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Fractions for Exact Categories (**) (***)

ABSTRACT. — This paper deals with fractions for exact categories, in the sense of Puppe-Mitchell. If A is exact, we construct the *exact* categories of fractions of A by means of a three-arrow calculus and characterize the sets of morphisms whose inversion produces them; such categories of fractions coincide with the quotients of A modulo a thick subcategory.

Frazioni per categorie esatte

SOMMARIO. — Si studiano le categorie di frazioni per categorie esatte, nel senso di Puppe-Mitchell. Se A è esatta, si costruiscono le, categorie *esatte* di frazioni di A mediante un calcolo ternario e si caratterizzano gli insiemi di morfismi la cui inversione produce tali categorie di frazioni; queste ultime coincidono con i quozienti di A modulo una sottocategoria spessa.

INTRODUCTION

Categories of fractions, appearing informally in the *langage modulo C* of Serre [Se], were formalised for abelian categories by Grothendieck [Gr] and Gabriel [Ga] and extended to general category theory in Gabriel-Zisman [GZ]. Quite recently, Bérnabou [Be] gave a thorough study of this topic for various categorical frames, among which the category **REG** of regular categories.

We consider here the case of exact categories and exact functors, in the sense of Puppe-Mitchell: a situation where homological algebra can be developed to a good extent [GV1, 2], with various advantages on the more restricted and rigid frame of abelian categories; in particular, the notion of distributive homological algebra and the tool of universal models for spectral sequences [G3].

Given an exact functor $F: A \to B$ between exact categories, consider its *isokernel* Ikr F, i.e. the set of the morphisms of A transformed by F into isomorphisms and its *annihilation kernel* Ker F, i.e. the set of the objects of A annihilated by F. Trivially,

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these two sets determine each other:

(1) $\quad \mathrm{Ikr}\, F = \{f \mid \mathrm{Ker}\, f,\ \mathrm{Cok}\, f \in \mathrm{Ker}\, F\}, \qquad \mathrm{Ker}\, F = \{A \mid (0\colon A \to A) \in \mathrm{Ikr}\, F\};$

equivalently, we also consider the full subcategory $\mathbf{Ker}\, F$ of A whose objects are in $\mathrm{Ker}\, F$.

A set Σ of morphisms of the exact category A is said to be an *exact isokernel* if it satisfies the closure conditions (ik.0-3) of § 3. The isokernel of any exact functor from A is trivially so. Conversely, if Σ is an exact isokernel of A, the category of fractions $\Sigma^{-1}A$ has a *three-arrow calculus* (§ 5) by whose means we can prove that this category and the projection functor $P\colon A \to \Sigma^{-1}A$ are exact, with $\mathrm{Ikr}\, P = \Sigma$ (Thm. 7). Thus, the exact isokernels coincide with the isokernels of exact functors, while the exact categories of fractions are precisely those whose isokernel is exact.

On the other hand, the annihilation kernels of the functors coincide with the *thick subcategories*, as in the abelian case (Thm. 11). If K is a thick subcategory of A and Σ is the associated exact isokernel, the exact category $A/K = \Sigma^{-1}A$ solves the universal problem of annihilating the obejcts of K. This also proves that an exact category of fraction of an abelian category is abelian, as it happens in the frame of regular categories [Be, 2.6].

Finally, the whole category $\mathbf{EX}$ of exact categories (in a universe) and exact functors has kernels and cokernels, *with respect to* the ideal of functors which annihilate all the objects (§ 12): the kernel of the exact functor $F\colon A \to B$ is its annihilation kernel $\mathbf{Ker}\, F$; its cokernel is the quotient $B/\mathbf{Nim}\, F$, where $\mathbf{Nim}\, F$ (the normal image of F) is the least thick subcategory of B containing $F(A)$.

The construction of the exact categories of fractions is made by means of *three-map* diagrams, of type (2), or equivalently of type (3), where the dot-marked arrows are in Σ:

(2) $\qquad A \xleftarrow{\ \bullet\ } \!<\, \bullet \longrightarrow \bullet \xleftarrow{\ \bullet\ } B$

(3) $\qquad A \longrightarrow \bullet \xleftarrow{\ \bullet\ }\!< \bullet \rightarrowtail B$

It derives from a three-arrow construction of the category of relations $\mathrm{Rel}\, A$, here recalled in § 1: indeed $\Sigma^{-1}A$ can be obtained as the (ordinary) quotient A'/R of a subcategory $A' = \check{\Sigma}A$ of $\mathrm{Rel}\, A$ (generated by A and by the reversed arrows of Σ), modulo a congruence R described in 5.2 for the diagrams of type (2).

A denotes always an exact category and B a category.

1. - Relations in exact categories

For the basic theory of exact categories (in the sense of Puppe-Mitchell), see [Pu; Mt; HS]; we just recall here shortly their definition and their categories of relations, as well as their transfer functor for subobjects (in § 2).

An *exact* category can be defined as a category with zero-object (initial and terminal), such that every morphism has a kernel and a cokernel and factors (uniquely) through a conormal epi and a normal mono. As a consequence, every mono is normal and every epi is conormal; finite products and sums need no exist: if they do, the category is abelian, by means of a unique additive structure; the main diagrammatic properties of homological algebra hold: *e.g.* the 3 × 3 lemma, the five lemma, the connecting morphism lemma.

An *exact* functor between exact categories has to preserve kernels and cokernels, or equivalently the short exact sequences, or also the exact ones; it also preserves the zero object.

An exact category A is canonically embedded in its category of relations Rel A [T1-T2; BP; G1]; this is provided with a *regular* involution $a \mapsto \bar{a}$ ($a = a\bar{a}a$, for every morphism a) and with an order relation $a \leq b$ on parallel maps, consistent with composition and involution. A relation $a\colon A \to B$ determines up to isomorphism a diagram of A:

(1) 

whose two squares are bicartesian in A (¹); this diagram contains the following «main» factorisations of the relation a (terminology as in [G1]):

(2) $a = \bar{p}\cdot f\cdot \bar{m}$ (*ternary* factorisation, along the solid path, with $f = n'q'$),

(3) $a = n\cdot \bar{g}\cdot q$ (*coternary* factorisation, along the dotted path, with $g = m'p'$),

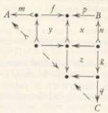
(4) $a = n\bar{p}'q'\bar{m} = \bar{p}n'\bar{m}'q$ (*quaternary* and *coquaternary* factorisations, along the lower and upper path).

Actually, the category Rel A is usually constructed by means of four-map diagrams in A [T1-T2; BP; G1], of the quaternary type (or equivalently, of the coquaternary one); it could also be constructed by means of three-map diagrams of ternary or coternary type, with the inconvenience that these two types are not stable under involution, but turned the one into the other. Nevertheless, the three-arrow factorisations (2) and (3) are more suitable for the calculus of fractions (where there is no involution) and here we use only them.

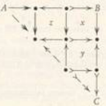
The composition of relations is obtained by limits and colimits existing in A: inverse images (pullbacks) of monos and direct images (pushouts) of epis. Using ternary factorisations, the composition of the relations $a = \bar{p}\cdot f\cdot \bar{m}\colon A \to B$ and

(¹) In an exact category a square consisting of two parallel monos and two parallel epis is a pullback iff it is a pushout.

$b = \bar{q}\cdot g\cdot\bar{n}: B \to C$ is the slanting path of the following diagram:

(5) 

where the square (x) is commutative (epi-mono factorisation of pn), (y) is a pulback and (z) a pushout. Analogously, for coternary factorisations, use the diagram (6):

(6) 

A zero-preserving functor $F: \mathbf{A} \to \mathbf{B}$ between exact categories is exact iff it extends to an involution preserving functor $\mathrm{Rel}\,F: \mathrm{Rel}\,\mathbf{A} \to \mathrm{Rel}\,\mathbf{B}$ [G1].

2. - Direct and inverse images in exact categories [G2]

For every object A the ordered sets $\mathrm{Sub}\,A$ and $\mathrm{Quo}\,A$ of subobjects and quotients of A are *modular* lattices (with 0 and 1), anti-isomorphic via cokernels and kernels.

A morphism $f: A \to B$ determines two transfer mappings of subobjects, the *direct and inverse images via f*:

(1) $$f_*: \mathrm{Sub}\,A \to \mathrm{Sub}\,B, \quad f_*(m) = \mathrm{im}\,(fm).$$

(2) $f^*: \mathrm{Sub}\,B \to \mathrm{Sub}\,A$, $f^*(n) = \ker((\mathrm{cok}\,n)\cdot f) =$ pullback of n along f,

which form a *modular connection* $\mathrm{Sub} f = (f_*, f^*): \mathrm{Sub}\,A \to \mathrm{Sub}\,B$, i.e.:

(3) $f_*: \mathrm{Sub}\,A \to \mathrm{Sub}\,B$ and $f^*: \mathrm{Sub}\,B \to \mathrm{Sub}\,C$ are increasing mappings,

(4) $f^*f_*(x) = x \vee f^*0 \geq x$, $\quad f_*f^*(y) = y \wedge f_*1 \leq y \qquad$ (for $x \in \mathrm{Sub}\,A$, $y \in \mathrm{Sub}\,B$).

In particular, (f_*, f^*) is a Galois connection $(f_* \dashv f^*)$. Globally, we have an exact functor, the *transfer functor* of A:

(5) $$\mathrm{Sub} = \mathrm{Sub}_A : A \to \mathit{Mlc}, \quad A \mapsto \mathrm{Sub}\,A, \quad f \mapsto \mathrm{Sub} f = (f_*, f^*),$$

from A to the exact category Mlc of (small) modular latices (with 0 and 1) and modular connections.

The direct and inverse images of monos extend in the obvious way to relations: if $a = \bar{p} \cdot f \cdot \bar{m} = n \cdot \bar{g} \cdot q$: $A \to B$, we get two increasing mappings:

(6) $$a_* : \mathrm{Sub}\,A \to \mathrm{Sub}\,B, \quad a_* = p^* f_* m^* = n_* g^* q_*,$$

(7) $$a^* : \mathrm{Sub}\,B \to \mathrm{Sub}\,A, \quad a^* = m_* f^* p_* = q^* g_* n^*,$$

which no longer form a Galois connection, but verify properties that can be seen in [G2]; here we only need the fact that also these transformations are consistent with the composition of relations: $(ba)_* = b_* a_*$ and $(ba)^* = a^* b^*$.

These transformations can be seen as a description of the functor $\mathrm{Rel\,Sub}_A$: $\mathrm{Rel}\,A \to \mathrm{Rel}\,\mathit{Mlc}$, whose codomain $\mathrm{Rel}\,\mathit{Mlc}$ has a concrete realization, by means of suitable pairs of increasing mappings between modular lattices [G2].

Occasionally, we shall also need the transfer mappings of quotients along $f: A \to B$, related to the previous ones by the ker-cok anti-isomorphism:

(8) $$f_0 : \mathrm{Quo}\,A \to \mathrm{Quo}\,B,$$
$$f_0(p) = \mathrm{cok}\,(f \cdot \ker p) = (\text{pushout of } p \text{ along } f) = \mathrm{cok}\, f_*(\ker p),$$

(9) $$f^0 : \mathrm{Quo}\,B \to \mathrm{Quo}\,A, \quad f^0(q) = \mathrm{coim}\,(qf) = \mathrm{cok}\, f^*(\ker q).$$

3. - Isokernels and exact isokernels

Given a functor $F: A \to B$, defined on an exact category, consider its *isokernal* $\mathrm{Ikr}\,F$, i.e. the set of morphisms of A which are turned by F into isomorphisms of B, and let $\mathbf{Ikr}\,F$ be the full subcategory of A^2 whose objects are in $\mathrm{Ikr}\,F$.

A set Σ or morphisms of A will be said to be an *exact isokernel* if:

(ik.0) Σ contains all the isomorphisms of A,

(ik.1) *composition and decomposition*: given $h = gf$ in A, if two morphisms are in Σ the third is too,

(ik.2) *factorisation*: if $f = mp$ is an epi-mono factorisation, f belongs to Σ iff p and m do; further, given a commutative square (1), if $m \in \Sigma$ then $n \in \Sigma$, if $p \in \Sigma$ then $q \in \Sigma$:

(1) 

(ik.3) if (2) is a *pullback* and (3) a *pushout*: $f \in \Sigma \Rightarrow g \in \Sigma$ (in both), $n \in \Sigma \Rightarrow$ $\Rightarrow m \in \Sigma$, $p \in \Sigma \Rightarrow q \in \Sigma$;

(2) 

(3) 

Clearly, the isokernel $\mathrm{Ikr}\, F$ of an exact functor F between exact categories is an exact isokernel; we shall prove that also the converse is true (Thm. 8) and that $\mathbf{Ikr}\, F$ is an exact subcategory of A^2 (Thm. 11). A morphism of Σ will usually be denoted by a dot-marked arrow: $\rightarrowtail$

4. - Some properties of exact isokernels

An exact isokernel Σ of A is trivially the set of morphisms of a subcategory of A, containing all the objects of A (ik.0, 1). Moreover:

a) (ik.1*a*) a composition of two monos (resp. epis) belongs to Σ iff each term does,
(ik.1*b*) in the commutative square 3.1: m and q belong to Σ iff n and p do,

b) if $m \geqslant n$ in $\mathrm{Sub}\, A$ and $n \in \Sigma$ then $m \in \Sigma$; if $p \geqslant q$ $\mathrm{Quo}\, A$ and $q \in \Sigma$ then $p \in \Sigma$,

c) given a direct image square 3.2 ($n = f_*(m)$): if $f \in \Sigma$ then $g \in \Sigma$; if m and f are in Σ, so are g and n,

*c**) given an inverse image square 3.3 ($p = f^{\circ}(q)$): if $f \in \Sigma$ then $g \in \Sigma$; if q, f are in Σ, so are g and p,

d) in the presence of the other axioms, (ik.1*a*, b) implies (ik.1).

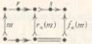
Proof: *a*) and *b*). For (ik.1*a*), note that the square (4) is a pullback:

(4)

thus, if $n \in \Sigma$, also $b \in \Sigma$ (ik.3) and m too (ik.1); the property *b*) follows trivially. As to (ik.1*b*): if m and q are in Σ, also n is (ik.2) and p too (ik.1).

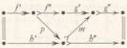
c) Factor $f = sr$ (epi-mono) and apply (ik.1, 2) to the diagram (5):

(5)



d) Consider a composition $h = gf$ and factor epi-mono all these morphisms $f = f''f'$ and so on:

(6)



now, if f and g are in Σ, so are f', f'', g', g'' (ik.2), whence also p and m (ik.2), h' and h'' (ik.1*a*) and h (ik.2); instead, if h and f are in Σ, so are h', h'', f', f''; then p, m and g'' too (ik.1*a*); last, also g' is in Σ because of (ik.1*b*); the last case follows by duality.

5. - Fractions

We construct now the category of fractions $P: \Sigma^{-1} \to \Sigma^{-1}A$, where Σ is an exact isokernel of A.

A) Consider the subcategory $A' = \hat{\Sigma}A$ of Rel$A$ generated by A and by the reversed arrows $\bar{h}$ of Σ: each morphism φ of A' has a ternary factorisation:

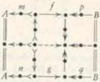
(1)

$$\varphi = \bar{p}\cdot f\cdot \bar{m} \qquad (m, p \in \Sigma)$$

because such maps are easily seen to be stable under the composition of relations (1.5).

Let R be the following relation between parallel maps of A': $\varphi R \psi$ if there is a commutative diagram in A:

(2)



$$\varphi = \bar{p} \cdot f \cdot \bar{m}$$

(η)

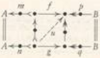
$$\psi = \bar{q} \cdot g \cdot \bar{n}\,,$$

i.e. if there is some $\eta < \varphi, \psi$ (the dotted morphism of A'): the relation $\eta < \varphi$ is described in the upper half of the diagram.

We want to prove that R is a category-conguence and that $\Sigma^{-1}A = A'/R$.

B) It will be useful to remark that, given a commutative diagram:

(3)



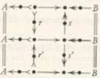
$$\varphi = \bar{p} \cdot f \cdot \bar{m}$$

$$\psi = \bar{q} \cdot g \cdot \bar{n},$$

then $\varphi R \psi$: indeed the intermediate morphism $\bar{p}u\bar{n}$ precedes both φ and ψ, with respect to $<$.

C) Σ is transitive. Since $<$ is trivially so, it suffices to prove that, if $\varphi < \zeta$ and $\psi < \zeta$ in A':

(4)



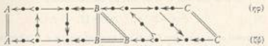
(φ)

(ζ)

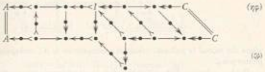
(ψ)

there is some η preceding both φ and ψ in A': construct η by means of the pullback of r, r' and of the pushout of s, s'.

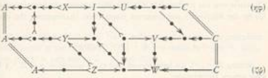
D) Σ is consistent with composition: this is not trivial, as $<$ is not so. It suffices to prove that, if $\varphi < \psi$ and $\eta < \zeta$ then $\eta\varphi \, R \, \zeta\psi$; the hypotheses supply the diagram:

(5)


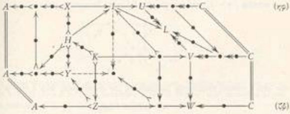
where the twist of the «planes» is functional to the constructions to follow. The composition of the rows will be obtained in two steps; first, by canonical factorisation of the central part:

(6)


second, by three pullbacks of monos (X, Y, Z) on the left side and three pushouts of epis (U, V, W) on the right one:

(7)


The factorisation of $\eta\varphi$ (on the upper path) and $\zeta\psi$ (on the lower path) is now complete. By means of two new pullbacks (H, K) and one pushout (L):

(8)


we interpose two intermediate morphisms σ (the path A, H, I, L, C) and τ

(the path A, H, K, V, C) of A', proving the thesis: $\eta\varphi \mathrel{R} \sigma \mathrel{R} \tau \mathrel{R} \zeta\psi$; note that the last two statements follow from remark B).

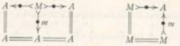
E) It is now easy to verify that the natural functor:

(9) $$P: A \to \Sigma^{-1}A, \quad P(f) = [f] = \bar{1}\cdot f\cdot \bar{1},$$

composed of an embedding into A' and a natural projection, is indeed the category of fractions of A determined by Σ.

First, P takes every motphism of Σ into an isomorphism; by (ik.2), it suffices to prove this fact for monos and epis; now, if $m \in \Sigma$ is a monomorphism, the relations $[m]\cdot[\bar{m}] = [m\cdot\bar{m}] = 1$ and $[\bar{m}]\cdot[m] = 1$ follow from the diagrams:

(10)



where the second (a pullback) calculates a composition in A'; analogously for epimorphisms $p \in \Sigma$.

Further, a functor $F: A \to B$, with values in an arbitrary category and taking every map of Σ into an isomorphism, factors uniquely through P, via the functor:

(11) $$G: \Sigma^{-1}A \to B, \quad G[\bar{p}\cdot f\cdot \bar{m}] = (Fp)^{-1}\cdot(Ff)\cdot(Fm)^{-1}.$$

Of course one has to show that this definition is consistent (transforming the diagram (2) by the functor F) and preserves composition (transforming the diagram 1.5).

6. - Factorisations

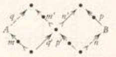
Writing $Pf = [f]$, we have proved that every morphism α in $\Sigma^{-1}A$ has the following *ternary* factorisation:

(1) $$\alpha = [p]^{-1}\cdot[f]\cdot[m]^{-1},$$

with f in A and p, m in Σ (resp. epi and mono).

Factoring $f = n'q'$ and forming the following bicartesian squares:

(2)



one gets a second three-arrow factorisation, the dotted path, with $b = m'p' \in \Sigma$:

(3) $$\alpha = [n]\cdot[b]^{-1}\cdot[q],$$

with b in Σ and q, n in A (resp. epi and mono).

Of course, the diagram (2) contains also a quaternary factorisation (the lower path) and a coquaternary one (the upper path), which we are not going to use here.

7. - Exactness properties

THEOREM: $\Sigma^{-1}A$ is exact as well as the functor P and the isokernel $\Sigma = \mathrm{Ikr}\, P$; if B is an exact category and the functor $F\colon A \to B$ takes every map of Σ into an isomorphism, then F is exact iff the unique functor $G\colon \Sigma^{-1}A \to B$ such that $F = GP$ is so.

Given a morphism α and its factorisations 6.1-3:

(1) $$\alpha = [\varphi] = [p]^{-1}\cdot[f]\cdot[m]^{-1} = [n]\cdot[b]^{-1}\cdot[q],$$

we have

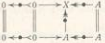
(2) $$\ker\alpha = [\varphi^*(0)] = [\ker q], \qquad \mathrm{coim}\,\alpha = [q],$$
$$\mathrm{cok}\,\alpha = [\mathrm{cok}\, n], \qquad \mathrm{im}\,\alpha = [\varphi_*(1)] = [n],$$

(3) α is iso iff n and q are in Σ, iff f is in Σ; in this case:

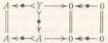
$$\alpha^{-1} = [m]\cdot[f]^{-1}\cdot[p] = [q]^{-1}\cdot[b]\cdot[n]^{-1}.$$

PROOF: A) 0 is an initial object in $\Sigma^{-1}A$: it suffices to prove that any morphism $0 \to A$ of A' is equivalent to the morphism: $\bar{1}\cdot 0_{0A}\cdot\bar{1}$, as it is shown below in (4); analogously, to prove that 0 is terminal in $\Sigma^{-1}A$, use (5):

(4)



(5)

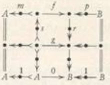


B) Using the images of subobjects along relations, recalled in 2.6-7, we prove now that $\alpha = [\varphi]\colon A \to B$ is a zero morphism in $\Sigma^{-1}A$ iff $\varphi^*(0) \in \Sigma$; according to our two factorisations of φ, the latter is given by:

(6) $$\varphi^*(0) = m_*f^*p_*(0) = m_*f^*(0), \qquad \varphi^*(0) = q^*b_*n^*(0) = q^*(0) = \ker q.$$

Let $\alpha = [\varphi]$ be a zero morphism, so that $\varphi \mathrel{R} \bar{1} \cdot 0_{AB} \cdot \bar{1}$:

(7)



$\varphi = \bar{p} \cdot f \cdot \bar{m}$

(c)

$\psi = \bar{1} \cdot 0 \cdot \bar{1}$,

from the lower middle square, $g = 0$; from the upper one:

$$(8) \qquad f^*(0) \geq f^* r^*(0) \geq s_*(s^*(f^* r^*(0))) = s_*(g^*(0)) = s_*(1) = s \in \Sigma,$$

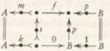
so that $f^*(0) \in \Sigma$ (by 4*b*)) and $\varphi^*(0) = m_* f^*(0)$ too (by 4*c*)).

Conversely, assume that $k = \varphi^*(0) = m_* f^*(0) \in \Sigma$; since $k \leq m$, $k = ms$ in A, and $fs = 0$, as:

$$(9) \qquad s^* f^*(0) = s^* m^* m_* f^*(0) = s^* m^*(k) = k^*(k) = 1 .$$

Thus we can form the commutative diagram of A:

(10)



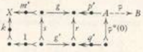
$\varphi = \bar{p} \cdot f \cdot \bar{m}$

$[\psi] = [\bar{1} \cdot 0 \cdot \bar{k}] = 0$,

where $s \in \Sigma$, because of (ik.1); thus, by the Remark 5*B*), $[\varphi] = [\psi] = 0$.

C) Now, it is easy to show that $\ker[\varphi] = [\ker q]$, and the other properties in (2): indeed, $[\varphi] \cdot [\ker q] = 0$; if $[\varphi] \cdot [\psi] = 0$, then $(\varphi\psi)^*(0) = \psi^* \varphi^*(0) \in \Sigma$ and we form the diagram:

(11)



$\psi = \bar{m}' \cdot g \cdot \bar{p}'$

from right to left, by direct and inverse images of monos: $r = p'_*(\varphi^*(0))$, $s = g^*(r)$, $k = m'_*(s) = \psi^* \varphi^*(0) \in \Sigma$; since P takes the reversed horizontal arrows ($\leftarrow\!\!\bullet$) into isomorphism, it is easy to check that $[\psi]$ factors through $[\varphi^*(0)]$: $[\varphi^*(0)] \cdot [\bar{q}' \cdot g' \cdot \bar{k}] = [\psi]$.

Dually, the cokernels exist and are calculated as in (2). This proves also that $\Sigma^{-1} A$ is an exact category, as every morphism factors through a normal epi, an isomorphism and a normal mono: $[\varphi] = [n] \cdot [g]^{-1} \cdot [q]$.

P is exact by the previous characterization of kernels and cokernels in $\Sigma^{-1} A$. More generally, every functor G such that $F = GP$ is exact is also so: if $\varphi = n \cdot \bar{g} \cdot q$ in A':

(12) $$\ker G[\varphi] = \ker(Fn \cdot (Fg)^{-1} \cdot Fq) = \ker Fq = F(\ker q) = G[\ker q] = G[\ker \varphi].$$

D) We prove now our characterization of isomorphisms; the conditions given in (3) are clearly sufficient, as we already now that all the morphisms of Σ become iso in $\Sigma^{-1}A$. Conversely, note first that, if $\bar{p} \cdot f \cdot \bar{m} \colon A \to A$ is R-equivalent to 1_A in A', then $f \in \Sigma$:

(13)

$\bar{p} \cdot f \cdot \bar{m}$

$\bar{1} \cdot 0 \cdot \bar{1}$,

as g is in Σ by composition, and f too by decomposition (ik.1). By diagram 6.2, it follows easily that, if an endomorphism of A' in coternary factorisation $\varphi = n \cdot \bar{g} \cdot q$ is R-equivalent to 1, then q and n belong to Σ.

Consider now two morphisms φ, ψ of A', such that $\psi\varphi \, \mathrm{R}\, 1$ and $\varphi\psi \, \mathrm{R}\, 1$: using the composition of their coternary factorisations (1.6) and the previous result, it is easy to deduce that φ and ψ verify the conditions (3). It follows at once that $\mathrm{Ikr}\, P = \Sigma$.

8. - Characterization Theorem

As an outline of the previous results, the following conditions on a set Σ of morphisms of A are equivalent:

a) Σ is an exact isokernel of A,

b) there exists an exact functor $F \colon A \to B$ (with values in some exact category) whose isokernel is Σ,

c) the category of fractions $\Sigma^{-1}A$ is exact, the natural functor $P \colon A \to \Sigma^{-1}A$ also and $\mathrm{Ikr}\, P = \Sigma$.

9. - Factorisation structure for exact functors

Let $F \colon A \to B$ be an exact functor and $\Sigma = \mathrm{Ikr}\, F$; then F factors through the category of fractions $P \colon A \to \Sigma^{-1}A$:

(1) $$F = GP = (A \to \Sigma^{-1}A \to B),$$

by means of an exact functor G which is *conservative* (i.e., reflects the isomorphisms): indeed, if $G[\bar{p} \cdot f \cdot \bar{m}] = (Fp)^{-1} \cdot (Ff) \cdot (Fm)^{-1}$ is iso, so is Ff, whence $f \in \Sigma$ and $[\bar{p} \cdot f \cdot \bar{m}]$ is iso in $\Sigma^{-1}A$.

Thus, every exact functor factors $F = GP$, through a «projection» P on an exact category of fractions and a conservative exact functor G; such a factorisation is determined up to isomorphism of exact categories, as the conservative property of G gives: $\mathrm{Ikr}\,F = \mathrm{Ikr}\,GP = \mathrm{Ikr}\,P$.

10. - Annihilation kernels and Serre quotients

Given an exact functor $F\colon \boldsymbol{A} \to \boldsymbol{B}$, consider its *annihilation kernel* $\mathrm{Ker}\,F$, i.e. the set of all the objects annihilated by F and let $\mathbf{Ker}\,F$ be the full subcategory of $\boldsymbol{A}$ determined by these objects.

Plainly, $\mathrm{Ker}\,F$ and $\mathrm{Ikr}\,F$ determine each other:

(1) $\quad \mathrm{Ikr}\,F = \{f \mid \mathrm{Ker}\,f,\ \mathrm{Cok}\,f \in \mathrm{Ker}\,F\}, \qquad \mathrm{Ker}\,F = \{A \mid (0\colon A \to A) \in \mathrm{Ikr}\,F\}.$

In order to characterize the annihilations kernels, say that a *thick subcategory* $\boldsymbol{K}$ of an exact category $\boldsymbol{A}$ is defined the following conditions, as in the abelian case [Gr, Ga]:

(tk.0) $\boldsymbol{K}$ is a full subcategory of $\boldsymbol{A}$, containing all the zero objects,

(tk.1) given a short exact sequence $A' \rightarrowtail A \twoheadrightarrow A''$ of $\boldsymbol{A}$, A belongs to $\boldsymbol{K}$ iff both A' and A'' do.

As a consequence, $\boldsymbol{K}$ is an exact, invariant subcategory of $\boldsymbol{A}$, i.e.: it is exact in its own right, it embeds exactly in $\boldsymbol{A}$ and every object of $\boldsymbol{A}$ isomorphic to some objects of $\boldsymbol{K}$ belongs to the latter.

Again as in the abelian case we write $\boldsymbol{A}/\boldsymbol{K}$ (the *Serre quotient* of $\boldsymbol{A}$ modulo $\boldsymbol{K}$) the solution of the universal problem of annihilating $\boldsymbol{K}$, among exact functors from $\boldsymbol{A}$ into some exact category; we show below that the solution exists and is the associated category of fractions.

Equivalently, the *thick set of objects* $\underline{K} = \mathrm{Ob}\,\boldsymbol{K}$ satisfies analogous axioms: just forget in (tk.0) the full-subcategory condition.

11. - Characterization of thick subcategoties

Theorem: In the exact category $\boldsymbol{A}$, the following transformations between sets of morphisms Σ and sets of object $\underline{K}$

(1) $\quad \Sigma \mapsto \underline{K}(\Sigma) = \{A \mid (0\colon A \to A) \in \Sigma\}, \qquad \underline{K} \mapsto \Sigma(\underline{K}) = \{f \mid \mathrm{Ker}\,f,\ \mathrm{Cok}\,f \in \underline{K}\},$

establish a biunivocal correspondence between exact isokernels and thick sets of objects. Further, if Σ is an exact isokernel of $\boldsymbol{A}$, the full subcategory $\underline{\Sigma}$ of $\boldsymbol{A}^2$ whose objects are the morphisms of Σ is an exact subcategory of $\boldsymbol{A}^2$.

The following conditions on a set $\underline{K}$ of objects of A are equivalent:

a) $\underline{K}$ is thick in A,

b) there exists an exact functor $F\colon A \to B$, with values in some exact category, such that $\mathrm{Ker}\, F = \underline{K}$,

c) $\Sigma(\underline{K})^{-1}A$ is exact, the functor $P\colon A \to \Sigma(\underline{K})^{-1}A$ also and $\mathrm{Ker}\, F = \underline{K}$.

In this case P satisfies the universal problem of annihilating K: $A/K = \Sigma(\underline{K})^{-1}A$.

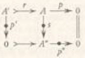
Every thick subcategory and every exact category of fractions of an abelian category is abelian.

Proof: A) Let Σ be an exact isokernel: we have to prove that $\underline{K} = \underline{K}(\Sigma) = \{A \mid (0\colon A \to A) \in \Sigma\}$ is a thick set of objects in A. First, $\underline{K}$ contains all the zero-objects, because Σ contains all the isos. Second, let be given the short exact sequence $A' \rightarrowtail A \twoheadrightarrow A''$ of A and consider the commutative diagram:

(2) 

If A is in $\underline{K}$, p and m are in Σ (ik.2), hence so are p' (ik.2) and m' (ik.1a); thus A' belongs to $\underline{K}$; dually, also A'' does.

Conversely, assume that A' and A'' are in $\underline{K}$, so that $p', m', p'', m'' \in \Sigma$; form

(3) 

the pushout of (r, p'), which is given by the cokernel of r, i.e. s; thus $s \in \Sigma$ (ik.3), $p'' \in \Sigma$ by hypothesis and $p \in \Sigma$ by composition; dually, also $m\colon 0 \to A$ is in Σ, whence $A \in \underline{K}$.

B) Now $\underline{K}$ is a thick set of objects and we prove that $\Sigma = \Sigma(\underline{K}) = \{f \mid \mathrm{Ker} f,\ \mathrm{Cok} f \in \underline{K}\}$ is an exact isokernel of A; we make use of § 4d), replacing the condition (ik.1) with (ik.1a, b).

The conditions (ik.0, 2) hold trivially. For (ik.1a), consider the composition of two monomorphisms:

(4) $$A'' \to A' \to A,$$

and the associated short exact sequence of A:

(5) $$A'/A'' \to A/A'' \twoheadrightarrow A/A'$$

thus (tk.1) proves that the two monomorphisms (4) are in Σ (i.e., A/A' and A'/A'' are in $\underline{K}$) iff their composition is so (i.e., A/A'' is in $\underline{K}$). Dually for epimorphisms.

(ik.1b) Given the commutative square occupying the lower left position in (6):

$$
\begin{array}{ccccc}
H \wedge L & \rightarrowtail & L & \twoheadrightarrow & L/(H \wedge L) \\
\downarrow & & \downarrow & & \downarrow \\
H & \rightarrowtail & A & \twoheadrightarrow & A/H \\
\downarrow & & \downarrow & & \downarrow \\
(H \vee L)/L & \rightarrowtail & A/L & \twoheadrightarrow & A/(H \vee L)
\end{array}
\tag{6}
$$

complete the diagrams by two cokernels (A/H and $A/(H \vee L)$) and three kernels: the upper row is short exact, by the 3×3 lemma. Now, if the two arrows from H are in Σ, $H \wedge L$ and A/H are in $\underline{K}$, whence so are $L/(H \wedge L)$ and $A/(H \vee L)$, and finally L (tk.1); thus the two arrows ending in A/L are in Σ.

(ik.3) We prove the pullback case, which we split in two. A pullback of a mono along a mono appears in the left upper square of (6), which again we can complete. Now, if $L \rightarrowtail A$ is in Σ, A/L is in $\underline{K}$, whence also $(H \vee L)/L$ is so and thesis follows. Last, a pullback of a mono along an epi appears in the lower left square of (7), which again is commutative with short exact rows and columns

$$
\begin{array}{ccccc}
L & = & L & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
H & \rightarrowtail & A & \twoheadrightarrow & A/H \\
\downarrow & & \downarrow & & \| \\
H/L & \rightarrowtail & A/L & \twoheadrightarrow & A/H
\end{array}
\tag{7}
$$

It is now easy to ckeck that if $H/L \rightarrowtail A/L$ (resp. $A \twoheadrightarrow A/L$) is in Σ, so is $H \rightarrowtail A$ (resp. $H \twoheadrightarrow H/L$).

C) Let Σ be an exact isokernel of A and $\underline{K}$ the associated thick set of objects: we prove that $\underline{\Sigma}$ is an exact subcategory of A^2. Since $\underline{\Sigma}$ contains clearly the zero-object $0 \rightarrow 0$, it suffices to consider a morphism $f: a \rightarrow b$ of $\underline{\Sigma}$ and prove that its kernel-object and cokernel-object in A^2 belong to Σ; we can assume that the Σ-morphisms $a: A' \rightarrow A''$ and $b: B' \rightarrow B''$ are both *monic*; form the diagram (8), commutative with exact rows and columns, and the associated exact sequence (9) produced by the connecting morphism d:

$$
(8)\qquad
\begin{array}{ccccccc}
K' & \rightarrowtail & A' & \xrightarrow{f'} & B' & \longrightarrow & C' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
K'' & \rightarrowtail & A'' & \xrightarrow{f''} & B'' & \longrightarrow & C'' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
K_0'' & \rightarrowtail & A_0'' & \longrightarrow & B_0'' & \longrightarrow & C_0''
\end{array}
$$

$$
(9)\qquad K' \rightarrowtail K'' \longrightarrow K_0'' \xrightarrow{d} C' \longrightarrow C'' \longrightarrow C_0'' .
$$

Now, A_0'' is in $\underline{K}$ by hypothesis, whence also its subobject K_0'' and $\mathrm{Ker}\, d$; thus the exact sequences:

$$
(10)\qquad K' \rightarrowtail K'' \twoheadrightarrow \mathrm{Ker}\, d ,
$$

$$
(11)\qquad \mathrm{Im}\, d \rightarrowtail C' \to C'' \twoheadrightarrow C_0'' ,
$$

respectively prove that $K' \rightarrowtail K''$ and $C' \to C''$ are in Σ.

D) The rest follows now trivially from the fact that, for every exact functor $F: \boldsymbol{A} \to \boldsymbol{B}$, the isokernel and the annihilation kernel are related by the correspondence (1). A thick subcategory or a Serre quotient of an abelian category modulo a thick subcategory are known to be abelian [Gr, Ga].

12. - The structure of EX

Finally, we sketch some topics which will be developed elsewhere, in a work studying *semiexact* and *homological* categories with respect to an *assigned* ideal of null morphisms. Consider the category **EX** of exact $\mathcal{U}$-categories and exact functors, for some universe $\mathcal{U}$ (²), together with its full subcategory **AB** of abelian $\mathcal{U}$-categories.

EX (or **AB**) has no zero-object, as its terminal object **1** is just bi-initial; but it has a natural ideal of «null morphisms»: *assume* that an exact functor is null if it annihilates all the objects, or also if it factors through some exact category whose objects are all zero (equivalent to **1**).

Every exact functor $F: \boldsymbol{A} \to \boldsymbol{B}$ has a *kernel* with respect to this ideal, satisfying the obvious universal property: namely its annihilation kernel $\mathbf{Ker}\, F$, i.e. the full subcategory of $\boldsymbol{A}$ formed by the objects annihilated by F. F has also a cokernel with respect to this ideal, which can be obtained as the Serre quotient of $\boldsymbol{B}$ modulo the normal image $\mathbf{Nim}\, F$, i.e. the last thick subcategory of $\boldsymbol{B}$ containing $F(\boldsymbol{A})$.

All this proves that **EX** is a semiexact category (with respect to the ideal of null functors); by the last statement in thm. 11, **AB** is a semiexact subcategory of **EX**.

(²) I.e.: every object and every morphism belongs to $\mathcal{U}$, as well as the set of subobjects of any object; as the hom-functors do not play a relevant role in the general theory of exact categories, we do not ask the hom-sets to be small.

It is easy to see that the category **EX** is not *generalized exact* with respect to the above ideal: in other words an exact functor $F: A \to B$ need not be an «exact morphism», i.e. need not induce an isomorphism from its normal coimage ($A/\mathbf{Ker}\,F$) to its normal image ($\mathbf{Nim}\,F$). A counterexample is given by the transfer functor $F = \text{Sub}: Ab \to Mlc$ of the category of abelian groups: F is not faithful [G2] but $\mathbf{Ker}\,F$ is plainly the full subcategory of zero groups; thus the normal coimage of F is Ab, and F determines a non-faithful functor into its normal image, whatever it be.

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