Second Order Elliptic Equations in Weighted Sobolev Spaces on Unbounded Domains

**SUMMARY.** — In this paper we study the Dirichlet problem for a class of second order linear elliptic partial differential equations with discontinuous coefficients in weighted Sobolev spaces on unbounded domains of $\mathbb{R}^n$. We obtain some existence and uniqueness results.

**Equazioni ellittiche del secondo ordine in spazi di Sobolev con peso su aperti non limitati**

**RIASSUNTO.** — In questo lavoro si studia il problema di Dirichlet per una classe di equazioni differenziali lineari ellittiche del secondo ordine a coefficienti discontinui in spazi di Sobolev con peso su aperti non limitati di $\mathbb{R}^n$. Si ottengono alcuni teoremi di esistenza ed unicità.

**INTRODUCTION**

Let $\Omega$ be an unbounded and sufficiently regular open subset of $\mathbb{R}^n$, $n \geq 2$.

We consider in $\Omega$ the uniformly elliptic second order linear differential operator

$$Lu = -\sum_{i,j=1}^{n} a_{ij} u_{ij} + \sum_{i=1}^{n} a_{i} u_{i} + au.$$  

We suppose that the coefficients of $L$ are real functions and satisfy the following *basic* hypotheses:

\begin{align*}
(2) & \quad a_{ij} = a_{ji} \in L^\infty(\Omega), \quad i, j = 1, \ldots, n; \\
(3) & \quad a_{i} \in L^1_{\text{loc}}(\overline{\Omega}), \quad \sup_{x \in \Omega} \|a_{i}\|_{L^1(\Omega \cap B(x,1))} < +\infty, \quad i = 1, \ldots, n; \\
(4) & \quad a \in L^1_{\text{loc}}(\overline{\Omega}), \quad \sup_{x \in \Omega} \|a\|_{L^1(\Omega \cap B(x,1))} < +\infty.
\end{align*}

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Here $B(x, 1)$ is the open ball of radius 1 centered at $x$; $s > 2$ if $n = 2$, $s = n$ if $n > 2$; $t = 2$ if $2 \leq n < 4$, $t > 2$ if $n = 4$, $t = n/2$ if $n > 4$.

It is well known (see e.g. [5], [7]) that, if $n > 2$, the previous regularity hypotheses on the coefficients don’t assure that the problem

$$u \in W^2(\Omega) \cap \hat{W}^1(\Omega), \quad Lu = f, \quad f \in L^2(\Omega)$$

is uniquely solvable. Additional regularity hypotheses on the $a_{ij}$ are needed to obtain this.

In the case of an unbounded open subset $\Omega$, problem (5) has been largely studied in some recent papers (see e.g. [11]-[14], [4]). In these papers some existence and uniqueness results have been obtained, supposing that the coefficients of $L$ have a suitable behaviour to the infinity and the $a_{ij}$ satisfy some regularity hypotheses like those of Miranda (see [9]) and Chicco (see [2], [3]).

In particular, in [14] the following regularity hypotheses on the $a_{ij}$ are considered:

$$a_{ij} = a_{ij}^0 + a_{ij}^w, \quad a_{ij}^0 \in L^\alpha(\Omega) \cap W^{1,1}_0(\Omega), \quad a_{ij}^w \in C(\overline{\Omega}),$$

where $C(\overline{\Omega})$ is the class of uniformly continuous bounded functions in $\Omega$.

In this paper we extend the results of [14] to the problem:

$$u \in W^2_q(\Omega) \cap \hat{W}^1_q(\Omega), \quad Lu = f, \quad f \in L^2_q(\Omega),$$

where $W^2_q(\Omega)$, $\hat{W}^1_q(\Omega)$ and $L^2_q(\Omega)$ denote some suitable classes of weighted Sobolev spaces (see sect. 2 for the definitions).

In the same hypotheses of [14] on $\Omega$ and on the coefficients of $L$, we prove that (7) is an index problem with index equal to zero. If the following further condition

$$\underset{\Omega}{\text{ess inf}} a > 0$$

is verified, we obtain that problem (7) is uniquely solvable.

These results are obtained using some results from [14] and a suitable regularity theorem which is proved in sect. 4, arguing like in [10] to establish a similar result in other situations.

1. - Some Preliminary Facts on a Class of Weight Functions

We set

$$B(x, r) = \{ y \in R^n \mid \| y - x \| < r \}, \quad B_r = B(0, r), \quad \| u \|_{p, A} = \| u \|_{L^p(\Omega)}.$$

Let $\Omega$ be an unbounded open subset of $R^n$, $n \geq 2$. 
We denote by \( \mathcal{A}(\Omega) \) the class of all measurable functions \( \rho: \Omega \to \mathbb{R}_+ \) such that:

\[
\sup_{x, y \in \Omega, \|x - y\| < \rho(y)} \left| \log \frac{\rho(x)}{\rho(y)} \right| < +\infty.
\]

Clearly, \( \rho \) verifies (1.1) if and only if there exists \( \gamma \in \mathbb{R}_+ \) such that:

\[
\gamma^{-1} \rho(y) \leq \rho(x) \leq \gamma \rho(y) \quad \forall y \in \Omega \quad \text{and} \quad \forall x \in \Omega \cap B(y, \rho(y)).
\]

We note that the class of all functions \( \rho: \Omega \to \mathbb{R}_+ \) which are Lipschitz continuous in \( \Omega \) with Lipschitz coefficient \( \gamma < 1 \) is contained in \( \mathcal{A}(\Omega) \) (see [16]).

Moreover, we note that for any \( \rho \in \mathcal{A}(\Omega) \) there exist \( a \in \mathbb{R}_+ \) and \( b \in [0, 1] \) such that:

\[
\rho(x) \leq a + b|x| \quad \forall x \in \Omega
\]

(see (19) and (20) of [15]) and then \( \mathcal{A}(\Omega) \subseteq L_{\text{loc}}^{\infty}(\Omega) \).

Let \( \rho_0 \) be a function such that \( \rho_0 \in \mathcal{A}(\mathbb{R}^n) \) and

\[
\inf_{\Omega} \rho_0 > 0, \quad \lim_{|x| \to +\infty} \rho_0(x) = +\infty.
\]

For example

\[
\rho_0: \mathbb{R}^n \to \frac{1 + |x|}{2}
\]

We put

\[
\rho = \rho_0|_\Omega.
\]

For any \( a \in [0, 1] \) we set

\[
I_a(x) = \Omega \cap B(x, a\rho_0(x)) \ni x
\]

\[
\phi_a: (x, y) \in \mathbb{R}^n \times \mathbb{R}^n \to \begin{cases} 
1 & \text{if } y \in I_a(x), \\
0 & \text{if } y \notin I_a(x).
\end{cases}
\]

If \( q \in \mathbb{N} \), \( L_q^1(\Omega) \) denotes the space of all measurable functions \( g: \Omega \to \mathbb{R} \) such that:

\[
\|g\|_{L_q^1(\Omega)} = \int_{\Omega} \rho^q(x)|g(x)| \, dx < +\infty,
\]

normed by (1.3).

**Lemma 1.1:** For any \( q \in \mathbb{N} \) and \( a \in [0, 1] \), \( g \in L_q^1(\Omega) \) if and only if \( g \in L_{\text{loc}}^{\infty}(\Omega) \) and the function \( x \in \mathbb{R}^n \to \rho_0^{q-n}(x)|g|_{\Omega} \) belongs to \( L^1(\mathbb{R}^n) \).
Moreover there exist $c_1, c_2 \in \mathbb{R}_+$ such that:

$$c_1 \|g\|_{L^1_+}(\omega) \leq \int_{\mathbb{R}^n} |g(y)| \, dx \leq c_2 \|g\|_{L^1_+}(\omega) \quad \forall g \in L^1_+\Omega).$$

**Proof:** Using methods similar to those of [15] to establish Lemma 2, we have the result noting that

$$\int_{\mathbb{R}^n} |g(y)| \, dy = \int_{\mathbb{R}^n} |g(x)\phi_\sigma(x, y) \, dx$$

and that, because of (26), (27) of [15] and (1.2), there exist $c_1, c_2 \in \mathbb{R}_+$ such that:

$$c_1 \sigma_0 \leq \int_{\mathbb{R}^n} |g(y)| \, dy \leq c_2 \sigma_0 \quad \forall y \in \mathbb{R}^n.$$

2. - A class of weighted Sobolev spaces

For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ we put:

$$|\alpha| = \alpha_1 + \ldots + \alpha_n, \quad \mathcal{A}^\alpha = \mathcal{A}_{\alpha_1} \ldots \mathcal{A}_{\alpha_n}.$$

If $k \in \mathbb{N}$ and $q \in \mathbb{R}$, $W^k_q(\Omega)$ denotes the space of all functions $u: \Omega \to \mathbb{R}$ such that $\mathcal{A}^\alpha u \in L^2(\Omega)$ for $|\alpha| \leq k$, normed by:

$$\|u\|_{W^k_q(\Omega)} = \sum_{|\alpha| \leq k} \|\mathcal{A}^\alpha u\|_{L^2(\Omega)}.$$

$\hat{W}^k_q(\Omega)$ denotes the closure of $C(\Omega)$ in $W^k_q(\Omega)$.

Put

$$L^2_+ = W^0_1(\Omega),$$

$$W^k_+ = W^k_1(\Omega), \quad \hat{W}^k_+ = \hat{W}^k_1(\Omega).$$

If $\Omega$ has the segment property, then for any $k \in \mathbb{N}$ and for any $q \in \mathbb{R}$ $C(\Omega)$ is dense in $W^k_+$ (see e.g. Lemma 2.2 of [10]).

If $k \in \mathbb{N}$, we denote by $W^{k}_\text{loc}(\overline{\Omega})$ (resp. $\hat{W}^{k}_\text{loc}(\overline{\Omega})$) the space of all functions $u: \Omega \to \mathbb{R}$ such that:

$$\zeta u \in W^k(\Omega) \quad \text{(resp. } \zeta u \in \hat{W}^k(\Omega)) \quad \forall \zeta \in C^\infty_c(\Omega).$$
Remark 2.1: Clearly, for any $\Omega$, any $k \in N_0$ and any $q \in N$ we have:

(2.2) $W^k_q(\Omega) \subset W^k_{qc}(\Omega)$. 

(2.3) $\hat{W}^k_q(\Omega) \subset \hat{W}^k_{qc}(\Omega)$. 

Moreover we prove the following

Lemma 2.1: If $\Omega$ has the segment property, then for any $k \in N_0$ and for any $q \in R$ we have:

(2.4) $W^k_q(\Omega) \cap \hat{W}^k_{qc}(\Omega) \subset \hat{W}^k_q(\Omega)$. 

Proof: We fix $k \in N_0$, $q \in R$ and $u \in W^k_q(\Omega) \cap \hat{W}^k_{qc}(\Omega)$. Let $\zeta$ be a function in $\partial(R^n)$ such that:

$0 \leq \zeta \leq 1$, \quad $\zeta|_{B_1} = 1$, \quad $\text{supp} \zeta \subset B_2$. 

Write $\zeta_r(x) = \zeta(x/r)$, for every $r \in R_+$ and for every $x \in R^n$, then we have:

(2.5) $\zeta_r \zeta_{2r} = \zeta_r$, 

(2.6) $|\partial^\alpha \zeta_r(x)| \leq \frac{c}{r^{\alpha}}$ \quad for $|\alpha| \leq k$,

with $c \in R_+$ independent of $x$, $r$ and $\alpha$.

From (2.6) it follows that there exists $c_1 \in R_+$ such that:

(2.7) $\| (1 - \zeta_r) v \|_{W^k_q(\Omega)} \leq c_1 \| v \|_{W^k_q(\Omega)} \quad \forall v \in W^k_q(\Omega)$ and $\forall r \geq 1$.

Since $\partial(\Omega)$ is dense in $W^k_q(\Omega)$, there exists $\phi_r \in \partial(\Omega)$ such that:

(2.8) $\| u - \phi_r \|_{W^k_q(\Omega)} \leq \frac{\varepsilon}{2c_1}$.

Choose $r_0 \in [1, +\infty[$ such that

$\text{supp} (1 - \zeta_{r_0}) \cap \text{supp} \phi_{r_0} = \emptyset$, 

we have:

(2.9) $\| (1 - \zeta_{r_0}) u \|_{W^k_q(\Omega)} = \| (1 - \zeta_{r_0})(u - \phi_{r_0}) \|_{W^k_q(\Omega)}$.

Now we note that there exists $c(\varepsilon) \in R_+$ such that:

(2.10) $\| \zeta_{r_0} v \|_{W^k_q(\Omega)} \leq c(\varepsilon) \| v \|_{W^k_q(\Omega)} \quad \forall v \in W^k_q(\Omega)$. 

On the other hand, there exists $\psi_\varepsilon \in \partial(\Omega)$ such that:

(2.11) $\| \zeta_{r_0} u - \psi_\varepsilon \|_{W^k_q(\Omega)} \leq \frac{\varepsilon}{2c(\varepsilon)}$. 

From (2.5), (2.7)-(2.11) it follows that:

\[ \|u - \zeta_r \phi_{e}\|_{W^q_r(\Omega)} \leq \|1 - \zeta_r\|_{W^q_r(\Omega)} + \|\zeta_r (\zeta_{2r}, u - \phi_{e})\|_{W^q_r(\Omega)} \leq \]

\[ \leq c_1 \|u - \phi_{e}\|_{W^q_r(\Omega)} + c(\varepsilon) \|\zeta_{2r}, u - \phi_{e}\|_{W^q_r(\Omega)} \leq \varepsilon. \]

From (2.12) it follows that \( u \in \tilde{W}^k_q(\Omega) \) and then we have the result.

3. SOME KNOWN EMBEDDING LEMMAS

For any \( p \in [1, +\infty] \) \( M^p(\Omega) \) denotes the space of all functions \( f \in L^p_{\text{loc}}(\Omega) \) such that:

\[ \|f\|_{M^p(\Omega)} = \sup_{x \in \Omega} |f|_{p, \Omega \cap B(x, 1)} < +\infty, \]

normed by (3.1), and \( M^p_0(\Omega) \) denotes the subspace of \( M^p(\Omega) \) of all functions \( f \) such that:

\[ \lim_{|x| \to +\infty} |f|_{p, \Omega \cap B(x, 1)} = 0. \]

For some properties of the spaces \( M^p(\Omega) \) and \( M^p_0(\Omega) \) we refer to [11], [12].

For any \( b \in N^p_0 \) we put:

\[ \Theta^b u = \left( \sum_{|a| = b} (\Theta^a u)^2 \right)^{1/2}. \]

Let \( k, b, p \in \mathbb{R} \) such that:

\[ k \in \mathbb{N}, \quad b \in \{0, \ldots, k - 1\}, \quad p \in [2, +\infty[, \]

\[ p \geq \frac{n}{k - b}, \quad p > 2 \text{ if } n = 2(k - b) \]

and let \( g \) be a function such that:

\[ g \in M^p(\Omega). \]

From Theorem 3.1 of [6] we have the following

**Lemma 3.1:** If (3.3), (3.4), (3.5) hold and \( \Omega \) has the cone property, then, for a fixed \( q \in \mathbb{R}, \) for any \( u \in W^k_q(\Omega) \) we have \( g \Theta^b u \in L^q_{\rho}(\Omega) \) and:

\[ \|g \Theta^b u\|_{L^q_{\rho}(\Omega)} \leq c\|g\|_{M^p(\Omega)} \|u\|_{W^k_q(\Omega)}, \]

where \( c \in \mathbb{R}^+ \) is independent of \( u, g, \rho \) and \( q. \)

From Theorem 3.2 of [6] it follows:

**Lemma 3.2:** If the hypotheses of Lemma 3.1 are satisfied and in addition \( g \in M^p_0(\Omega), \)
then, for a fixed \( q \in \mathbb{R}, \) for any \( e \in \mathbb{R}, \) there exist a constant \( c(e) \in \mathbb{R}^+ \) and a bounded open
set $\Omega_\epsilon \subset \Omega$ such that:

$$
\|g^{\epsilon} u\|_{L^q_\epsilon(\Omega)} \leq \epsilon \|u\|_{W^{1}_q(\Omega)} + c(\epsilon) |u|_{L^2, \Omega}, \quad \forall u \in W^{1}_q(\Omega).
$$

From Lemma 3.2 and from known results we have the following (see also Theorem 3.3 of [6]):

**Lemma 3.3:** If the hypotheses of Lemma 3.2 hold, then, for a fixed $q \in R$, the operator:

$$
u \in W^{1}_q(\Omega) \rightarrow g^{\epsilon} u \in L^q_\epsilon(\Omega)
$$

is compact.

**4. - A REGULARITY THEOREM AND AN A PRIORI BOUND**

We consider the operator $L$ defined by (1).

We note that if (2), (3), (4) are verified and $\Omega$ has the cone property, it follows from Lemma 3.1 that, for every fixed $q \in R$, the position

$$
u \rightarrow L\nu
$$
defines a continuous linear operator from $W^{1}_q(\Omega)$ to $L^2_q(\Omega)$.

Now we want to study problem (7) assuming that the following hypotheses are satisfied:

i) $\Omega$ has the uniform $C^2$-regularity property in the sense of n. 4.6 of [1];

ii) the operator $L$ is uniformly elliptic in $\Omega$, the coefficients $a_{ij}$ satisfy (6) and in addition:

$$
(a_{ij})_{ii} \in M^0_0(\Omega), \quad \lim_{|x| \rightarrow +\infty} a_{ij}(x) = a_{ij}^0, \quad i, j, b = 1, \ldots, n,
$$
or

$$
\lim_{|x| \rightarrow +\infty} a_{ij}(x) = a_{ij}^0, \quad i, j = 1, \ldots, n;
$$

ii) the conditions

$$
a_i \in M^0_0(\Omega), \quad i = 1, \ldots, n,
$$

$$
a = a' + a'' \in M^0_0(\Omega), \quad a' \in M^0_0(\Omega), \quad \text{ess inf}_{\Omega} a'' > 0
$$

hold.

We fix a function $\beta : \Omega \rightarrow R_+$ such that:

$$
\beta, \beta^{-1} \in L^0_{\text{loc}}(\Omega), \quad \beta \in M^0_0(\Omega) \quad \text{and} \quad \exists \delta \in M^0_0(\Omega) \ni \beta \leq \beta_0.
$$
For example:

\[ \beta = 1 \quad \text{or} \quad \beta(x) = \frac{1}{(1 + |x|)^\tau}, \quad \tau > 0. \]

**Theorem 4.1:** If the hypotheses \( h_1 \), \( h_2 \) and \( h_3 \) are verified, then for every \( q, m \in \mathbb{R} \), \( \lambda \in \mathbb{R}_+ \) and every function \( u \) such that:

\[
\begin{align*}
\left\{ \begin{array}{l}
u \in W^2_{\text{loc}}(\Omega) \cap \bar{W}^1_{\text{loc}}(\Omega) \cap L^2_m(\Omega), \\
Lu + \lambda \beta u \in L^2_q(\Omega),
\end{array} \right.
\end{align*}
\]

we have: \( u \in W^2_q(\Omega) \) and there exist \( c \in \mathbb{R}_+ \) and a bounded open set \( \Omega_0 \subset \Omega \), independent of \( u \) and \( \lambda \), such that:

\[
\|u\|_{W^2_q(\Omega)} \leq c\left( \|Lu + \lambda \beta u\|_{L^2_q(\Omega)} + \|u\|_{L^2_q(\Omega)} \right).
\]

**Proof:** We choose \( r, r' \in \mathbb{R}_+ \), with \( r < r' \leq 1 \), and a function \( \phi \in \mathcal{O}(\mathbb{R}^n) \) such that:

\[
\phi|_{B_1} = 1, \quad \text{supp } \phi \subset B_{r'},
\]

\[
\sup_{\mathbb{R}^n} |\partial^\alpha \phi| \leq c_{\alpha} (r' - r)^{-|\alpha|}, \quad \forall \alpha \in \mathbb{N}^n_0,
\]

where \( c_{\alpha} \in \mathbb{R}_+ \) depends only on \( \alpha \).

For example:

\[
\phi: x \in \mathbb{R}^n \mapsto \left( \frac{r' - r}{2} \right)^{-n} \int_{B_{r'}(x)} \left( \frac{2(y - x)}{r' - r} \right) dy,
\]

where \( g \) is a function in \( \mathcal{O}(\mathbb{R}^n) \) such that \( \text{supp } g \subset B_1 \) and \( \int g(x) \, dx = 1 \).

We fix \( y \in \mathbb{R}^n \) and put:

\[
\psi = \psi_y : x \in \mathbb{R}^n \mapsto \phi \left( \frac{x - y}{\rho_0(y)} \right).
\]

Clearly we have that:

\[
\psi|_{B(y, \rho_0(y))} = 1, \quad \text{supp } \psi \subset B(y, r' \rho_0(y)),
\]

\[
\sup_{\mathbb{R}^n} |\partial^\alpha \psi| \leq c_{\alpha} \rho_0^{-|\alpha|}(y)(r' - r)^{-|\alpha|}, \quad \forall \alpha \in \mathbb{N}^n_0.
\]

Now we fix \( u \in W^2_{\text{loc}}(\Omega) \cap \bar{W}^1_{\text{loc}}(\Omega) \) and \( \lambda \in \mathbb{R}_+ \).

Since \( \psi u \in W^2(\Omega) \cap \bar{W}^1(\Omega) \), from Theorem 6 of [14] it follows:

\[
\|\psi u\|_{W^2(\Omega)} \leq c_1 \left( \|Lu + \lambda \beta u\|_{L^2_0(\Omega)} + \|u\|_{L^2_0(\Omega)} \right),
\]
where the constant $c_1 \in \mathbb{R}_+$ and the bounded open set $\Omega_1 \subset \Omega$ are independent of $\psi$, $u$, and $\lambda$.

Since $\rho \in L^\infty_{01}(\overline{\Omega})$, we have that:

$$|\psi u|_{2,0,1} \leq c_2 \rho^{-1}(y) |u|_{2,1,\lambda}(y),$$

where $c_2 \in \mathbb{R}_+$ is independent of $\psi$, $u$, $y$, $r$, and $r'$.

Using Theorem 3.1 of [6], from (4.8) and (4.9) we deduce the bound:

$$\|u\|_{W^2_{01}(\Omega)} \leq \|\psi u\|_{\mathbb{W}^2(\omega)} \leq c_3 (r' - r)^{-2} (|Lu + \lambda \beta u|_{2,1,\lambda}(y) + |u|_{\mathbb{L}^2_{01}(\Omega)} \rho^{-2}(y) |u|_{2,1,\lambda}(y))^{1/2} |u|_{\mathbb{L}^2_{01}(\Omega)}^{1/2},$$

where $c_3 \in \mathbb{R}_+$ is independent of $u$, $\lambda$, $y$, $r$, and $r'$.

By a well-known lemma of monotonicity of Miranda (see Lemma 3.1 of [8]), it follows from (4.10) that:

$$\|u\|_{W^2_{01}(\Omega)} \leq c_4 (|Lu + \lambda \beta u|_{2,1,\lambda}(y) + \rho^{-2}(y) |u|_{2,1,\lambda}(y))^{1/2} |u|_{\mathbb{L}^2_{01}(\Omega)}^{1/2},$$

where $c_4 \in \mathbb{R}_+$ is independent of $u$, $\lambda$, and $y$.

From (4.11) we have that:

$$\|u\|_{W^2_{01}(\Omega)} \leq c_5 (|Lu + \lambda \beta u|_{2,1,\lambda}(y) + \rho^{-2}(y) |u|_{2,1,\lambda}(y)),$$

where $c_5 \in \mathbb{R}_+$ is independent of $u$, $\lambda$, and $y$.

Now we fix $q$, $m \in \mathbb{R}$ and suppose that $u$ verifies (4.6).

From (4.12) it follows:

$$\int_{\mathbb{R}^n} \rho^{-2-n}(y) |u|_{\mathbb{L}^2_{01}(\Omega)} dy \leq$$

$$\leq c_6 \left( \int_{\mathbb{R}^n} \rho^{-2-n}(y) |Lu + \lambda \beta u|_{2,1,\lambda}(y) dy + \int_{\mathbb{R}^n} \rho^{-2-n-4}(y) |u|_{2,1,\lambda}(y) dy \right),$$

with $c_6 \in \mathbb{R}_+$ independent of $u$ and $\lambda$.

If $m \geq q - 2$, since

$$L^\infty_\omega(\Omega) \hookrightarrow L^2_{q-2}(\Omega),$$

from (4.13) and from Lemma 1.1 we have that $u \in \mathbb{W}^2_q(\Omega)$ and

$$\|u\|_{\mathbb{W}^2_q(\omega)} \leq c_7 (|Lu + \lambda \beta u|_{L^2_\omega(\Omega)} + |u|_{L^2_{q-2}(\omega)}),$$

with $c_7 \in \mathbb{R}_+$ independent of $u$ and $\lambda$.

If $m < q - 2$, we denote by $k$ the positive integer such that:

$$\frac{q - m}{2} - 1 \leq k < \frac{q - m}{2}.$$
Then, for $i = 1, \ldots, k$, we have that:

$$L_{q}^{2}(\Omega) \hookrightarrow L_{q+2i}^{2}(\Omega).$$

Therefore, proceeding as in the proof of (4.13) and (4.14) with $m + 2i$, $i = 1, \ldots, k$, instead of $q$, we have that $u \in W_{m+2}^{2}(\Omega), \ldots, u \in W_{m+2k}^{2}(\Omega).$

On the other hand we have that

$$W_{m+2k}^{2}(\Omega) \hookrightarrow L_{q-2}^{2}(\Omega)$$

and then, since $u \in L_{q-2}^{2}(\Omega)$, (4.13) holds. Thus $u \in W_{q}^{2}(\Omega)$ and satisfies (4.14).

Moreover, from Lemma 3.2 it follows that for every $\varepsilon \in R_{+}$ there exist $c(\varepsilon) \in R_{+}$ and a bounded open set $\Omega_{\varepsilon} \subset \Omega$, independent of $u$, such that:

$$\|u\|_{L_{q-2}^{2}(\Omega)} \leq \varepsilon \|u\|_{W_{q}^{2}(\Omega)} + c(\varepsilon)\|u\|_{L_{q}^{2}(\Omega)}.$$

(4.15)

From (4.14) and (4.15) it follows (4.7) and then we have the result.

**Corollary 4.1:** In the same hypotheses of Theorem 4.1, for a fixed $q \in R$, there exist $\lambda_{0}, \epsilon \in R_{+}$ such that:

$$\|u\|_{W_{q}^{1}(\Omega)} \leq c\|Lu + k\beta u\|_{L_{q}^{2}(\Omega)} \quad \forall u \in W_{q}^{2}(\Omega) \cap \tilde{W}_{q}^{1}(\Omega) \quad \forall \lambda \geq \lambda_{0},$$

(4.16)

**Proof:** Since $\beta^{-1}, \rho^{-1} \in L_{\text{loc}}^{\infty}(\Omega)$, by (4.7) we have that for every $\lambda \in R_{+}$ and every $u \in W_{q}^{2}(\Omega) \cap \tilde{W}_{q}^{1}(\Omega)$:

$$\lambda\|u\|_{L_{q}^{2}(\Omega)} \leq c_{1}\|\beta u\|_{L_{q}^{2}(\Omega)} \leq c_{2} (\|Lu + \lambda\beta u\|_{L_{q}^{2}(\Omega)} + \|Lu\|_{L_{q}^{2}(\Omega)}) \leq$$

$$\leq c_{1}\|Lu + \lambda\beta u\|_{L_{q}^{2}(\Omega)} + c_{2}\|u\|_{W_{q}^{1}(\Omega)} \leq c_{3}\|Lu + \lambda\beta u\|_{L_{q}^{2}(\Omega)} + c_{4}\|u\|_{L_{q}^{2}(\Omega)}$$

(4.17)

where $c_{1}, \ldots, c_{4} \in R_{+}$ are independent of $u$ and $\lambda$.

From (4.7) and (4.17) we deduce the result.

5. - Existence results

**Theorem 5.1:** In the same hypotheses of Theorem 4.1, for a fixed $q \in R$, there exists $\lambda_{0} \in R_{+}$ such that for every $\lambda \geq \lambda_{0}$ the problem

$$u \in W_{q}^{2}(\Omega) \cap \tilde{W}_{q}^{1}(\Omega),$$

$$Lu + \lambda\beta u = f, \quad f \in L_{q}^{2}(\Omega),$$

(5.1)

is uniquely solvable.

**Proof:** Firstly we prove that, for $\lambda$ sufficiently large, there exists a unique $u_{0} \in W_{q}^{2}(\Omega) \cap \tilde{W}_{q}^{1}(\Omega)$ which is a solution of the equation:

$$-\Delta u + (\alpha u + \beta u) f = f, \quad f \in L_{q}^{2}(\Omega).$$

(5.2)
We note that, by Theorem 3.2 of [16], there exists $\sigma_0 \in \mathfrak{A}(\mathbb{R}^n) \cap C^\infty (\mathbb{R}^n)$ such that:

$$\sigma_0 (x) \sim \rho_0 (x) \quad \forall x \in \mathbb{R}^n,$$

$$|\partial^s \sigma_0 (x)| \leq c_0 \sigma_0^{-1/s} (x) \quad \forall x \in \mathbb{R}^n \quad \text{and} \quad \forall x \in N_0^\circ,$$

where $c_0 \in R_+$ is independent of $x$.

Write $\sigma = \sigma_0^{-1/s}$. Then we have that $\sigma \in \mathfrak{A}(\Omega) \cap C^\infty (\overline{\Omega})$ and:

$$\sigma (x) \sim \rho (x) \quad \forall x \in \Omega,$$

$$|\partial^s \sigma (x)| \leq c_0 \sigma^{-1} (x) \quad \forall x \in \Omega \quad \text{and} \quad \forall x \in N_0^\circ.$$  

(5.3) 

(5.4)

Put:

$$a_\lambda (u, v) = \int_\Omega \left( \sum_{i=1}^n u_{x_i} (\sigma^2 v)_{x_i} + (a'' + \lambda \beta) u \sigma^{2q} v \right) \, dx \quad \forall u, \, v \in \tilde{W}^{1, q}_q (\Omega) \quad \text{and} \quad \forall \lambda \geq 0.$$  

From (5.3), (5.4) and Lemma 3.1 it follows that the bilinear form $a_\lambda$, $\lambda \geq 0$, is continuous.

Moreover, for any fixed $u \in \tilde{W}^{1, q}_q (\Omega)$ and $\lambda \geq 0$, we have that:

$$a_\lambda (u, u) = \int_\Omega \left( \sigma^{2q} \sum_{i=1}^n u_{x_i}^2 + 2q \sigma^{2q-1} u \sum_{i=1}^n u_{x_i} \sigma_{x_i} + (a'' + \lambda \beta) u^2 \sigma^{2q} \right) \, dx \geq$$

$$\geq c_1 \| u \|_{W^{1, q}_q (\Omega)}^2 + \lambda \int_\Omega \beta u^2 \sigma^{2q} \, dx - 2|q| \sum_{i=1}^n \int_\Omega \sigma^{2q-1} |u_{x_i} \sigma_{x_i}| \, dx,$$

where $c_1 \in R_+$ is independent of $u$ and $\lambda$.

On the other hand, from Lemma 3.2 we get:

$$\int_\Omega \sigma^{2q-1} |u \partial^1 u| \, dx \leq \| u \|_{L_q (\Omega)} \| \partial^1 u \|_{L_q (\Omega)} \leq \varepsilon_0 \| \partial^1 u \|_{L_{2q} (\Omega)}^2 + c (\varepsilon_0) \| u \|_{L_{2q} (\Omega)}^2 \leq$$

$$\leq c (\varepsilon) \| u \|_{W^{1, q}_q (\Omega)}^2 + c (\varepsilon) \| u \|_{L_{2q} (\Omega)}^2,$$

where $c_0, c_1 \in R_+$ and the bounded open set $\Omega_0 \subset \Omega$ are independent of $u$.

From (5.3)-(5.6) it follows:

$$a_\lambda (u, u) \geq c_2 \| u \|_{W^{1, q}_q (\Omega)}^2 + \lambda \int_\Omega \beta u^2 \sigma^{2q} \, dx - c_3 |u|_{L_{2q} (\Omega)},$$

where the constants $c_2, c_3 \in R_+$ and the bounded open set $\Omega_0 \subset \Omega$ are independent of $u$ and $\lambda$.  

(5.7)
Moreover, since $\sigma, \sigma^{-1}, \beta^{-1} \in L^\infty_\text{loc} (\Omega)$, we have that:

$$|u|_{2, \Omega_0}^2 \leq c_4 \int_{\Omega} \beta u^2 \sigma^{-2} dx,$$

where $c_4 \in \mathbb{R}_+$ is independent of $u$.

Therefore, from (5.7) and (5.8) it follows that there exist $c_0, \lambda_1 \in \mathbb{R}_+$ such that:

$$a_\lambda (u, u) \geq c_0 \|u\|_{W^1_q (\Omega)}^2 \quad \forall u \in \overset{\circ}{W}^1_q (\Omega) \text{ and } \forall \lambda \in [\lambda_1, + \infty[.$$

Then, from Lax-Milgram Lemma we deduce that, for every $f \in L^2_q (\Omega)$ and $\lambda \geq \lambda_1$, there exists a unique $u_0 \in \overset{\circ}{W}^1_q (\Omega)$ such that:

$$a_\lambda (u_0, v) = \int_{\Omega} f \sigma^{-2} v dx \quad \forall v \in \overset{\circ}{W}^1_q (\Omega).$$

Since $f \in L^2_\text{loc} (\Omega)$, from well-known results of local regularity of elliptic problems we have that $u_0 \in W^2_\text{loc} (\Omega)$ (see e.g. chap. 8 of [3]); moreover $u_0$ is a solution of the equation (5.2). Then, by Theorem 4.1, we have that $u_0 \in \overset{\circ}{W}^2_q (\Omega)$.

For every $\tau \in [0, 1]$, we put:

$$A_\tau = (1 - \tau)(-\Delta + a'' + \lambda \beta) + \tau (L + \lambda \beta) = -(1 - \tau) \Delta + \tau (L - a'') + a'' + \lambda \beta.$$ 

Since $A_\tau$ verifies the hypotheses of Corollary 4.1 uniformly with respect to $\tau \in [0, 1]$, for $\lambda$ sufficiently large we have:

$$\|u\|_{W^2_q (\Omega)} \leq c \|A_\tau u\|_{L^2_q (\Omega)} \quad \forall u \in \overset{\circ}{W}^2_q (\Omega) \cap \overset{\circ}{W}^1_q (\Omega),$$

where $c \in \mathbb{R}_+$ is independent of $u$ and $\tau$.

Using the method of continuity we obtain the result.

**Theorem 5.2:** If the hypotheses (i1), (i2) and (i3) hold, then, for every $q \in \mathbb{R}$, the operator

$$u \in W^2_q (\Omega) \cap \overset{\circ}{W}^1_q (\Omega) \rightarrow Lu \in L^2_q (\Omega)$$

is a Fredholm operator with index equal to zero.

**Proof:** We consider a function $\beta: \Omega \rightarrow \mathbb{R}_+$ which is in $M^0_1 (\Omega)$ and satisfies (4.5).

For example:

$$\beta: x \in \Omega \rightarrow \frac{1}{(1 + |x|)^\tau}, \quad \tau > 0.$$
We put:

\[ L = (L + \lambda_0 \beta) - \lambda_0 \beta, \]

where \( \lambda_0 \in \mathbb{R}_+ \) is the number defined in Theorem 5.1.

By Lemma 3.3 the operator

\[ u \in W^2_q (\Omega) \cap \tilde{W}^1_q (\Omega) \rightarrow \beta u \in L^2_q (\Omega) \]

is compact. Therefore from Theorem 5.1 and from well-known results we have the statement.

**Theorem 5.3:** If the hypotheses \( i_1, i_2, i_3 \) and the condition:

\[ \text{ess inf}_a a > 0 \]

hold, then, for every \( q \in \mathbb{R} \), problem (7) is uniquely solvable.

**Proof:** Because of Theorem 5.2, it is sufficient to prove that a uniqueness result for problem (7) holds.

We fix \( u \in W^2_q (\Omega) \cap \tilde{W}^1_q (\Omega) \) such that \( Lu = 0 \).

From Theorem 4.1 it follows that \( u \in W^2 (\Omega) \).

On the other hand, since \( u \in W^1 (\Omega) \cap \tilde{W}^0_{\text{loc}} (\Omega) \), from Lemma 2.1 we have that \( u \in \tilde{W}^1 (\Omega) \).

Thus, from Theorem 10 of [15] (see also Remark 5.1) we deduce that \( u = 0 \).

**Remark 5.1:** One of the Author (M. Tansirico) takes the opportunity of this paper to indicate some emendations to bring in two previous papers (see [13] and [14]) which were published jointly with another Author.

In [13] the following emendations must be done:

1) in (34) the condition

\[ \lim_{|x| \to +\infty} e_q^c (x) = e^c_q \]

must be added;

2) the condition \( \partial \Omega \) is bounded \( \Rightarrow \) must be added to the hypotheses of Theorem 9;

3) in the proof of Theorem 9 the \( e_q^c \) and \( a_q^c \), defined by (40) and (41) respectively, must be replaced by: \( e_q^c \) such that the matrix \((e_q^c) \in E (v, R^*) \) and \( e_q^c |_{\Omega} = e_q \);

\[ a_q^c = \begin{cases} ga_q^c & \text{in } \Omega, \\ e_q^c & \text{in } R^* \setminus \Omega. \end{cases} \]

In [14] Theorem 10, in the case that (31) is verified, cannot be deduced from Theorem 9 of [13] because \( \partial \Omega \) must be bounded there. The result is obtained like in Theorem 10, in the case that (45) is verified, considering the functions \( a^k_q \) studied in Lemma 3 instead of the functions \( a_q^c + a^k_q \).
REFERENCES