A Nowhere Dense but not Porous Set in the Space of Convex Bodies

ABSTRACT. — We give an example of a subset which is nowhere dense but not porous in the space of all convex bodies.

Un insieme raro ma non poroso nello spazio dei corpi convessi

RIASSUNTO. — La nozione di insieme poroso (σ-poroso) fu estesa ad uno spazio metrico qualsiasi nel 1976 da Zajíček. Tale nozione rappresenta un raffinamento della nozione di insieme raro (magro) in tutti quelli spazi in cui è possibile dare esempi di insiemi rari, ma non porosi (σ-porosi). Questo problema, risolto in uno spazio di Banach, è ancora aperto nello spazio C dei corpi convessi dotato della topologia indotta dalla metrica di Hausdorff. Nel presente lavoro si contribuisce alla soluzione di tale problema con un esempio in C di un insieme raro, ma non poroso.

The notion of porous set on the real line $\mathbb{R}$ was introduced by Dolženko in 1967 [2] and was generalized to a general metric space by Zajíček in 1976 [3].

In this paper we shall use the following definition of porous set [5].

Let $(X, d)$ be a metric space and $B(x, \varepsilon)$ denotes the ball of center $x \in X$ and radius $\varepsilon$. A subset $M$ of $X$ is porous (with coefficient $\alpha$) if there is a real number $\alpha > 0$ such that for each $x \in X$ and for each ball $B(x, \varepsilon)$ there exists an element $y \in B(x, \varepsilon)$ such that:

$$B(y, \alpha d(x, y)) \cap M = \emptyset.$$ 

A countable union of porous sets (all with coefficient $\alpha$) is called $\sigma$-porous (with coefficient $\alpha$).

It is obvious that a porous set is also nowhere dense and that a $\sigma$-porous set is mea-

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ger. The problem of finding nowhere dense sets which are not $\sigma$-porous is solved in the euclidean $d$-dimensional space $E^d$ and more generally in a Banach space but «it is not known at which more general metric space» such a problem can be solved ([4], page 322).

In this paper we give a contribution to the solution of this problem by showing an example of a subset which is nowhere dense but not porous in the space $C$ of all convex bodies endowed with the Hausdorff metric.

Eventually we recall that a convex body is a compact convex subset of $E^d$ with nonempty interior and that if $C, D \in C$ their Hausdorff distance $\delta(C, D)$ is defined in the following way:

$$\delta(C, D) = \max \left\{ \sup_{x \in C} \inf_{y \in D} d(x, y), \sup_{y \in D} \inf_{x \in C} d(x, y) \right\},$$

where $d$ is the usual euclidean distance.

If for $F \in C$ and for each positive real number $\rho$, we put

$$F_\rho = \{ x \in E^d : d(x, F) \leq \rho \}$$

we have also that

$$\delta(C, D) = \inf \{ \rho : C_\rho \supset D \text{ and } D_\rho \supset C \} .$$

Notations: The ball of $E^d$ of center a point $x$ and radius $\varepsilon$ will be denoted by $B(x, \varepsilon)$ while the ball of $C$ of center an element $C$ of $C$ and radius $\varepsilon$ will be denoted by $B(C, \varepsilon)$.

The abbreviations bd, int and conv stand for boundary, interior and convex hull.

The example

Choose on a straightline $R$ of the euclidean $d$-dimensional space $E^d$ a nowhere dense but not porous subset $M$ of real numbers and define:

$$M = \{ C \in C : (\text{bd } C) \cap M \neq \emptyset \} .$$

1. $M$ is not porous in $C$

Since $M$ is not porous in $R$, for each real number $\alpha$, $0 < \alpha \leq 1$, there are an element $x \in R$ and a positive $\varepsilon$ such that, for each $z \in B(x, \varepsilon)$,

$$B(z, \alpha d(x, z)) \cap M \neq \emptyset .$$

Let $C$ be a ball with center in a point $c$ of $R$, radius $\rho$ greater than $(5 + 3 \sqrt{2}) \varepsilon$ and such that $x \in \text{bd } C$. 
We shall show that, for each $D \in \mathcal{B}(C, \epsilon)$,

(1) \[ B(D, xD(D, C)) \cap M \neq \emptyset. \]

Firstly we prove that, if $z$ is the point of $(\text{bd } D) \cap \mathbb{R}$ belonging also to the interval $(x - \epsilon, x + \epsilon)$ of $\mathbb{R}$, then

(2) \[ d(x, z) \leq \delta(C, D). \]

Indeed, if $z \notin \text{int } C$ we have easily:

\[ d(z, x) = d(z, C) \leq \delta(D, C). \]

If $z \in \text{int } C$ we take a support hyperplane $\pi$ to $D$ at the point $z$. Afterwards we consider the hyperplane $\pi'$ parallel to $\pi$, tangent to $C$ such that the point $\nu$ common to $\pi$ and $\pi'$ belongs to the halfspace bounded by $\pi$ and not containing $c$. Therefore:

\[ d(z, x) \leq d(\nu, \pi) \leq d(\nu, D) \leq \delta(D, C) \]

and again (2) follows.

Now let $q$ be an element of $B(z, \alpha d(x, z)) \cap M$.

a) If $q \notin \text{int } D$, we set

\[ F = \text{conv } \{D \cup \{q\}\}. \]

Since $q \in (\text{bd } F) \cap M$ it follows that $F \in M$.

Moreover, using also (2):

\[ \delta(F, D) \leq d(q, D) \leq d(q, z) < \alpha d(x, z) \leq \alpha \delta(D, C). \]

Therefore $F \in \mathcal{B}(D, \alpha \delta(D, C))$ and (1) holds.

b) Let us assume now that $q \in \text{int } D$. Then we can choose a point $m$ of $\text{bd } D$ such that $d(q, \text{bd } D) = d(q, m)$. Afterwards we consider the hyperplane $\pi$ through $q$ and parallel to a support hyperplane $\pi'$ to $D$ at $m$. Then if $P$ is the closed halfspace bounded by $\pi$ and not containing $m$, we set

\[ F = D \cap P. \]

We shall prove that

(3) \[ \delta(F, D) \leq d(q, m). \]

If we put $d(q, m) = \gamma$, we have obviously that $D_{\gamma} \supset F$. So there is only to prove that a point $a$ of $D$ but not of $F$ belongs also to $F_{\gamma}$.

Let $a'$ be the point of $\pi$ such that the straightline $aa'$ is perpendicular to $\pi$. We claim that

(4) \[ a' \in (\text{bd } F) \cap \pi. \]

Indeed, if otherwise $a' \notin (\text{bd } F) \cap \pi$, we can choose a straightline $S$ of $\pi$ through $a'$ which meets $(\text{bd } F) \cap \pi$. We put $S \cap (\text{bd } F) \cap \pi = [b, e]$ and we choose the points $b$ and $e$ in such a way that $d(a', b) < d(a', e)$. 

Now we observe that from the definition of the points \( m, q \) and \( z \) we have that

\[
d(\pi, \pi') \leq d(q, m) < \alpha d(x, z) < \alpha \varepsilon < \varepsilon.
\]

Therefore, since \( \pi' \) does not intersect the ball of center \( c \) and radius \( \rho - \varepsilon \), the hyperplane \( \pi \) does not intersect the ball \( C' \) of center \( c \) and radius \( \rho - 2\varepsilon \). Since \( \rho > (5 + 3\sqrt{2})\varepsilon \), it follows then that the straightline through \( b \) and parallel to the straightline \( aa' \) cuts the boundary of \( C' \) in two points. If \( f \) is one of them,

\[
\text{conv} \{b, e, f\}
\]
is a convex set of the plane spanned by the points \( a, a', b \) and its interior points are also interior points of \( D \).

Then the straightline through the points \( a \) and \( b \) would contain the point \( b \) of \( \text{bd} D \), points of \( \text{int} D \) on one of the two half-lines bounded by \( b \) and points of \( D \) on the other one, which is a contradiction.

Then (4) holds.

Therefore:

\[
d(a, F) \leq d(a, a') \leq d(\pi, \pi') \leq \gamma,
\]

hence \( a \in F \), and also (3) holds.

Now, from (3) and (2), we can obtain that

\[
\delta(D, F) \leq d(m, q) \leq d(q, z) < \alpha d(x, z) \leq \alpha \delta(D, C)
\]
i.e.

\[
F \in B(D, \alpha \delta(D, C)).
\]

Since moreover \( q \in (\text{bd} F) \cap M \), we have that (1) is fulfilled and \( M \) is not porous in \( C \).

\[
2. \quad M \text{ IS NOWHERE DENSE IN } C
\]

We shall show that for each nonempty open set \( B(D, \varepsilon) \) of \( C(D \in C) \) it is possible to find another non empty open set \( B(E, \eta) \) contained in \( B(D, \varepsilon) \) and disjoint from \( M \). There are three cases.

i) \( (\text{bd} D) \cap R = \emptyset \).

Let \( \alpha \) be a positive real number such that

\[
2\alpha < \min (\varepsilon, d(D, R)).
\]

Since \( D \subset D_{\alpha} \subset (D_{\alpha})_{\alpha} \), from Lemma 12.9.13 of [1], vol. 3 page 139, there exists a positive \( \eta \) such that, if \( S \in C \) and \( \delta(D_{\alpha}, S) \leq \eta \), then \( D \subset S \subset (D_{\alpha})_{\alpha} \). Therefore
(bd $S$) $\cap R = \emptyset$ and

$$B(D, \eta)$$

is an open set contained in $B(D, \varepsilon)$ and disjoint from $M$.

ii) (bd $D$) $\cap R = \{x_1, x_2\}$ i.e. two points of $R$.

Since $D \subset D_{i/2}$ also bd $D_{i/2}$ intersects $R$ in exactly two points $y_1$ and $y_2$. We can assume that the two open intervals of $R$, $(y_1, x_1)$ and $(x_2, y_2)$, are disjoint from int $D$.

Since $M$ is nowhere dense there are two closed intervals $[b_1, a_1] \subset (y_1, x_1)$ and $[a_2, b_2] \subset (x_2, y_2)$ disjoint from $M$. We assume $b_1 \notin (a_1, x_1)$ and $b_2 \notin (x_2, a_2)$.

Afterwards we put

$$F = \text{conv} \{D \cup \{a_1\} \cup \{a_2\}\}$$

and we choose a positive real number $\alpha$ such that

$$2\alpha < \min (d(a_1, b_1), d(a_2, b_2), d(b_1, F), d(b_2, F)).$$

Obviously $\alpha < \varepsilon/4$.

Moreover since $D \subset F \subset (F_{a})_{\varepsilon}$; from the same Lemma used in (i) there is a positive $\eta$ such that if $S \subset C$ and $\delta(S, F_{a}) \leq \eta$, then $F \subset S \subset (F_{a})_{\varepsilon}$. We claim that

$$B(F_{a}, \eta)$$

is the open set we are looking for.

Firstly we observe that $b_1 \notin (F_{a})_{\varepsilon}$, since, otherwise, there would exist a point $y \in F$ such that $b_1 \in B(y, 2\alpha)$ and then $d(b_1, y) \leq 2\alpha < d(b_1, F)$, which is a contradiction.

Then $b_1 \notin (F_{a})_{\varepsilon}$, $a_1 \in \text{bd} F$ and analogously for $b_2$ and $a_2$.

Since, for each $S \subset B(F_{a}, \eta)$, we have that $F \subset S \subset (F_{a})_{\varepsilon}$, it follows that

$$(\text{bd } S) \cap R \subset [b_1, a_1] \cup [a_2, b_2]$$

and therefore $S \notin M$.

In order to show that $B(F_{a}, \eta) \subset B(D, \varepsilon)$ we put

$$\sigma = \max (d(a_1, D), d(a_2, D)).$$

Then, from $\sigma < \varepsilon/2$ and $F \subset D_{\varepsilon}$, it follows that

$$D \subset S \subset (F_{a})_{\varepsilon} \subset (D_{\varepsilon})_{2\alpha} = D_{\varepsilon} + 2\alpha \subset D_{\varepsilon}.$$

iii) If $R$ is a support straightline of $D$, i.e. (bd $D$) $\cap R$ is either a single point either a line segment, we can consider in the neighbourhood $B(D, \varepsilon)$ the convex body $D_{i/2}$. Then there is only to apply to the neighbourhood $B(D_{i/2}, \varepsilon/4)$ the procedure used in (ii).
REFERENCES