-1})}} > (R^* - \epsilon)^{1/(p-A)}, \]
or
\[ \log \left| \frac{d_{k-1}}{d_k} \right| < (\lambda_k - \lambda_{k-1}) \exp^{(p-2)} \left( \frac{\phi(\log^{(p-2)}(\lambda_{k-1}))}{(R^* - \epsilon)^{1/(p-A)}} \right). \]

Putting \( k = n_0, n_0 + 1, \ldots, n \) in above and adding the inequalities thus obtained, we get
\[ \log \left| \frac{d_{n-1}}{d_{n}} \right| < \sum_{k = n_0}^{n} (\lambda_k - \lambda_{k-1}) \exp^{(p-2)} \left( \frac{\phi(\log^{(p-2)}(\lambda_{k-1}))}{(R^* - \epsilon)^{1/(p-A)}} \right) < \]
\[ < \lambda_1 \phi(\lambda_{n-1}) - \lambda_{n-1} \phi(\lambda_{n-2}) - \sum_{k = n_0 + 1}^{n} (\phi(\lambda_{k-1}) - \phi(\lambda_{k-2})) \lambda_{k-1}, \]
where
\[ \log(q - 2) \phi(x) = \frac{\phi(\log(q - 2) x)}{(R^a - \epsilon)^{1/A}}. \]

Hence
\[ \log \left| \frac{a_{n-1}}{a_n} \right| < \lambda_n \phi(\lambda_n - 1) - \lambda_{n-1} \phi(\lambda_{n-1}) - \int_{\lambda_{n-1}}^{\lambda_n} x d[\phi(x)]. \]

Considering (3.2) for \((p, q) = (2, 1)\),
\[ \log \left| \frac{a_n}{a_{n-1}} \right| > \lambda_{n-1} \phi(\lambda_{n-1}) - \lambda_n \phi(\lambda_n - 1) + \int_{\lambda_{n-1}}^{\lambda_n} x d \left[ \log \frac{\phi(x)}{(R^a - \epsilon)^{1/p}} \right], \]
or
\[ \log |a_n| > O(1) - \lambda_n \phi(\lambda_n - 1) + \int_{\lambda_{n-1}}^{\lambda_n} \frac{x \phi'(x)}{\phi(x)} dx. \]

Using the property (1.2), we get
\[ \log |a_n|^{-1/\rho} < o(1) + \phi(\lambda_n - 1) - \frac{1}{\rho + \epsilon} \frac{\lambda_{n-1}}{\lambda_n} \leq o(1) + \log \frac{\phi(\lambda_n - 1)}{(R^a - \epsilon)^{1/p}} - \frac{\rho}{\rho + \epsilon}. \]

Hence
\[ e^{\rho/\epsilon + o(1)} (R^a - \epsilon) < (1 + o(1)) \left( \frac{\phi(\lambda_n - 1)}{|a_n|^{-1/\rho_n}} \right)^{\rho}. \]

Passing to limits and using [9, Thm. 1], we get
\[ e^{\rho - 1} R^a \leq \rho^a. \]

For the case \((p, q) = (2, 2)\), inequality (3.2) yields
\[ \log \left| \frac{a_n}{a_{n-1}} \right| > O(1) - \lambda_n \phi(\lambda_n - 1) + \int_{\lambda_{n-1}}^{\lambda_n} x d[\phi(x)] = \]
\[ O(1) - \lambda_n \phi(\lambda_n - 1) + \lambda_{n-1} \phi(\lambda_{n-1}) - \int_{\lambda_{n-1}}^{\lambda_n} \phi(x) dx = \]
\[ = O(1) - \lambda_n \phi(\lambda_n - 1) + \lambda_{n-1} \phi(\lambda_{n-1}) - \frac{\rho - 1}{\rho (R^a - \epsilon)^{1/(\rho - 1)}} \left[ \phi(x) \right]_{\lambda_{n-1}}^{\lambda_n}. \]
or
\[
\log |a_n|^{1/k_n} < \left(1 - \frac{\lambda_n - 1}{\lambda_n}\right) \frac{\phi(\lambda_n - 1)}{(R^* - \varepsilon)^{1/(\beta - 1)}} + (1 + o(1)) \frac{\varepsilon - 1}{\rho} \frac{\phi(\lambda_n - 1)}{(R^* - \varepsilon)^{1/(\beta - 1)}} \]
\[= \left(1 - \alpha + \frac{\varepsilon - 1}{\rho}\right) \frac{\phi(\lambda_n - 1)}{(R^* - \varepsilon)^{1/(\beta - 1)}}.
\]

Thus
\[R^* - \varepsilon < (1 + o(1)) \left(\frac{\varepsilon - 1}{\rho}\right)^{1/2} \left(\frac{\phi(\lambda_n - 1)}{\log |a_n|^{1/k_n}}\right)^{1/2}.
\]

Passing to limits as \(n \to \infty\) (again, in view of [9, Thm. 1]; we get
\[\alpha^{1/(\beta - 1)} \left(\frac{\varepsilon - 1}{\rho - \alpha}\right)^{1/2} R^* \leq t^*.
\]

Finally, let us consider the case when \((p, q) \neq (2, 1)\) and \((p, q) \neq (2, 2)\). In this situation (3.2) is reduced to
\[
\log \left|\frac{\sigma_{n+1}}{\sigma_n}\right| = O(1) + \lambda_n \phi(\lambda_n - 1) - \int_{\lambda_n - 1}^{\lambda_n} t d[\psi(t)] <
\]
\[< O(1) + \lambda_n \phi(\lambda_n - 1) - \lambda_n - 1 \phi(\lambda_n - 1) < O(1) + \lambda_n \phi(\lambda_n - 1) - \lambda_n - 1 \phi(\lambda_n - 1),
\]
or
\[
\log |a_n|^{1/k_n} < (1 + o(1)) \phi(\lambda_n - 1) = (1 + o(1)) \exp^{[\rho - 1]} \left(\frac{\phi(\log^{[\rho - 1]} \lambda_n - 1)}{(R^* - \varepsilon)^{1/\rho}}\right).
\]

Proceeding to limits as \(n \to \infty\), we get
\[R^* \leq t^*.
\]

This inequality together with (3.3) and (3.4) give \(YR^* \leq t^*\) for all index-pairs \((p, q)\).

In order to prove the third part of the inequality (3.1) we assume that \(Q^* < \infty\). Then, for given \(\varepsilon > 0\) and \(k > k_0\), we have
\[
\frac{\phi(\log^{[\rho - 1]} \lambda_k)}{\log^{[\rho - 1]} \left|\frac{\sigma_{k+1}}{\sigma_k}\right|^{1/(\lambda_k - \lambda_{k-1})}} < (Q^* + \varepsilon)^{1/\rho - \lambda_0},
\]
or

\[
\log \left| \frac{a_{k-1}}{a_k} \right| > (\lambda_k - \lambda_{k-1}) \exp^{(p-2)} \left( \frac{\phi(\log^{(p-2)} x)}{(Q^* + \varepsilon)^{1/(p-1)}} \right).
\]

Putting \( k = n_0, n_0 + 1, \ldots, n \) in above and adding the inequalities thus obtained we have

\[
\log \left| \frac{a_{n_0-1}}{a_{n_0}} \right| > \sum_{k=n_0}^{n} (\lambda_k - \lambda_{k-1}) \xi(\lambda_k) > \lambda_n \xi(\lambda_n) - \lambda_{n_0-1} \xi(\lambda_{n_0}) - \sum_{k=n_0+1}^{n} (\xi(\lambda_k) - \xi(\lambda_{k-1})) \lambda_{k-1},
\]

where

\[
\log^{(p-2)} \xi(x) = \frac{\phi(\log^{(p-2)} x)}{(Q^* + \varepsilon)^{1/(p-1)}}.
\]

Hence we have

\[(3.5) \quad \log \left| \frac{a_{n_0-1}}{a_{n_0}} \right| > \lambda_n \xi(\lambda_n) - \lambda_{n_0-1} \xi(\lambda_{n_0}) - \int_{\xi(\lambda_n)}^{\xi(\lambda_{n_0})} x d[\xi(x)].
\]

Consider (3.5) for \((p, q) = (2, 1)\). Then

\[
\log \left| \frac{a_{n_0-1}}{a_{n_0}} \right| > O(1) + \lambda_n \xi(\lambda_n) - \int_{\xi(\lambda_n)}^{\xi(\lambda_{n_0})} \frac{\phi(x)}{(Q^* + \varepsilon)^{1/p}} > O(1) + \lambda_n \log \left( \frac{\phi(\lambda_n)}{(Q^* + \varepsilon)^{1/p}} \right) - \frac{1}{p - \varepsilon} (\lambda_n - \lambda_{n_0+1}),
\]

or

\[
\log |a_n|^{-1/p} > o(1) + \log \left( \frac{\phi(\lambda_n)}{(Q^* + \varepsilon)^{1/p}} \right) - \frac{1}{p - \varepsilon},
\]

which implies

\[
|a_n|^{-1/p \cdot e^{1/p - \varepsilon}} > (1 + o(1)) \frac{\phi(\lambda_n)}{(Q^* + \varepsilon)^{1/p}}.
\]
Proceeding to limits as \( n \to \infty \), we have

\[
Q^n \geq \frac{1}{\varepsilon} \lim_{n \to \infty} \sup \left( \frac{\phi(\lambda_n)}{|a_n|^{-1/n}} \right)\]

Next, for \((p, q) = (2, 2)\), we observe that

\[
\log \left| \frac{d_{\lambda_n} - 1}{a_n} \right| > \lambda_n \tilde{\eta}(\lambda_n) - \lambda_n \tilde{\eta}(\lambda_n - 1) - \int_{\lambda_n}^{\lambda_n} s d[\tilde{\xi}(x)] > O(1) + \int_{\lambda_n}^{\lambda_n} \tilde{\xi}(x) dx.
\]

Hence

\[
\log |a_n|^{-1} > O(1) + \frac{1}{(Q^n + \varepsilon)^{1/\lambda_n - 1}} \int_{\lambda_n}^{\lambda_n} \tilde{\xi}(x) dx,
\]

or

\[
\log |a_n|^{-1} > O(1) + \frac{\varepsilon - 1}{\varepsilon (Q^n + \varepsilon)^{1/\lambda_n - 1}} \left[ \{\tilde{\xi}(x)\}^{\lambda_n} \right]
\]

or

\[
\log |a_n|^{-1/\lambda_n} > o(1) + \frac{(\varepsilon - 1) \tilde{\eta}(\lambda_n)}{\varepsilon (Q^n + \varepsilon)^{1/\lambda_n - 1}} \left( \frac{\phi(\lambda_n)}{\lambda_n} - \frac{\phi(\lambda_n)}{\lambda_n \phi(\lambda_n)} \right),
\]

which further implies that

\[
\log |a_n|^{-1/\lambda_n} > o(1) + \frac{(\varepsilon - 1) \tilde{\eta}(\lambda_n)}{\varepsilon (Q^n + \varepsilon)^{1/\lambda_n - 1}}
\]

i.e.,

\[
Q^n + \varepsilon > (1 + o(1)) \left( \frac{\varepsilon - 1}{\varepsilon} \right)^{1 - \frac{1}{\lambda_n}} \left( \frac{\phi(\lambda_n)}{\log |a_n|^{-1/\lambda_n}} \right)^{1 - \frac{1}{\lambda_n}}
\]

Taking limits, we have

\[
Q^n \geq \left( \frac{\varepsilon - 1}{\varepsilon} \right)^{1 - \frac{1}{\lambda_n}} \lim_{n \to \infty} \sup \left( \frac{\phi(\lambda_n)}{\log |a_n|^{-1/\lambda_n}} \right)^{1 - \frac{1}{\lambda_n}}
\]

Finally, if the index-pair \((p, q)\) of the function \( f(x) \) is such that \( 3 \leq p < \infty \), then
by (3.5)

\[ \log \left| \frac{a_n - 1}{a_n} \right| > \lambda_n \xi(\lambda_n) - \lambda_{n-1} \xi(\lambda_{n-1}) - \int_{\lambda_{n-1}}^{\lambda_n} \int \overline{a}(z) \overline{\xi(z)}\,dz > 0(1) + \lambda_n \xi(\lambda_n) + o(\lambda_n), \]

so that

\[ \log \left| \frac{a_n}{a_{n+1}} \right|^{-1/\lambda_n} > o(1) + \xi(\lambda_n), \]

or

\[ \log^{(\rho - 1)} \left| \frac{a_n}{a_{n+1}} \right|^{-1/\lambda_n} > \frac{\rho(\log^{(\rho - 1)} \lambda_n)}{(Q^* + \varepsilon)^{1/\rho}}. \]

Passing to limits as \( n \to \infty \), we get \( Q^* \geq T^* \) for \( p \geq 3 \). Since this inequality is seen to be true by (3.6) and (3.7) when the index-pair of the function is \( (2, 1) \) or \( (2, 2) \), we have \( XQ^* \geq T^* \) and this completes the proof.

**Corollary 3:** Let \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) be an entire function having \((p, q)\)-order \( \rho(\rho < p, q < \infty) \), generalized \((p, q)\)-type \( T^* \) and generalized lower \((p, q)\)-type \( t^* \) such that

(i) \( \log^{(\rho - 1)} \lambda_{k-1} = \log^{(\rho - 1)} \lambda_k \) as \( k \to \infty \),

(ii) \( R^* = Q^* \).

Then \( f(z) \) is of perfectly regular \((p, q)\)-growth with respect to \( \rho_{p, q}(r) \), and \( T^* = t^* = XR^* \).

**Theorem 5:** Let \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) be an entire function of \((p, q)\)-order \( \rho(\rho < p, q < \infty) \) and generalized \((p, q)\)-type \( T^* \) and suppose that \( \left| \frac{a_k}{a_{k+1}} \right|^{1/(\lambda_{k+1} - \lambda_k)} \) forms a nondecreasing function of \( k \) for \( k > k_0 \), then

\[ T^* \geq MQ^*. \]

**Proof:** Let

\[ \theta(k) = \left| \frac{a_k}{a_{k+1}} \right|^{1/(\lambda_{k+1} - \lambda_k)}. \]

Then

\[ \log \left| \frac{a_k}{a_{k+1}} \right| = (\lambda_{k+1} - \lambda_k) \log \theta(k). \]
Adding these equations for \( k = n_0, n_0 + 1, \ldots, n - 1 \), we get, since \( \theta(k) \) is nondecreasing,

\[
\log \left| \frac{a_k}{a_n} \right| = \sum_{k=n_0}^{n-1} (\lambda_{k+1} - \lambda_k) \log \theta(k) < 
\]

\[
< \log \theta(n-1) \sum_{k=n_0}^{n-1} (\lambda_{k+1} - \lambda_k) = (\lambda_n - \lambda_{n_0}) \log \left| \frac{a_{n-1}}{a_n} \right|^{1/(\lambda_n - \lambda_{n_0})},
\]

or

\[
\log |a_k|^{-1/\lambda_k} < (1 + o(1)) \log |a_{n-1}|^{1/(\lambda_n - \lambda_{n_0})}.
\]

Hence

\[
\left[ \frac{\phi(\log^{(p-2)/n} \lambda_k)}{\log^{(p-2)/n} |a_k|^{-1/\lambda_k}} \right]^{-\lambda_k} > \left[ \frac{\phi(\log^{(p-2)/n} \lambda_n)}{\log^{(p-2)/n} |a_{n-1}|^{-1/\lambda_n}} \right]^{-\lambda_n}.
\]

Passing to limits we get the desired result (2.8) on using [5, Thm. 1].

**Remark 1:** If we assume \( \rho_{p,q}(r) = \rho \) for all \( r > n_0 \) and define \( \phi(x) = x^{1/(\rho - A)} \) then Theorems 4 and 5 include the results of Dudeja [3], Srivastava and Singh [16] for \((p, q) = (2, 1)\) and Bajpai et al. [1], for \((p, q) = (p, 1)\).

**Theorem 6:** Let \( f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k} \) be an entire function with index-pair \((p, q)\) and \( \mu^* \) and \( \delta^* \) be \((p, q)\)-growth number and lower \((p, q)\)-growth number, respectively of \( f(z) \) with respect to a proximate order \( \rho_{p,q}(r) \). Then

\[
(3.9) \quad \delta^* \leq \mu^* \liminf_{|z| \to \infty} \frac{\log^{(p-2)/n} \lambda_k}{\log^{(p-2)/n} \lambda_{k+1}}.
\]

Further, if \( \theta(k) = |a_k/a_{k+1}|^{1/\lambda_k} \) forms a strictly increasing sequence for \( k > k_0 \), then

\[
(3.10) \quad \mu^* = Q^* \quad \text{and} \quad \delta^* = R^*,
\]

where \( Q^* \) and \( R^* \) are defined in Thm. 1.
Proof: Let $r_1$ be the value of $r$ at which $v(r)$ jumps from a value less than or equal to $\lambda_{(0)}$ to a value greater than or equal to $\lambda_{(0)+1}$. Then

$$z^* \leq \lim_{k \to \infty} \inf \left( \frac{\log^{(p-k)} v(r_1 - 0)}{\log^{(p-k)} v(r_1 + 0)} \right) \leq \lim_{k \to \infty} \sup \left( \frac{\log^{(p-k)} v(r_1 + 0)}{\log^{(p-k)} v(r_1 - 0)} \right) \leq \mu^* \lim_{k \to \infty} \inf \left( \frac{\log^{(p-k)} \lambda_k}{\log^{(p-k)} \lambda_{k+1}} \right).$$

This proves (3.9). For proving (3.10) we have, from (0.2), for given $\epsilon > 0$ and sufficiently large values of $k$,

$$\log^{(p-k)} v(r) < (\mu^* + \epsilon)(\log^{(p-k)} v)^{\mu^*(\epsilon - A)},$$

Since $\{\theta(k)\}$ forms a strictly increasing sequence of $k$, the $k$-th term will be the maximum term for $|z| = r$, if and only if

$$v(r) = \lambda_k \quad \text{and} \quad \mu(r) = \frac{\lambda_k}{r^{\lambda_k}}, \quad \text{for } \theta(k-1) \leq r < \theta(k).$$

Thus, in view of (3.11), we have

$$\log^{(p-k)} \lambda_k < (\mu^* + \epsilon)(\log^{(p-k)} v)^{\mu^*(\epsilon - A)},$$

or

$$\frac{\Phi(\log^{(p-k)} \lambda_k)}{\mu^* + \epsilon} < \Phi(\log^{(p-k)} v)^{\mu^*(\epsilon - A)}.$$\)

Using (1.1) and the property (1.3), we have

$$\frac{\Phi(\log^{(p-k)} \lambda_k)}{(\mu^* + \epsilon)^{1/(\epsilon - A)}} < \log^{(p-k)} v.$$\)

Hence

$$\frac{\Phi(\log^{(p-k)} \lambda_k)}{\log^{(p-k)} v} < (\mu^* + \epsilon)^{1/(\epsilon - A)},$$

which on taking limits gives

$$\lim_{k \to \infty} \sup \left[ \frac{\Phi(\log^{(p-k)} \lambda_k)}{\log^{(p-k)} v} \right]^{\epsilon - A} \leq \mu^*,$$

and hence

$$Q^* \leq \mu^*.$$
Further, from (0.2) we have
\[ \log^{b-2\epsilon} n(r) > (\mu_{x} - \epsilon)(\log^{a+1} n(r) n^{a+\lambda}) \]
for a sequence of values of \( r = r_{1}, r_{2}, \ldots, r_{k} \to \infty \). Thus (3.11), for \( k \)'s corresponding to these values of \( n_{k} \)'s yields
\[ \log^{a+1} n_{k} > (\mu_{y} - \epsilon)(\log^{a+1} n_{k} n_{k}^{a+\lambda}) \]
or
\[ \left( \phi(\log^{a+1} n_{k}) \frac{n_{k}^{a+\lambda}}{a+1} \right)^{\frac{a+1}{a+\lambda}} > \mu_{x} - \epsilon. \]
Since \( \rho(n_{k}) \to \rho \) as \( n \to \infty \), on taking limits and combining the result with (3.12), we get
\[ Q^{*} = \mu^{*}. \]
The case \( z^{*} = R^{*} \) can be handled in a similar fashion.

Remark 2: For our studies in this paper we have preferred \( (\rho, \varphi) \)-growth to \( (\alpha, \beta) \)-growth which was introduced by Seremata [14] and later, on extensively discussed by Balasov [2] and Shah [15].

Let \( L^{0} \) denote the class of functions \( b \) satisfying the following conditions (H, i) and (H, ii):

(H, i) \hspace{1cm} \( b(x) \) is defined on \( [a, \infty) \) and is positive strictly increasing, differentiable and tends to \( \infty \) as \( x \to \infty \).

(H, ii) \hspace{1cm} \( \lim_{x \to \infty} \frac{b((1 + 1/\psi(x))^{-1})}{b(x)} = 1 \),

for every function \( \psi(x) \) such that \( \psi(x) \to \infty \) as \( x \to \infty \).

Let \( \Lambda \) denote the class of functions \( b \) satisfying conditions (H, i) and (H, iii):

(H, iii) \hspace{1cm} \( \lim_{x \to \infty} \frac{b(x)}{b(x)} = 1 \),

for every \( \epsilon > 0 \).

Let \( f(x) \) be any entire function and suppose that \( \alpha(x) \in \Lambda \), \( \beta(x) \in L^{0} \). Write
\[ \rho(\alpha, \beta, f) = \sup \{ \log M(r, f) \} \]
\[ \lambda(\alpha, \beta, f) = \lim_{r \to \infty} \inf \frac{\beta(\log r)}{\alpha(\log r)}. \]

Then \( \rho(\alpha, \beta, f) \) is called the generalized order of \( f \) and \( \lambda(\alpha, \beta, f) \) the generalized lower order of \( f \).

It has been observed that for \( \alpha = \beta \) the results of these authors are not valid (cf.
Kapoor and Nautiyal [6]). Hence to study entire functions of slow growth the functions $\alpha$ and $\beta$ are defined in a different way for which independent discussion is required and interestingly, $(p, q)$-scale covers all cases simultaneously.

REFERENCES


