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Inversion Methods for Fourier Transform (**)

SUMMARY. — A general method which allows to reconstruct a periodic function, knowing only an approximation of its Fourier coefficients, is presented in [2] in its main aspects. In this paper the method is widely illustrated. Proofs of theorems stated in [2] are given and large classes of approximate units are shown which can be employed in the method and make it effective. These classes include classical and radial kernels, besides special kernels used in [1].

Metodi di inversione per la trasformata di Fourier

RIASSUNTO. — Nella Nota viene ampiamente illustrato il metodo generale presentato in [2] per la ricostruzione di una funzione periodica di cui siano noti soltanto valori approssimati dei coefficienti di Fourier. Vengono dimostrati i teoremi enunciati in [2] e vengono presentate ampie classi di unità approssimate che possono essere utilizzate nel metodo e che lo rendono operativo. In tali classi rientrano i nuclei classici, i nuclei radiali e i nuclei impiegati in [1].

1. - It is well known that the Fourier transform $F(f)$ of a function f in $L^p(T^N)$ is in $l^{q_0}(Z^N)$, where q_0 is the conjugate exponent of $p_0 = \min(p, 2)$ and unfortunately the map F between these spaces is not surjective unless $p = 2$. Then F^{-1} is not continuous and this implies that (except for L^2) the reconstruction of a periodic function from an approximation of its Fourier coefficients (obviously very important for the applications) is an ill-posed problem.

First Tikhonov and Arsénine [10] suggested a method to solve this problem for particular families of continuous functions.

Recently we constructed in [1] some classes of regularizing operators for the problem, whatever is p, which turn out to be of interest also for numerical applications (see *e.g.* [3], [4], [6], [7]). The construction is based on the properties of a special approximate unity.

The method in [1] can be developed in the general setting of approximations to the identity and leads to similar results only with some very light hypotheses on the kernels, stated in [2].

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Here we prove these results and moreover we give large and significant classes of approximate units which fall under the scope of the theorems.

2. - Let us now introduce some standard notations.

If $N \geqslant 1$, let Z^N be the lattice of integer points of R^N and $T^N = R^N/Z^N$ the N-dimensional torus. Let us name B, indifferently, the Lebesgue space $L^p(T^N)$, $1 \leqslant p < +\infty$ or the space of continuous functions $C(T^N)$ and denote their norm by $\| \, \|_B$. For our convenience, we identify T^N with $[-1/2, 1/2)^N = Q^N$

Let $p_0 = \min(p, 2)$ (for $B = C(T^N)$ we set $p = +\infty$) and q_0 such that $1/p_0 + 1/q_0 = 1$ and let $b = l^{q_0}(Z^N)$ with the usual norm.

If $f \in L^1(T^N)$ or $L^1(R^N)$ we denote by $\hat{f}$ respectively the sequence of its Fourier coefficients

$$\hat{f}_n = \int_{T^N} f(t) \exp[-2\pi int]\, dt, \qquad n \in Z^N$$

or its Fourier transform

$$\hat{f}(x) = \int_{R^N} f(y) \exp[-2\pi ixy]\, dy.$$

Finally, let $G \in L^1(R^N)$ such that $\hat{G}(0) = 1$ and, for every $\lambda = \{\lambda_n\}_{n \in Z^N}$, $\lambda_n \in C\ \forall n$, let us set

$$R_\sigma \lambda \sim \sum_{n \in Z^N} \hat{G}(\sigma n)\, \lambda_n \exp[2\pi int], \qquad t \in T^N.$$

3. - With the previous notations, the following theorem holds.

THEOREM 1: *If* $\{\hat{G}(\sigma n)\} \in l^{p_0}(Z^N)$ *for every* $\sigma > 0$ *and* $\sigma = \sigma(\delta): R^+ \to R^+$ *satisfies*

$$\lim_{\delta \to 0} \sigma(\delta) = 0, \qquad \lim_{\delta \to 0} \delta \| \{\hat{G}(\sigma(\delta)\, n)\} \|_2^{1-2/p|} = 0, \tag{3.1}$$

then, if $f \in B$, *for every* $\varepsilon > 0$ *there exists* $\delta_0 = \delta_0(\varepsilon, f)$ *such that, if* $\delta < \delta_0$

$$\lambda \in b \quad \text{and} \quad \|\lambda - \hat{f}\|_b \leqslant \delta \Rightarrow \|f - R_{\sigma(\delta)}\lambda\|_B < \varepsilon. \tag{3.2}$$

In order to prove this result, we need the following

LEMMA 1: *If* $\{\hat{G}(\sigma n)\} \in l^{p_0}(Z^N)$, *then*

$$\|R_\sigma\|_{[b,B]} \leqslant a_p \| \{\hat{G}(\sigma n)\} \|_2^{1-2/p|} \tag{3.3}$$

where $a_p = \|G\|_1^{1-|1-(2/p)|}$.

Indeed, if $b=l^2$ and $B=L^2$ by Plancherel formula we have

$$\|R_\sigma\|_{[l^2,L^2]} \leqslant \sup_n |\hat{G}(\sigma n)| \leqslant \|G\|_1 .$$

If $b=l^2$ and $B=C$ we have

$$\|R_\sigma \lambda\|_\infty \leqslant \sum |\lambda_n| \, |\hat{G}(\sigma n)| \leqslant \|\lambda\|_2 \cdot \| \{\hat{G}(\sigma n)\} \|_2 .$$

If $b=l^\infty$ and $B=L^1$ we have

$$\|R_\sigma \lambda\|_1 \leqslant \|R_\sigma \lambda\|_2 \leqslant \|\lambda\|_\infty \cdot \| \{\hat{G}(\sigma n)\} \|_2 .$$

Then, by Riesz-Thorin interpolation theorem, we obtain (3.3).

Proof of Theorem 1: From (3.3) and (3.1) for every $\varepsilon>0$ and $\delta \leqslant \delta_1(\varepsilon)$, if $\|\lambda - \hat{f}\|_b \leqslant \delta$ we have

$$\|R_{\sigma(\delta)}(\lambda - \hat{f})\|_B < \varepsilon/2 \tag{3.4}$$

whatever is $f \in B$.

Then, we have only to prove that if $\sigma \leqslant \sigma_0(\varepsilon, f)$

$$\|f - R_\sigma \hat{f}\|_B < \varepsilon/2 . \tag{3.5}$$

In order to do this, we use (as in [1]) some periodization techniques (see *e.g.* [8]).

Let us set

$$G_\sigma(x) = \sigma^{-N} G\left(\frac{x}{\sigma}\right).$$

Since $G \in L^1(\boldsymbol{R}^N)$, for every $\sigma>0$ $\hat{G}_\sigma$ is a multiplier of $L^p(\boldsymbol{R}^N)$ $(1 \leqslant p \leqslant +\infty)$ and its norm is bounded by $\|G\|_1$. Then if we set

$$K_\sigma(t) = \sum_{n \in \boldsymbol{Z}^N} G_\sigma(t+n) \qquad t \in \boldsymbol{T}^N,$$

$K_\sigma \in L^1(\boldsymbol{T}^N)$ and $\hat{K}_\sigma(n) = \hat{G}_\sigma(n) = \hat{G}(\sigma n)$ ([8], p. 251, Th. 2.4).

By periodization theorem ([8], p. 260, Th. 3.8), $\hat{K}_\sigma$ is a multiplier of B for every $\sigma>0$ and its norm is bounded by $\|G\|_1$. Since $\hat{K}_\sigma(n) \to 1$ for every n if $\sigma \to 0^+$, (3.5) holds for trigonometric polynomials and, by density, for every $f \in B$, q.e.d..

Let us now remark that obviously $G \in L^1(\boldsymbol{R}^N)$ does not imply $\{\hat{G}(\sigma n)\} \in$ $\in l^{p_0}(\boldsymbol{Z}^N)$.

In the following proposition we give a sufficient condition on G in order that $\{\hat{G}(\sigma n)\} \in l^2(\boldsymbol{Z}^N)$ for every $\sigma>0$. Moreover we also give an estimate of its norm (involved in (3.1)).

Let us set

$$M(x) = \sup_{|y| \geqslant |x|} \text{ess}\, |G(y)|\,.$$

PROPOSITION 1: *If $G \in L^1(\mathbf{R}^N) \cap L^2(\mathbf{R}^N)$ and $M \in L^1(\mathbf{R}^N)$, then $\{\hat{G}(\sigma n)\} \in l^2(\mathbf{Z}^N)$ for every $\sigma > 0$ and if $\sigma \to 0^+$*

$$\|\{\hat{G}(\sigma n)\}\|_2 \leqslant (\|G\|_2 + o(1))\,\sigma^{-N/2}\,. \tag{3.6}$$

Moreover, if $G(x) \geqslant 0$ a.e. in $\mathbf{R}^N$ the equality sign holds in (3.6).

PROOF: We have

$$\|\{\hat{G}(\sigma n)\}\|_2 = \left(\int_{T^N} \left|\sum_n G_\sigma(t+n)\right|^2 dt\right)^{1/2} =$$

$$= \sigma^{-N/2}\left(\int_{Q^N/\sigma} \left|\sum_n G\left(x+\frac{n}{\sigma}\right)\right|^2 dx\right)^{1/2} \leqslant \sigma^{-N/2} \sum_n \left\{\int_{Q^N/\sigma} \left|G\left(x+\frac{n}{\sigma}\right)\right|^2 dx\right\}^{1/2} =$$

$$= \sigma^{-N/2}\left\{\left(\int_{Q^N/\sigma} |G(x)|^2\, dx\right)^{1/2} + \sum_{n\neq 0}\left(\int_{Q^N/\sigma} \left|G\left(x+\frac{n}{\sigma}\right)\right|^2 dx\right)^{1/2}\right\} \leqslant$$

$$\leqslant \sigma^{-N/2}\left(\|G\|_2 + 2^N\left(\frac{1}{\sigma}\right)^{N/2} \sum_{n\in Z^{N+}} M\left(\frac{n}{\sigma} - \frac{\nu_n}{2\sigma}\right)\right),$$

where Z^{N^+} is the cone of $\mathbf{Z}^N$ with nonnegative entries without the origin and $\nu_n = \nu_{(n_1,\dots,n_N)}$ is the element in Z^{N+} such that the K-coordinate is 0 if $n_K = 0$ and 1 otherwise. Since

$$\sum_{n\in Z^{N+}} M\left(\frac{n}{\sigma} - \frac{\nu_n}{2\sigma}\right) = (4\sigma)^N \sum_{n\in Z^{N+}} M\left(\frac{n}{\sigma} - \frac{\nu_n}{2\sigma}\right)\cdot(4\sigma)^{-N} \leqslant c\sigma^N \int_{R^N - (Q^N/2\sigma)} M(x)\, dx$$

where c is a suitable constant depending only on N, we have

$$\|\{\hat{G}(\sigma n)\}\|_2 \leqslant \sigma^{-N/2}\{\|G\|_2 + o(1)\}\,. \tag{3.7}$$

On the other hand, if $G \geqslant 0$ it is easy so see that

$$\|\{\hat{G}(\sigma n)\}\|_2 \geqslant \sigma^{-N/2}\left\{\int_{Q^N/\sigma} G^2(x)\, dx\right\}^{1/2} = \sigma^{-N/2}\{\|G\|_2 + o(1)\}\,.$$

Remarks: 1) $G \in L^1(\boldsymbol{R}^N) \cap L^2(\boldsymbol{R}^N)$ generally does not imply $\{\hat{G}(\sigma n)\} \in l^2(\boldsymbol{Z}^N)$. Indeed if $N = 1$ and

$$G(x) = \begin{cases} 1 & \text{if } |x - n| < n^{-2}, \quad n = 1, 2, \ldots, \\ 0 & \text{elsewhere}, \end{cases}$$

it is easy to show that $G \in L^1 \cap L^2$ and that if $\sigma = 1/m$ $(m = 1, 2, \ldots)$ $\{\hat{G}(\sigma n)\} \notin l^2$. (It sufficies to remark that $\sum_m G_{1/m}(x+n) = (|x|^{-2}/\sqrt{m})(1 + o(1))$ when $x \to 0$).

2) It is easy to see that the functions G connected with the classical kernels (Féjer, Weierstrass, Poisson) verify the conditions of Theorem 1 and Proposition 1. This is also true for the kernels constructed in [1].

3) A large class of radial functions in $\boldsymbol{R}^N$ which satisfy Theorem 1 and Proposition 1 can be obtained as follows.

If F is a radial function, let us denote by $\tilde{F}: R^+ \to R$ its «radial trace» on $[0, +\infty)$, that is $\tilde{F}(|x|) = F(x)$. Then we have the following proposition.

Proposition 2: *Let F be a radial $L^1(\boldsymbol{R}^N)$ function. Then $\hat{F}$ is a radial $L^1(\boldsymbol{R}^N)$ function and $\tilde{\hat{F}}(|y|)$ is the Fourier transform of a radial function $f_s \in L^1(\boldsymbol{R}^s)$ for every $s = 1, 2, \ldots, N$ if and only if F has the form*

$$F(x) = \int_0^{+\infty} \exp[-\lambda |x|^2]\, d\mu(\lambda) \tag{3.8}$$

where μ is a positive finite Radon measure which satisfies the condition

$$\int_0^{+\infty} \lambda^{-N/2}\, d\mu(\lambda) < +\infty\,. \tag{3.9}$$

Proof: We recall (see *e.g.* [9], p. 61, (α)) that for every $\delta > 0$ and for every $y \in \boldsymbol{R}^N$

$$\int_{R^N} \exp[-\pi\delta |x|^2] \exp[-2\pi i x y]\, dx = \exp[-\pi |y|^2/\delta]\, \delta^{-N/2}\,. \tag{3.10}$$

Then we have

$$\int_{R^N} \left| \int_0^{+\infty} \exp[-\lambda |x|^2]\, d\mu(\lambda) \right| dx = \int_0^{+\infty} \left(\int_{R^N} \exp[-\lambda |x|^2]\, dx \right) d\mu(\lambda) = \int_0^{+\infty} \left(\frac{\lambda}{\pi} \right)^{-N/2} d\mu(\lambda)\,.$$

Therefore if F has the form (3.8):

$$F \in L^1(\boldsymbol{R}^N) \Leftrightarrow \mu \text{ satisfies } (3.9)\,. \tag{A}$$

If this is the case, by (3.10) an easy calculation gives

$$\hat{F}(y) = \int_0^{+\infty} \exp\,[-\pi^2 |y|^2/\lambda] \left(\frac{\lambda}{\pi}\right)^{-N/2} d\mu(\lambda)\,. \tag{3.11}$$

By Schoenberg theorem (see *e.g.* [5], p. 205) the functions F of the form (3.8) are the only functions with the property that $\tilde{F}(|y|)$ is the Fourier-Stieltjes transform of a positive finite Radon measure. Since $\hat{F}$ has the form (3.8) with respect to a suitable positive finite Radon measure satisfying (3.9), then $\hat{F} \in L^1(\mathbf{R}^N)$ and $\tilde{\hat{F}}(|y|)$ has the requested property.

Conversely, suppose that $\hat{F}$ and $\tilde{\hat{F}}$ have these properties. Again by Schoenberg theorem $\hat{F}$ has the form (3.8) with respect to a suitable μ and this μ satisfies (3.9) because of (A). Since $(\hat{F})^\wedge = F$, this completes the proof.

Now let confine ourselves to the class of radial functions G of the form (3.8) with

$$\int_0^{+\infty} \left(\frac{\lambda}{\pi}\right)^{-N/2} d\mu(\lambda) = 1 \tag{3.12}$$

(in order to have $\hat{G}(0) = 1$). These are radial continuous $L^1(\mathbf{R}^N)$ functions with $\tilde{G}$ monotonic, and their Fourier transform $\hat{G}$ given by (3.11) is a $L^1(\mathbf{R}^N)$ function.

In particular, if μ is the Dirac measure δ_π we recover the Weierstrass kernel.

Obviously, all these functions G satisfy the hypotheses of Th. 1 and Prop. 1 and it is also possible to give an estimate of $\|G\|_2$. By Hölder integral inequalities (see *e.g.* [9], p. 271, A.1) and (3.10) we obtain

$$\|G\|_2 = \left\{ \int_{\mathbf{R}^N} \left(\int_0^{+\infty} \exp\,[-|x|^2 \lambda]\, d\mu(\lambda) \right)^2 dx \right\}^{1/2} \leqslant$$

$$\leqslant \int_0^{+\infty} \left(\int_{\mathbf{R}^N} \exp\,[-2|x|^2 \lambda]\, dx \right)^{1/2} d\mu(\lambda) = \int_0^{+\infty} \left(\frac{2\lambda}{\pi}\right)^{-N/4} d\mu(\lambda)\,.$$

4. - In this section we give an «a priori» estimate of the error $\|f - R_\sigma \lambda\|_B$ for functions in Lipschitz classes. Such an estimate is not possible for the whole B.

We denote, as usual, by $K\,\mathrm{Lip}\,(\alpha, B)$, $0 < \alpha \leqslant 1$, the class of $f \in B$ such that for every $u \in T^N$ the function $\Delta_u f(t) = f(t+u) - f(t)$ satisfies

$$\|\Delta_u f\|_B \leqslant K|u|^\alpha\,.$$

Then, if

$$\mu_{f,B}(\delta,\sigma) = \sup\{\|f - R_\sigma\lambda\|_B : \lambda \in b \wedge \|\lambda - \hat{f}\|_b \leqslant \delta\}$$

we have

THEOREM 2: *If* $\{\hat{G}(\sigma n)\} \in l^{p_0}(Z^N)$ *for every* $\sigma > 0$ *and* $\int_{R^N} |x|^\alpha |G(x)|\,dx = c_\alpha < +\infty$, *then for every* $\delta > 0$, $\sigma > 0$, *if* $f \in K\,\mathrm{Lip}(\alpha, B)$ *we have*

$$\mu_{f,B}(\delta,\sigma) \leqslant K c_\alpha \sigma^\alpha + a_p \|\{\hat{G}(\sigma n)\}\|_2^{1-2/p|}\delta \tag{4.1}$$

where $a_p = \|G\|_1^{1-|1-2/p|}$.

PROOF: Let f^* be the periodic continuation of f in R^N. We have

$$R_\sigma \hat{f}(x) = K_\sigma * f(x) = \int_{T^N} K_\sigma(u) f(x-u)\,du = \int_{R^N} G_\sigma(u) f^*(x-u)\,du\,.$$

Then

$$\|f - R_\sigma \hat{f}\|_B = \left\|\int_{R^N} (f^*(x-u) - f^*(x))\,G_\sigma(u)\,du\right\|_B \leqslant \int_{R^N} \|\Delta_u f\|_B |G_\sigma(-u)|\,du \leqslant$$

$$\leqslant \int_{R^N} K|u|^\alpha |G_\sigma(-u)|\,du = \frac{K}{\sigma^N}\int_{R^N} |u|^\alpha \left|G\left(\frac{-u}{\sigma}\right)\right| du \leqslant K\sigma^\alpha \int_{R^N} |x|^\alpha |G(x)|\,dx\,.$$

Moreover, Lemma 1 gives

$$\|R_\sigma(\lambda - \hat{f})\|_B \leqslant a_p \|\hat{G}_\sigma\|_2^{1-2/p|} \|\hat{f} - \lambda\|_b\,.$$

Then from triangular inequality (4.1) follows.

REMARKS: 1) If $G(x) \geqslant 0$ *a.e.* on R^N, obviously $a_p = 1$ and moreover if $\delta = 0$, $B = C(T^N)$ (4.1) is sharp.

Indeed, let $f(t) = |t|^\alpha$ if $0 \leqslant |t| \leqslant 1/4$, $f(t) = (1/4)^\alpha$ elsewhere in T^N. We have

$$\|f - R_\sigma \hat{f}\|_\infty \geqslant |f(0) - R_\sigma \hat{f}(0)| \geqslant \left|\int_{R^N} f^*(-u)\,G_\sigma(u)\,du\right| \geqslant \int_{|u| \leqslant 1/4} |u|^\alpha \sigma^{-N} G\left(\frac{u}{\sigma}\right) du =$$

$$= \int_{|x| \leqslant 1/4\sigma} \sigma^\alpha |x|^\alpha G(x)\,dx = c_\alpha \sigma^\alpha (1 + o(1))\,.$$

2) As for Theorem 1, the functions G connected with the classical kernels and the kernels of [1] satisfy the hypotheses of Theorem 2 (except the Féjer kernel when $\alpha = 1$).

3) If G is a radial function of the form (3.8), by (3.10) we have

$$c_\alpha = \int_{R^N} |x|^\alpha G(x)\, dx = \int_0^{+\infty} \lambda^{-(N+\alpha)/2} d\mu(\lambda) \int_{R^N} |t|^\alpha \exp[-|t|^2]\, dt .$$

Then $c_\alpha < +\infty$ if and only if $\mu(\lambda)$ satisfies the further condition (independent of (3.11))

$$\int_0^{+\infty} \lambda^{-(N+\alpha)/2} d\mu(\lambda) = d_\alpha < +\infty . \tag{4.2}$$

In this case we have

$$c_\alpha = d_\alpha m(S_N) \int_0^{+\infty} r^{N+\alpha-1} \exp[-r^2]\, dr$$

where $m(S_N)$ is the surface measure of the unit ball of R^N. Then

$$c_\alpha = \frac{\pi^{N/2}}{\Gamma\left(\frac{N}{2}\right)} d_\alpha \int_0^{+\infty} t^{((N+\alpha)/2)-1} \exp[-t]\, dt = \pi^{N/2} d_\alpha \frac{\Gamma\left(\frac{N+\alpha}{2}\right)}{\Gamma\left(\frac{N}{2}\right)} .$$

5. - In this section we point out the pointwise convergence aspects of the above procedure. Of course, first we have to restrict ourselves to the Lebesgue points of f and further to make heavier hypotheses than in Theorem 1.

THEOREM 3: *If* $\{\hat{G}(\sigma n)\} \in l^{p_0}(Z^N)$ *for every* $\sigma > 0$, $M \in L^1(R^N)$ *and* $\sigma = \sigma(\delta)$: $R^+ \to R^+$ *satisfies*

$$\lim_{\delta\to 0} \sigma(\delta) = 0, \qquad \lim_{\delta\to 0} \delta \| \{\hat{G}(\sigma(\delta)\, n)\} \|_{p_0} = 0 , \tag{5.1}$$

then, if $f \in B$ *and* t *is a Lebesgue point of* f, *for every* $\varepsilon > 0$ *there exists* $\delta_0 = \delta_0(\varepsilon, f, t)$ *such that if* $\delta \leq \delta_0$

$$\lambda \in b \quad \text{and} \quad \|\lambda - \hat{f}\|_b \leq \delta \Rightarrow |f(t) - R_{\sigma(\delta)} \lambda(t)| < \varepsilon . \tag{5.2}$$

PROOF: We suppose $t = 0$. We have

$$|f(0) - R_\sigma \hat{f}(0)| = |f(0) - K_\sigma * f(0)| = \tag{5.3}$$

$$= \left| \int_{Q^N} K_\sigma(x)(f(-x) - f(0))\, dx \right| \leq \left| \int_{Q^N} G_\sigma(x)(f(-x) - f(0))\, dx \right| +$$

$$+ \left| \int_{Q^N} \sum_{n \neq 0} G_\sigma(x+n)(f(-x) - f(0))\, dx \right| = I_1 + I_2 .$$

First we evaluate I_2.

$$I_2 \leqslant \sum_{n \neq 0} \int_{Q^N} M_\sigma(x+n)|f(-x)-f(0)|\,dx \leqslant$$

$$\leqslant \left(\int_{Q^N} |f(x)|\,dx + |f(0)| \right) 2^N \sum_{n \in Z^{N+}} \frac{1}{\sigma^N} M\left(\frac{n}{\sigma} - \frac{\nu_n}{2\sigma} \right) \leqslant$$

$$\leqslant (\|f\|_1 + |f(0)|)\, c \int_{R^N - Q^N/2\sigma} M(x)\,dx \qquad (c = c(N))$$

$$= o(1) \qquad \text{when } \sigma \to 0^+ .$$

Let us now set $\tilde{f}(x) = f(x)$ if $x \in Q$, $\tilde{f}(x) = 0$ if $x \notin Q$. We have

$$I_1 \leqslant \int_{R^N} M_\sigma(x)|\tilde{f}(-x) - f(0)|\,dx \leqslant$$

$$\leqslant \int_{|x| \leqslant \rho} M_\sigma(x)|\tilde{f}(x) - f(0)|\,dx + \int_{|x| > \rho} M_\sigma(x)|\tilde{f}(x) - f(0)|\,dx = J_1(\rho) + J_2(\rho) .$$

Let

$$S(r) = \int_{|x| = r} |\tilde{f}(x) - f(0)|\,ds$$

where ds is the surface area element of the sphere $|x| = r$ and

$$F(r) = \int_0^r S(z)\,dz .$$

For every $\eta > 0$, since $t = 0$ is a Lebesgue point, we have

$$F(r) < \eta r^N \qquad \forall r \leqslant \rho_0(\eta) .$$

Then if $\rho < \rho_0$

$$J_1(\rho) \leqslant \int_0^\rho M_\sigma(r)\, S(r)\,dr \leqslant F(r)\, M_\sigma(r)|_0^\rho - \int_0^\rho F(r)\,dM_\sigma(r) \leqslant$$

$$\leqslant \eta r^N M_\sigma(r)|_0^\rho - \eta \int_0^\rho r^N\,dM_\sigma(r) \leqslant \eta N \int_0^\rho r^{N-1} M_\sigma(r)\,dr \leqslant$$

$$\leqslant \eta N \int_0^{+\infty} r^{N-1} M(r)\,dr \leqslant \eta A$$

since M is monotonic and $M \in L^1(\boldsymbol{R}^N)$.

Now, if χ_ρ is the characteristic function of the set $|x| > \rho$ and q is the conjugate exponent of p we have

$$J_2(\rho) \leqslant (\|f\|_p + |f(0)|) \|M_\sigma \chi_\rho\|_q = (\|f\|_p + |f(0)|) \|M\chi_{\rho/\sigma}\|_q = o(1) \quad \text{when } \sigma \to 0^+ .$$

From the above estimates and (5.3) we have for every $\varepsilon > 0$ and $\sigma \leqslant \sigma_0(\varepsilon)$

$$|f(0) - R_\sigma \hat{f}(0)| < \varepsilon/2 .$$

Since

$$|R_\sigma(\hat{f} - \lambda)(0)| = \left| \sum_{n \in Z^N} \hat{G}(\sigma n)(\hat{f}_n - \lambda_n) \right| \leqslant \|\{\hat{G}(\sigma n)\}\|_{p_0} \delta \tag{5.4}$$

the theorem easily follows from (5.1).

For the next theorem we have to introduce the following classes of functions.

Let $t \in T^N$ and $0 < \alpha \leqslant 1$. We say that a function f of $L^1(T^N)$ belongs to the Lebesgue class $K\,\mathrm{Leb}(\alpha; t)$ at the point t if for every r, $0 < r \leqslant \sqrt{N}/2$

$$\int_{|x-t| \leqslant r} |f^*(x) - f(t)|\, dx \leqslant K r^{N+\alpha}$$

where f^* (as in the proof of Theorem 2) is the periodic continuation of f in R^N.

Now, if $f \in B \cap K\,\mathrm{Leb}(\alpha; t)$ let

$$\mu_{f,t}(\delta, \sigma) = \sup \{ |f(t) - R_\sigma \lambda(t)| : \lambda \in b \wedge \|\lambda - \hat{f}\|_b \leqslant \delta \} .$$

We have

Theorem 4: *If* $\{\hat{G}(\sigma n)\} \in l^{p_0}(Z^N)$ *for every* $\sigma > 0$ *and* $\int_{R^N} |x|^\alpha M(x)\, dx = \gamma_\alpha < +\infty$, *then for every* $\delta > 0$, $\varepsilon > 0$, $0 < \sigma \leqslant \sigma_0(\varepsilon)$, *if* $f \in B \cap K\,\mathrm{Leb}(\alpha; t)$ *we have*

$$\mu_{f,t}(\delta, \sigma) \leqslant K \bar{c}_\alpha \sigma^\alpha + \delta \|\{\hat{G}(\sigma n)\}\|_{p_0} \tag{5.5}$$

where

$$\bar{c}_\alpha = \left(\frac{N+\alpha}{2} + \varepsilon \right) \pi^{-N/2} \Gamma\left(\frac{N}{2} \right) \gamma_\alpha .$$

Proof: As in Theorem 3 we evaluate $|f(0) - R_\sigma \hat{f}(0)|$ by (5.3). We have, with the same notations of the proof of Prop. 1

$$I_2 \leqslant \sum_{n \neq 0} \int_{Q^N} M_\sigma(x+n) |f(-x) - f(0)|\, dx \leqslant 2^N \sum_{n \in Z^{N+}} M_\sigma\left(n - \frac{\nu_n}{2}\right) \int_{Q^N} |f(-x) - f(0)|\, dx \leqslant$$

$$\leqslant K\left(\frac{\sqrt{N}}{2}\right)^{N+\alpha} 2^N \sum_{n\in Z^{N+}} \frac{1}{\sigma^N} M\left(\frac{n}{\sigma}-\frac{\nu_n}{2\sigma}\right) \leqslant Kc \int_{R^N - Q^N/2\sigma} M(x)\,dx \qquad (c=c(N))$$

$$\leqslant Kc \int_{R^N - Q^N/2\sigma} |4\sigma x|^\alpha M(x)\,dx = o(\sigma^\alpha) \quad \text{when } \sigma \to 0^+ .$$

As for the estimate of J_1 in Theorem 3 we have

$$I_1 \leqslant \int_0^{\sqrt{N}/2} M_\sigma(r)\,S(r)\,dr \leqslant$$

$$\leqslant F(r)\,M_\sigma(r)\big|_0^{\sqrt{N}/2} - \int_0^{\sqrt{N}/2} F(r)\,dM_\sigma(r) \leqslant Kr^{N+\alpha} M_\sigma(r)\big|_0^{\sqrt{N}/2} - K\int_0^{\sqrt{N}/2} r^{N+\alpha}\,dM_\sigma(r) \leqslant$$

$$\leqslant K(N+\alpha)\int_0^{\sqrt{N}/2} r^{N+\alpha-1} M_\sigma(r)\,dr \leqslant K(N+\alpha)\,\sigma^\alpha \int_0^{+\infty} \rho^{N+\alpha-1} M(\rho)\,d\rho =$$

$$= K(N+\alpha)\,\sigma^\alpha \frac{\gamma\alpha}{m(S_N)} = K\frac{N+\alpha}{2}\pi^{-N/2}\Gamma\left(\frac{N}{2}\right)\gamma_\alpha \sigma^\alpha .$$

From the above estimates and (5.4) we obtain (5.5).

Remarks: 1) If $G(x) \geqslant 0$ a.e. it is not possible to get for the all class of functions a better behaviour with respect to σ.

Indeed, if f is the same function as in Remark 1 of Theorem 2, with the same calculations we have

$$|R_\sigma \hat{f}(0) - f(0)| \geqslant c_\alpha \sigma^\alpha (1 + \sigma(1)) .$$

2) The functions G connected with the classical kernels and kernels of [1] satisfy the hypotheses of Theorem 3. This is also true for Theorem 4 with the same restriction of Theorem 2 for the Féjer kernel.

The radial functions G with $\tilde{G}$ of the form (3.8) with $\mu(\lambda)$ satisfying also (4.2) fall under the hypotheses of Theorems 3 and 4, since $M(x) = \tilde{G}(|x|)$.

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