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Existence and Uniqueness Results for the Dirichlet Problem in Unbounded Domains (**) (***)

SUMMARY. — In this paper we study the Dirichlet problem for a class of second order linear elliptic partial differential equations with discontinuous coefficients in unbounded domains of $\mathbb{R}^n$. We obtain some existence and uniqueness results.

Teoremi di esistenza ed unicità per il problema di Dirichlet in aperti non limitati

RIASSUNTO. — In questo lavoro si studia il problema di Dirichlet per una classe di equazioni differenziali lineari ellittiche del secondo ordine a coefficienti discontinui in aperti non limitati di $\mathbb{R}^n$. Si ottengono alcuni teoremi di esistenza ed unicità.

INTRODUCTION

We consider in an open subset $\Omega$ of $\mathbb{R}^n$, $n \geq 2$, the uniformly elliptic linear differential operator

(1) \[ Lu = - \sum _{i,j=1} ^n a_{ij} \partial _{x_i x_j} u + \sum _{i=1} ^n a_i \partial _x u + au \]

with real coefficients.

Suitable regularity hypotheses and behaviour to the infinity of the coefficients $a_i$ ($i = 1, \ldots, n$) and $a$ (see n. 3) are given, while

(2) \[ a_{ij} = a_{ji} \in L^\infty (\Omega), \quad i,j = 1, \ldots, n. \]

We are concerned with the problem

(3) \[ u \in W^2 (\Omega) \cap \overset{0}{\bar{W}}^1 (\Omega), \quad Lu = f, \quad f \in L^2 (\Omega). \]

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It is well known that, if \( n > 2 \), condition (2) doesn’t assure uniqueness for problem (3), whatever hypotheses on other data (see e.g. O. A. Ladyzhenskaya - N. N. Ural’tseva [8], D. Gilbarg - N. S. Trudinger [6]).

In some recent papers (see e.g. [10], [13], [14]) M. Troisi and one of the Authors studied problem (3) in an open unbounded and sufficiently regular \( \Omega \), with additional hypotheses on the \( a_{ij} \). Some a priori bounds and existence and uniqueness results, extending to unbounded domains of \( \mathbb{R}^n \) some classical results for the bounded domains (see e.g. [8], C. Miranda [9], M. Chicco [2], [3], [4], [5]), are given.

In particular in [10] hypotheses like those of C. Miranda [9] on the \( a_{ij} \) are given, namely

\[
a_{ij} \in W^{1,1}_{C, loc} (\Omega), \quad i, j = 1, \ldots, n,
\]

where \( s > 2 \) if \( n = 2 \) and \( s = n \) if \( n > 2 \), together with the condition to the infinity

\[
\lim_{|x| \to +\infty} \|(a_{ij})_{x_i}\|_{L^1(\Omega \cap B(x, 1))} = 0, \quad i, j, b = 1, \ldots, n,
\]

where \( B(x, 1) = \{ y \in \mathbb{R}^n \mid |y - x| < 1 \} \). In [13], less restrictive regularity hypotheses on the \( a_{ij} \) are imposed, but together with the condition to the infinity

\[
\lim_{|x| \to +\infty} a_{ij}(x) = a_{ij}^0.
\]

In [14] the following case has been dealt with:

\[
a_{ij} = a_{ij}^l + a_{ij}^r, \quad a_{ij}^l = a_{ij}^r \in L^\infty(\Omega) \cap W^{1,1}_{C, loc} (\Omega), \quad a_{ij}^0 = a_{ij}^0 \in C(\overline{\Omega})
\]

and

\[
\lim_{|x| \to +\infty} \|(a_{ij}^l)_{x_i}\|_{L^1(\Omega \cap B(x, 1))} = 0, \quad \lim_{|x| \to +\infty} a_{ij}^{l0}(x) = a_{ij}^{r0}
\]

or

\[
\lim_{|x| \to +\infty} a_{ij}(x) = a_{ij}^0,
\]

where \( C(\overline{\Omega}) \) is the space of uniformly continuous bounded functions on \( \Omega \).

Successively (see [15]) the same Authors studied problem (3) in more general hypotheses on the \( a_{ij} \), and extended an a priori bound that in the previous papers is basic for some existence and uniqueness results.

In these hypotheses, like is pointed out by the Authors in [15], is not possible, with the methods of [10], [13], [14], to deduce from the a priori bound the theorems of existence and uniqueness.

In this paper we consider problem (3) in the hypotheses of [15], and in addition we suppose that \( \partial \Omega \) is bounded. In such hypotheses we prove that problem (3) is uniquely solvable.

To obtain this result we use the a priori bound established in [15] and a new a pri-
ori bound proved in this paper for the solutions of the problem

\[ u \in W^2(\Omega_r) \cap \overset{\circ}{W}^1(\Omega_r), \quad Lu = f, \quad f \in L^2(\Omega). \]

Here \( \Omega_r = \{ x \in \Omega | |x| < r \} \) and the constant of the bound doesn’t depend on \( r \in [r_0, +\infty[ \), where \( r_0 \) is a sufficiently large, fixed value.

1. Some preliminary facts

We set

\[ B(x, r) = \{ y \in \mathbb{R}^n | |y - x| < r \}, \quad B_r = B(0, r), \]

\[ |u|_{p, E} = \| u \|_{L^p(E)}, \quad u_x = \left( \sum_{i=1}^{n} u_{x_i}^2 \right)^{1/2}, \quad u_{xx} = \left( \sum_{i,j=1}^{n} u_{x_i x_j}^2 \right)^{1/2}. \]

Let \( E \) be an open subset of \( \mathbb{R}^n, n \geq 2 \).
We denote by \( W^m(E) , m \in \mathbb{N}, \) the usual Sobolev space \( W^{m,2}(E) \) and by \( \overset{\circ}{W}^m(E) \) the closure of \( \mathcal{O}(E) \) in \( W^m(E) \).
Moreover we set \( W^m_0(E) \) for \( L^2(E) \).
If \( E \) is unbounded, for any \( p \in [1, +\infty[ \) \( M^p(E) \) denotes the space of all functions \( f \in L^p_\text{loc}(E) \) such that

\[ \| f \|_{M^p(E)} = \sup_{x \in E} |f|_{p, E \cap B(x, 1)} < +\infty, \]

normed by (1.1), and \( M^p_0(E) \) denotes the subspace of \( M^p(E) \) of all functions \( f \) such that

\[ \lim_{|x| \to +\infty} |f|_{p, E \cap B(x, 1)} = 0. \]

For some properties of the spaces \( M^p(E) \) and \( M^p_0(E) \) we refer to [10], [11].
Let \( s, t \) be two real numbers such that

\[ s > 2 \quad \text{if} \quad n = 2, \quad s = n \quad \text{if} \quad n > 2, \]

\[ t = 2 \quad \text{if} \quad 2 \leq n < 4, \quad t > 2 \quad \text{if} \quad n = 4, \quad t = n/2 \quad \text{if} \quad n > 4. \]

If \( A \subset E \) is an open subset, \( k \in \mathbb{N} \) and \( v \in \mathbb{R}_+ \), we write \( E_k(v, A) \) for the class of the \( k \times k \) square matrices \( (e_{ij}) \) such that

\[ e_{ij} = e_{ij} \in L^\infty(A) \cap \overset{\circ}{W}^1_\text{loc}(A), \quad i, j = 1, \ldots, k, \]

\[ \sum_{i,j=1}^{k} e_{ij} \eta_i \eta_j \geq v |\eta|^2 \quad \text{a.e. in} \ A, \quad \forall \eta \in \mathbb{R}^k. \]

Moreover, we denote by \( G(A) \) the class of all functions \( g \in L^\infty(A) \) such that

\[ \text{ess inf}_A g \geq 0. \]

We consider in \( E \) the second order linear differential operator \( L \) defined by (1)
and write

\[ L_0 u = - \sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} \].

We say that \( L_0 \) is of Chicco type in \( E \) (see [3]-[5] and [13]-[15]) if

\[ a_{ij} = a_{ji} \in L^\infty(E), \quad i,j = 1, \ldots, n, \]

and there exist \( v \in R_+ \), a matrix \((e_{ij}) \in E_n(v, E)\) and a function \( g \in G(E) \) such that

\[ \text{ess sup}_{E} \sum_{i,j=1}^{n} (e_{ij} - ga_{ij})^2 < v^2. \]

\[ 2. \text{ Preliminary Lemmas} \]

**Lemma 2.1:** Let \( r_1 \in R_+ \). For each \( r \in [r_1, +\infty[ \) there exists a continuous linear operator

\[ q_r : W^2(B_r) \rightarrow W^2(R^n) \]

such that

\[ q_r u|_{B_{r}} = u, \]

\[ \|q_r u\|_{W^k(R^n)} \leq c \|u\|_{W^k(B_r)}, \quad k = 0, 1, 2, \]

where \( c \in R_+ \) is independent of \( r \) and \( u \).

**Proof:** It is well known that there exists \( q_r \), satisfying these properties (see R. A. Adams [1]).

We note that, given \( r \in [r_1, +\infty[ \) and \( u \in W^2(B_r) \) the function

\[ v(y) = u\left(\frac{r}{r_1}y\right) \in W^2(B_{r_1}) \]

and hence \( q_r v \in W^2(R^n) \).

Write \( w_r(x) = (q_r v)(\frac{r_1}{r} x) \), for every \( x \in R^n \), then we define

\[ q_r : u \in W^2(B_r) \rightarrow q_r u = w_r \in W^2(R^n). \]

If \( x \in B_{r_1} \), we have

\[ w_r(x) = v\left(\frac{r_1}{r} x\right) = u(x) \]

and then (2.1) is proved.
Moreover, we have that

\begin{equation}
|w_r|_{L^2(R^*)} \leq \left( \int_{R^*} \left( \frac{r}{r_1^x} x \right)^2 \, dx \right)^{1/2} =
\end{equation}

\begin{equation}
\leq c \left( \int_{B_{r_1}} \frac{r}{r_1} \frac{x^2}{\theta} \, dy \right)^{1/2} = c|u|_{L^2(B)},
\end{equation}

with \( c \in R_+ \) independent of \( r \) and \( u \), then (2.2) for \( k = 0 \) is obtained.

On the other hand we get

\begin{equation}
|(w_r)_r|_{L^2(R^*)} \leq c(r^{-1} |u|_{L^2(B)} + |u|_{L^2(B)}),
\end{equation}

\begin{equation}
|(w_r)_{xx}|_{L^2(R^*)} \leq c(r^2 |u|_{L^2(B)} + r^{-1} |u|_{L^2(B)} + |u|_{L^2(B)}),
\end{equation}

with \( c \in R_+ \) independent of \( r \) and \( u \).

From (2.3), (2.4) and (2.5) the bound (2.2) for \( k = 1, 2 \) easily follows.

**Lemma 2.2.** Let \( \epsilon_1, r_1 \in R_+ \). Then there exists a constant \( c \in R_+ \) such that

\begin{equation}
\int_{B_{r}} u_{\epsilon}^2 \, dx \leq c \left( \epsilon \int_{B_{r}} u_{\epsilon r}^2 \, dx + \frac{1}{\epsilon} \int_{B_{r}} u_{\epsilon}^2 \, dx \right),
\end{equation}

\( \forall \epsilon \in ]0, \epsilon_1[, \forall r \in [r_1, +\infty[ \text{ and } \forall u \in W^2(B_r). \)

**Proof:** We fix \( \epsilon \in ]0, \epsilon_1[, r \in [r_1, +\infty[, u \in W^2(B_r) \) and put

\[ v(y) = u(\epsilon y). \]

Since \( v \in W^2(B_1) \), for any \( \epsilon_0 \in \left[0, \frac{\epsilon_1}{r_1^2}\right] \) we obtain

\begin{equation}
\int_{B_1} v_{\epsilon}^2 \, dy \leq c(B_1) \left( \epsilon_0 \int_{B_1} v_{\epsilon r}^2 \, dy + \epsilon_0 \int_{B_1} v_{\epsilon}^2 \, dy \right),
\end{equation}

where \( c(B_1) \in R_+ \) is independent of \( \epsilon_0 \) and \( v \) (see e.g. [1], pag. 75).

(2.6) follows now from (2.7) when \( y = \frac{x}{r} \) and \( \epsilon_0 = \frac{\epsilon}{r} \).

Now we assume:

i) let \( \Omega \) be an unbounded open subset of \( R^n, n \geq 2 \), of class \( C^2 \) with bounded boundary.

Put

\[ \Omega_r = \Omega \cap B_r, \quad \forall r \in R_+. \]
Choose \( r_1 \in R \), such that

\[
\partial \Omega \subset B_{r_1}
\]

and let \( \psi \) be a function in \( \mathcal{O}(R^n) \) such that

\[
\psi|_{R^n \setminus \Omega} = 1, \quad \text{supp} \psi \subset B_{r_1}.
\]

**Lemma 2.3:** For any \( r \in [r_1, +\infty[ \) there exists a continuous linear operator

\[
p_r: W^2(\Omega_r) \to W^2(\Omega)
\]

such that

\[
p_r u|_{\Omega_r} = u,
\]

\[
\|p_r u\|_{W^k(\Omega)} \leq c \|u\|_{W^k(\Omega)}, \quad k = 0, 1, 2,
\]

where \( c \in R_+ \) is independent of \( r \) and \( u \).

**Proof:** Let \( r \in [r_1, +\infty[ \) and \( u \in W^2(\Omega_r) \).

Define

\[
v = \begin{cases} \psi u & \text{in } \Omega, \\ 0 & \text{in } \Omega \setminus \Omega_r, \end{cases}, \quad w = \begin{cases} (1 - \psi) u & \text{in } \Omega, \\ 0 & \text{in } B_r \setminus \Omega_r, \end{cases}, \quad p_r u = v + (q, w)|_{\Omega},
\]

where \( q \) is the operator in Lemma 2.1.

From (2.1) and (2.2) we easily obtain (2.8) and (2.9), and the result follows.

**Lemma 2.4:** For any \( \epsilon \in R_+ \) there exists a constant \( c(\epsilon) \in R_+ \) such that

\[
\sum_{\Omega} u^2 \leq \epsilon \int_{\Omega} u^2 \leq 2 \left( \int_{\Omega} (\psi u)^2 \right) + c(\epsilon) \int_{\Omega} u^2 \leq \forall r \in [r_1, +\infty[ \) and \( \forall u \in W^2(\Omega_r) \).
\]

**Proof:** Let \( r \in [r_1, +\infty[ \) and \( u \in W^2(\Omega_r) \).

If \( w \) is the map defined in the proof of Lemma 2.3, we have

\[
\sum_{\Omega} u^2 \leq 2 \left( \int_{\Omega} (\psi u)^2 \right) + c_1(\epsilon) \int_{\Omega} u^2 \leq 2 \left( \int_{\Omega} (\psi u)^2 + \int_{B_r} u^2 \right).
\]

It is well known that

\[
\sum_{\Omega} u^2 \leq c(\epsilon) \int_{\Omega} u^2 \leq 2 \left( \int_{\Omega} (\psi u)^2 \right) + c_1(\epsilon) \int_{\Omega} u^2,
\]

with \( c_1(\epsilon) \in R_+ \) independent of \( u \).
Moreover, from Lemma 2.2 it follows that

\[ \int B_{b_2} u_2^2 \, dx \leq \varepsilon_2 \int B_{b_2} u_2^2 \, dx + c_2 (\varepsilon_2) \int B_{b_2} u^2 \, dx, \]

with \( c_2 (\varepsilon_2) \in R_+ \) independent of \( r \) and \( u \).

From (2.11), (2.12), and (2.13) we obtain easily (2.10), as required.

3. - AN A PRIORI BOUND

We consider in \( \Omega \) the second order linear differential operators \( L \) and \( L_0 \) defined respectively by (1) and (1.7).

Now we assume the following:

i3) \( L_0 \) is of Chicco type in \( \Omega \); moreover

\[ a_i \in D'_{0} (\Omega) \quad i = 1, \ldots, n, \]

\[ a = a' + a'' \quad a' \in D'_{0} (\Omega) \quad a'' \in D' (\Omega); \]

i3) there exist \( \mu, \mu_0, \eta_0 \in R_+ \), \( (a_{xy}) \in E_{\eta} (\mu, \Omega \setminus \overline{B}_{\eta_0}), \quad ((\alpha)) \in E_1 (\mu_0, \Omega \setminus \overline{B}_{\eta_0}), \quad \eta \in G (\Omega) \) such that

\[ \sum_{i,j=1}^{n} (a_{xy} - \eta a_{xy})^2 \quad \mu_0^{-2} \quad \text{ess sup} \quad (x - \eta a'')^2 < 1. \]

We note that the last condition is equivalent to condition i3) defined in n. 3 of [15], where two examples of operators for which it holds are quoted (see examples 4 and 5 of [15]).

Next example contains the two quoted examples of [15].

**Example 3.1:** If we suppose:

\[ \sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j \geq a_0 |x|^2 \quad \text{a.e. in } \Omega, \quad \forall \xi \in R^n, \]

\[ a_{xy} = b_{xy} + c_{xy}, \quad b_{xy} = b_{xy} \in L^\infty (\Omega), \quad (b_{xy})_{xy} \in D'_{0} (\Omega), \]

\[ \lim_{|x| \to +\infty} c_{xy} (x) = c_{xx}, \quad c_{xx} = c_{xx}^0, \quad a'' \in G (\Omega), \]

then, clearly, i3) holds with

\[ \mu = \frac{a_0}{2}, \quad \mu_0 = \text{ess sup} \quad a'', \quad a_{xy} = b_{xy} + c_{xy}^0, \quad \alpha = \mu_0, \quad \eta = 1. \]
and \( r_0 \in R_+ \) such that
\[
\sum_{i,j=1}^n a_{ij} \xi_i \xi_j \geq \mu |\xi|^2 \quad \text{a.e. in } \Omega \setminus \overline{B}_{r_0}, \forall \xi \in \mathbb{R}^n,
\]
\[
\text{ess sup}_{\Omega \setminus \overline{B}_r} \sum_{i,j=1}^n (c_{ij} - c_{ij})^2 < \mu \left( \text{ess inf}_{\Omega} a''_{\alpha_0} \right) \cdot
\]

**Remark 3.1:** From Example 3.1 we deduce that \( i_2 \) and \( i_3 \) are satisfied by the hypotheses, mentioned in the introduction, contained in the papers [10, 13, 14].

Let \( \beta \) denote a mapping \( \Omega \to R_+ \) such that
\[
\beta \in M^t(\Omega) \quad \text{and} \quad \exists \delta \in M_0^d(\Omega) \quad \exists \beta \leq \beta^t,
\]
\[
\beta, \beta^{-1} \in L^\infty(\overline{\Omega}).
\]

Let \( r_1 \in R_+ \) such that
\[
r_1 > r_0, \quad \partial \Omega \subset B_{r_1},
\]
where \( r_0 \) is the number defined in hypothesis \( i_3 \).

**Lemma 3.1:** If \( i_1 \), \( i_2 \) and \( i_3 \) hold, then there exist a constant \( c \in R_+ \) and a bounded open set \( \Omega_0 \subset \Omega \) such that:
\[
\|u\|_{W^2(\Omega_0)} \leq c(\|Lu + \lambda \eta^{-1} \beta u\|_{2,\Omega} + |u|_{2,\Omega_0} \cap \Omega_0)
\]
\[
\forall \lambda \geq 0, \forall r \in [r_1, +\infty] \quad \text{and} \quad \forall u \in W^2(\Omega_r) \cap W^1(\overline{\Omega}_r).
\]

**Proof:** Assume firstly that \( L = L_0 + a'' \), and fix \( r_0 \in [0, r_1] \) and \( \phi \in C_0(\mathbb{R}^n) \) such that \( \partial \Omega \subset B_{r_0} \), \( \phi|_{B_{r_0}} = 1 \) and \( \text{supp } \phi \subset B_{r_1} \).

We first prove (3.8) for \( \phi u \).

Since the operator \( \eta^{-1} L_0 \) is of Chicco type in \( \Omega_n \), from \([4]\) we obtain
\[
\|\phi u\|_{W^2(\Omega_r)} = \|\phi u\|_{W^2(\Omega_r)} \leq c(r_1) \|L_0 + a''(L_0 + a'')\phi u + \lambda \eta^{-1} \beta \phi u\|_{2,\Omega_n} + |\phi u|_{2,\Omega_n} \|u\|_{2,\Omega_n} \leq
\]
\[
\leq c'(r_1) \|L_0 + a''\phi u + \lambda \eta^{-1} \beta \phi u\|_{2,\Omega_n} + |\phi u|_{2,\Omega_n} =
\]
\[
= c'(r_1) \|L_0 + a''\phi u + \lambda \eta^{-1} \beta \phi u\|_{2,\Omega_n} + |\phi u|_{2,\Omega_n} \cap \Omega_n,
\]
where \( c(r_1), c'(r_1) \in R_+ \) are independent of \( \lambda, r \) and \( u \).

We prove now (3.8) for \( v = (1 - \phi) u \).

Write \( E = \Omega_n \setminus \overline{\Omega}_{r_0} \), and consider in \( E \), the operator
\[
Av = - \sum_{i,j=1}^n a_{ij} v_i v_j.
\]

Note that \( \text{supp } v \subset \overline{\Omega}_{r_0} \setminus \Omega_n \). Then, from well-known results (see [8], pag. 152,
162), using methods similar to those in [15] to establish (36), (37) and (38), it follows that

$$(3.10) \quad \mu^2 \int_{E_r} v_{xx}^2 \, dx \leq \int_{E_r} (Av)^2 \, dx + \varepsilon_1 \int_{E_r} v_{xx}^2 \, dx + c(\varepsilon_1) \int_{E_r} \sum_{i,j=1}^n (\alpha_{ij}^2 + \alpha_x^2 + \delta^2) v_x^2 \, dx,$$

where $c(\varepsilon_1) \in R_+$ is independent of $r$ and $\nu$.

On the other hand we have:

$$(3.11) \quad \int_{E_r} (Av)^2 \, dx \leq \int_{E_r} (Av + (\alpha + \lambda \beta) \nu)^2 \, dx - (1 - \varepsilon_2) \int_{E_r} (\alpha + \lambda \beta)^2 \nu^2 \, dx + c(\varepsilon_2) \int_{E_r} \left( \sum_{i,j=1}^n (\alpha_{ij}^2 + \alpha_x^2 + \delta^2) \right) v_x^2 \, dx,$$

where $c(\varepsilon_2) \in R_+$ is independent of $\lambda$, $r$ and $\nu$ (see (41)-(43) of [15]).

Since

$$\text{ess inf}_{E_r} (\alpha + \lambda \beta) \geq \mu_0 \quad \forall \lambda \geq 0,$$

from (3.10) and (3.11) we obtain that

$$(3.12) \quad (\mu^2 - \varepsilon_1) \int_{E_r} v_{xx}^2 \, dx + (\mu_0^2 - \varepsilon_3) \int_{E_r} v^2 \, dx \leq \int_{E_r} (Av + (\alpha + \lambda \beta) \nu)^2 \, dx + c(\varepsilon_1, \varepsilon_3) \int_{E_r} \left( \sum_{i,j=1}^n (\alpha_{ij}^2 + \alpha_x^2 + \delta^2) \right) v_x^2 \, dx,$$

where $c(\varepsilon_1, \varepsilon_3) \in R_+$ is independent of $\lambda$, $r$ and $\nu$.

Put

$$g = \left( \sum_{i,j=1}^n (\alpha_{ij}^2 + \alpha_x^2 + \delta^2) \right)^{1/2}.$$

Since $g \in M_0^0(\Omega \setminus \overline{B}_a)$, from Theorem 3.2 of [7] and from Lemma 2.3 we get

$$(3.13) \quad \int_{E_r} g^2 v_x^2 \, dx \leq \int_{\Omega \setminus \overline{B}_a} g^2 (p, v)_x^2 \, dx \leq$$

$$\leq \varepsilon_4 \int_{\Omega} (p, v)_x^2 \, dx + c(\varepsilon_4) \int_{\Omega \cap \partial(\varepsilon_4)} (p, v)_x^2 \, dx \leq \varepsilon_4 \left\| p \right\|_{W^2(\Omega)} + c(\varepsilon_4) \int_{\Omega \cap \partial(\varepsilon_4)} (p, v)^2 \, dx,$$

where the constants $c, c(\varepsilon_4) \in R_+$ and the bounded open set $\Omega(\varepsilon_4) \subset \Omega$ are independent of $r$ and $\nu$. 
From (3.12), (3.13) and Lemma 2.4, we have that for any \( \varepsilon' \in ]0, \mu[ \) and for any \( \varepsilon'' \in ]0, \mu_0[ \) the following holds:

\[
(\mu - \varepsilon')^2 |v_{xx}|^2_{1, \Omega} + (\mu_0 - \varepsilon'')^2 |v|_{1, \Omega}^2 \leq \\
\leq |\lambda v + (\alpha + \lambda \beta) v|_{2, \Omega}^2 + c(\varepsilon', \varepsilon'') |p, v|_{2, \Omega \cap \Omega(\varepsilon', \varepsilon'')},
\]

where the constant \( c(\varepsilon', \varepsilon'') \in R_+ \) and the bounded open set \( \Omega(\varepsilon', \varepsilon'') \subset \Omega \) are independent of \( \lambda, r \) and \( v \).

With the same proof of [15] to deduce (33) from (45), from (3.14) we have

\[
|v_{xx}|_{2, \Omega} + |v|_{2, \Omega} \leq c_1 \left( |(L_0 + d'') v + \lambda \eta^{-1} \beta v|_{2, \Omega} + |p, v|_{2, \Omega \cap \Omega^*} \right),
\]

where the constant \( c_1 \in R_+ \) and the bounded open set \( \Omega^* \subset \Omega \) are independent of \( \lambda, r \) and \( v \).

Now we choose \( \sigma \in ]1, +\infty[ \) such that \( \Omega^* \subset \Omega_\sigma \).

From Lemma 2.3 we have

\[
|p, v|_{2, \Omega \cap \Omega_\sigma} \leq c|v|_{2, \Omega \cap \Omega_\sigma}.
\]

(3.8) for \( v \) follows now from (3.15), (3.16) and Lemma 2.4.

Then we have

\[
\|u\|_{W^2(\Omega)} \leq \|\phi u\|_{W^2(\Omega)} + \|(1 - \phi) u\|_{W^2(\Omega)} \leq \\
\leq c(\|L_0 \phi u + d'' \phi u + \lambda \eta^{-1} \beta \phi u|_{2, \Omega} + |\phi u|_{2, \Omega \cap \Omega_\sigma} + \\
+ |L_0 (1 - \phi) u + d''(1 - \phi) u + \lambda \eta^{-1} \beta (1 - \phi) u|_{2, \Omega} + |(1 - \phi) u|_{2, \Omega \cap \Omega_\sigma}) \leq \\
\leq c_2 (\|L_0 u + d'' u + \lambda \eta^{-1} \beta u|_{2, \Omega} + |u|_{2, \Omega \cap \Omega_\sigma} + \|u_{x_i} + u_{x_i} + u_{x_{i+1}} + u_{x_{i+1}}|_{2, \Omega_i} + |u_{x_{i+1}}|_{2, \Omega_i}),
\]

with \( c_2 \in R_+ \) independent of \( \lambda, r \) and \( u \).

On the other hand, from Lemma 2.4 it follows:

\[
\int_{\Omega_{\sigma_1}} u^2 \, dx \leq c \int_{\Omega_{\sigma_1}} u^2 \, dx + c(\varepsilon) \int_{\Omega_{\sigma_1}} u^2 \, dx.
\]

From (3.17) and (3.18) we deduce (3.8) for \( L = L_0 + d'' \).

In the general case for \( L \) we have then:

\[
\|u\|_{W^2(\Omega)} \leq c \left( \|L u + \lambda \eta^{-1} \beta u|_{2, \Omega} + |u|_{2, \Omega \cap \Omega_\sigma} + \sum_{i=1}^n \alpha_i u_{x_i} + a' u \right)_{2, \Omega}.
\]

On the other hand, since \( u \in M_{\alpha}^{\beta}(\Omega) \), \( i = 1, \ldots, n \), and \( u' \in M_{\alpha}^{\beta}(\Omega) \), from Theorem
3.2 in [7] and from Lemma 2.3 we get

\[
\sum_{i=1}^{n} a_i u_i + a'u \leq e_5 \|p, u\|_{W^1(\Omega)} + c(e_5) \int_{\Omega \cap \hat{\Omega}(e_5)} (p, u)^2 \, dx \leq e_5 \|u\|_{W^2(\Omega)} + c(e_5) \int_{\Omega \cap \hat{\Omega}(e_5)} (p, u)^2 \, dx,
\]

where the constant \(c(e_5) \in R^+\) and the bounded open set \(\hat{\Omega}(e_5) \subset \Omega\) are independent of \(r\) and \(u\).

From (3.19) and (3.20) we deduce that:

\[
\|u\|_{W^2(\Omega)} \leq c_3 \left( \|Lu + \lambda \eta^{-1} g\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega \cap \hat{\Omega}(e_5))} + \|p, u\|_{L^2(\Omega \cap \hat{\Omega}(e_5))} \right),
\]

where the constant \(c_3 \in R^+\) and the bounded open set \(\Omega^{**} \subset \Omega\) are independent of \(\lambda\), \(r\) and \(u\).

Now we choose \(\tau \in ]\eta_1, +\infty[\) such that \(\Omega^{**} \subset \Omega_\tau\); then

\[
|p, u|_{L^2(\Omega_\tau)} \leq c_4 |u|_{L^2(\Omega_\tau)},
\]

where \(c_4 \in R^+\) is independent of \(r\) and \(u\).

The result follows from (3.21) and (3.22).

4. - Existence results

Let \(G\) be a bounded open subset of class \(C^2\) of \(R^n, n \geq 2\).

We consider in \(G\) the second order linear differential operators \(L\) and \(L_0\) defined in (1) and (1.7) respectively.

We consider the problem

\[
u \in W^2(G) \cap \overset{\circ}{W}^1(G), \quad Lu = f, \quad f \in L^2(G).
\]

**Lemma 4.1:** If \(L_0\) is of Chicco type in \(G\), \(a_i \in L^1(G)\) \((i = 1, \ldots, n)\), \(a \in L^1(G)\) and

\[
es\inf_G a > 0,
\]

then the problem (4.1) is uniquely solvable. Moreover, if \(f \in L^\infty(G)\), then the solution \(u \in L^\infty(G)\) and satisfies

\[
|u|_{L^\infty(G)} \leq (\text{ess \, inf}_G (ga))^{-1} |gf|_{L^\infty(G)},
\]

where \(g\) is the function defined as in the hypothesis of Chicco type.

**Proof:** The existence and uniqueness result for the problem (4.1) follows from a theorem of M. Chicco (see [5]). The last statement is proved, as in the proof of the theorem of [5], noting that, if \(f \in L^\infty(G)\), the solution \(u\) is the weak limit in \(W^2(G)\) of a
sequence \((u_k)_{k \in \mathbb{N}}\) of functions such that:

\[
|u_k|_{\infty, C} \leq (\text{ess inf}_{C} (ga))^{-1} |gf|_{\infty, C} \quad \forall k \in \mathbb{N}.
\]

Concerning problem (3), we prove

**Theorem 4.1:** If i), i), and i) hold, then (3) is an index problem with index zero.

If moreover we suppose that

\[
\text{ess inf}_{a} a > 0,
\]

then problem (3) is uniquely solvable.

**Proof:** We suppose that i), i), i), and (4.6) hold.

Fix a strictly increasing sequence \((\tau_k)_{k \in \mathbb{N}}\) of positive real numbers, with \(\tau_1\) satisfying (3.7).

Suppose firstly that \(f \in L^2(\Omega) \cap L^\infty(\Omega)\) and consider, for every \(k \in \mathbb{N}\) the problem

\[
u \in \mathcal{W}^2(\Omega) \cap C(\overline{\Omega}), \quad Lu = f.
\]

From Lemmas 3.1 and 4.1 it follows that the solution \(u_k, k \in \mathbb{N}\), of problem (4.7) belongs to \(L^\infty(\Omega)\) and satisfies

\[
\|u_k\|_{\mathcal{W}^2(\Omega)} \leq c(|f|_{2,\Omega} + |u_k|_{2,\Omega \cap \Omega_0}),
\]

\[
|u_k|_{\infty, \Omega_0} \leq (\text{ess inf}_{\Omega_0} (ga))^{-1} |gf|_{\infty, \Omega_0},
\]

where the constant \(c \in \mathbb{R}^+\) and the bounded open set \(\Omega_0 \subset \Omega\) are independent of \(k\).

From (4.8) and (4.9) we get

\[
\|u_k\|_{\mathcal{W}^2(\Omega)} \leq c(|f|_{2,\Omega} + (\text{mis}(\Omega_0 \cap \Omega_0))^{1/2} (\text{ess inf}_{\Omega_0} (ga))^{-1} |gf|_{\infty, \Omega_0}) \leq c(|f|_{2,\Omega} + (\text{mis}(\Omega_0))^{1/2} (\text{ess inf}_{\Omega_0} (ga))^{-1} |gf|_{\infty, \Omega}) \quad \forall k \in \mathbb{N}.
\]

Write \(w_k = p_n u_k, k \in \mathbb{N}\), where \(p_n\) is the operator defined in Lemma 2.3.

From Lemma 2.3 and from (4.10) it follows that \(w_k \in \mathcal{W}^2(\Omega), k \in \mathbb{N}\), and there exists a constant \(c_0 \in \mathbb{R}^+\) such that

\[
\|w_k\|_{\mathcal{W}^2(\Omega)} \leq c_0 \quad \forall k \in \mathbb{N}.
\]

From (4.11) we deduce the existence of a subsequence of \((w_k)_{k \in \mathbb{N}}\) weakly convergent in \(\mathcal{W}^2(\Omega)\) to a function \(u \in \mathcal{W}^2(\Omega) \cap \mathcal{W}^1(\Omega)\). Since \(u_k\) is solution of (4.7) for every \(k \in \mathbb{N}\), with standard considerations we prove that \(u\) is solution of problem (3) with \(f \in L^2(\Omega) \cap L^\infty(\Omega)\).

From Theorem 3 of [15] and well-known results, it follows that the range \(R(L)\) of
the operator

\[ I: u \in W^2(\Omega) \cap \overset{0}{W}^1(\Omega) \rightarrow Iu \in L^2(\Omega) \]

is a closed subspace of \( L^2(\Omega) \). Moreover, from above result we have that \( L^2(\Omega) \cap L^\infty(\Omega) \subset R(L) \).

On the other hand, \( L^2(\Omega) \cap L^\infty(\Omega) \) is dense in \( L^2(\Omega) \); then we have:

\[ (4.12) \quad R(L) = L^2(\Omega) . \]

Suppose now that \( i_1 \), \( i_2 \), \( i_3 \) hold, but \( (4.6) \) doesn’t hold.
Consider a function \( \beta: \Omega \rightarrow R_+ \) of class \( M^0(\Omega) \) and satisfying (3.5) and (3.6).
For example

\[ \beta: x \in \Omega \rightarrow \frac{1}{(1 + |x|)^{\tau}} \quad , \quad \tau \in R_+ . \]

We note now that hypothesis \( i_3 \) implies:

\[ b_0 = \operatorname{ess\, inf}_{\Omega \setminus \overline{B}_0} a'' > 0 . \]

Fix \( \zeta \in C(\mathbb{R}^n) \) such that \( 0 < \zeta \leq 1 \), \( \zeta|_{B_0} = 1 \), \( \operatorname{supp} \zeta \subset B_{2\varepsilon_0} \) and put:

\[ b = \zeta b_0 + (1 - \zeta) a'' . \]

Clearly we have

\[ \operatorname{ess\, inf}_{\Omega} b \geq b_0 , \quad a - b = a' + \zeta(a'' - b_0) \in M^0(\Omega) , \]

\[ \mu^{-2} \operatorname{ess\, sup}_{\Omega \setminus \overline{B}_0} \sum_{i,j=1}^{n} (a_{ij} - \eta a_{ij})^2 + \mu_0^{-2} \operatorname{ess\, sup}_{\Omega \setminus \overline{B}_0} (a - \eta b)^2 < 1 . \]

We consider the operator

\[ A_\lambda: u \in W^2(\Omega) \cap \overset{0}{W}^1(\Omega) \rightarrow -\sum_{i,j=1}^{n} a_{ij}u_{ij,x} + \sum_{i=1}^{n} a_i u_{ix} + (b + \lambda \sigma^{-1} \beta) u \in L^2(\Omega) . \]

From above results we have:

\[ (4.13) \quad R(A_\lambda) = L^2(\Omega) \quad \forall \lambda > 0 . \]

On the other hand, with the same proof of [10] to deduce Corollary 4.2 from Theorem 4.4, from Theorem 3 of [15] it follows that there exists \( \lambda_0 \in R_+ \) such that

\[ (4.14) \quad N(A_\lambda) = \{0\} \quad \forall \lambda \geq \lambda_0 , \]

where \( N(A_\lambda) \) is the kernel of the operator \( A_\lambda \).

From (4.13) and (4.14) it follows that for every \( \lambda \geq \lambda_0 \) \( A_\lambda \) is a bijective operator.
Since the operator:

\[ u \in W^2(\Omega) \rightarrow (a - b - \lambda \eta^{-1} \beta) u \in L^2(\Omega) \]

is compact (see Lemma 3.4 of [10]) and:

\[ Lu = A_3 u + (a - b - \lambda \eta^{-1} \beta) u , \]

we deduce, from well-known results, that (3) is an index problem with index zero.

If also (4.6) holds, from above and from (4.12) we obtain that (3) is uniquely solvable.

REFERENCES


