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Wiener Estimates at Boundary Points for Parabolic Degenerate Equations (**)

SUMMARY. — We give regularity conditions and estimates of continuity modulus and energy decay for solutions of degenerate parabolic equations.

Stime di Wiener al bordo per equazioni paraboliche degeneri

RIASSUNTO. — Si danno condizioni di regolarità e stime del modulo di continuità e del decadimento dell'energia in un punto di bordo per soluzioni di equazioni paraboliche degeneri.

1. - INTRODUCTION

In the following paper we are concerned with the behaviour at the boundary of a weak solution of a second order degenerate parabolic equation in an open set $Q \subseteq \boldsymbol{R}^{N+1}$.

The Wiener condition for the regularity of a boundary point in the elliptic case is well known [11], and in [7, 13] estimates of the rate of convergence are also obtained by the so called «Wiener integral». The results in [12] have been extended to elliptic degenerate case with a weight in the A_2 Muckenhoupt's class in [1] and estimates analogous to those in [13] have been proved in [3].

The Wiener condition in the parabolic case has been obtained by Evans, Gariepy [4], for the heat equation and by Garofalo, Lanconelli [9], Fabes, Garofalo, Lanconelli [5], for equations with smooth coefficients; moreover Lanconelli has proved a Wiener type condition (sufficient for the regularity of a boundary point) for linear equations with bounded measurable coefficients [10]. In the preciding papers the «thermal» capacity and the Perron-Wiener solution are considered.

Wiener type sufficient conditions have been proved in linear or nonlinear case by Gariepy, Ziemer [7, 14], using the «thermal» or the Γ-capacity and the weak solution, which is supposed to have the time derivative in $L^2(Q)$. Moreover in [2] Biroli, Mosco have extended the result to general weak solutions using the Γ-capacity; in this

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paper estimates of the modulus of continuity at a boundary point are also obtained.

The purpose of the paper is to deal with the weak solution of a degenerate parabolic problem; we will prove a sufficient Wiener type condition and an estimate of the modulus of continuity at a boundary regular point by the Wiener integral using a suitable generalization of the Γ-capacity.

REMARK 1: We consider here the linear case only for sake of simplicity, but the result can be extended, under natural assumptions to the nonlinear case with quadratic growth in the spatial gradient.

2. - NOTATIONS

Let $Q \subseteq \mathbf{R}^{N+1}$ be a bounded open set with boundary ∂Q. We let $(x_1, \ldots, x_N)$ denote a point in $\mathbf{R}^N$ and the gradient of a function v will be denoted by $D_x v$.

In the following w will be a weight defined in $\mathbf{R}^N$, which belongs to the Muckenhoupt's class $A_{1+2/N}$ (we refer to [6] for the definition); with this assumption we obtain that there exists $\sigma > 0$ such that w is in the class $A_{1+2/N-\sigma}$.

Let $Z(Q; w)$ be the space $\{u \in D'(Q);\ u = \operatorname{div} g,\ g/w \in L^2(Q)\}$ endowed with the norm $\|u\|_Z = \inf_{u = \operatorname{div} g} \|g\|_{L^2(Q;w)}$.

Let $H^{0,1}(Q, w)$ be the space $\{u \in L^2(Q; w); D_x u \in L^2(Q; w)\}$ endowed with the norm

$$\|u\|_{H^{0,1}} = \left\{ \int_Q |u|^2 w\, dx\, dt + \int_Q |D_x u|^2 w\, dx\, dt \right\}^{1/2}$$

and $W^1(Q; w) = \{u \in H^{0,1}(Q; w), D_t u \in Z(Q; w)\}$.

We use also functions from the space from the space $V^2(Q; w)$ which is given by

$$V^2(Q; w) = \{u \in H^{0,1}(Q; w),\ \|u\|_{L^2(\Omega_t)} \in L^\infty(\mathbf{R})\},$$

where $\Omega_\tau = Q \cap \{t = \tau\}$, endowed with the norm

$$\|u\|_{V^2} = \operatorname*{ess\,sup}_t \left(\int_{\Omega_t} |u|^2 dx \right)^{1/2} + \left(\int_Q |D_x u|^2 w\, dx\, dt \right)^{1/2}.$$

We denote

$$B(r, x_0) = \{x; |x - x_0| < r\}.$$

We define

$$h_x(r) = \left(\int_{B(r;x_0)} w^{N/2} dx \right)^{2/N}; \tag{2.1}$$

since w is in $A_{1+2/N}$ we obtain

$$h_x(r) \approx \frac{r^2}{\int_{B(r;x_0)} w\,dx} \tag{2.2}$$

and taking into account that $\int_{B(r;x_0)} w\,dx \geq CR^{N+2-\sigma N}$ (this is a consequence of $w \in A_{1+2/N-\sigma}$) we have $h_x(r) \geq CR^{\sigma N}$.

Finally we denote

$$Q(r;z_0) = Q(r;x_0,t_0) = \{(x,t);\, |x-x_0| < r,\ \ |t-t_0| < h_{x_0}(r)\}\,.$$

Let $z_0 \in \partial Q$ we say that

$$u(z_0) \leq d \text{ weakly } (u \in W^1(Q;w)) \tag{2.3}$$

if $\forall k \geq d$ there exists $\tau > 0$ and a sequence $\{u_m\}$ of Lipschitz continuous functions such that $u_m \to u$ in $W^1(Q \cap Q(\tau;z_0);w)$ and $\operatorname{supp}(\eta(u_m-k)^+) \subseteq Q \cap Q(\tau;z_0)$ whenever $\eta \in C_0^\infty(Q(\tau;z_0))$.

The definition of

$$u(z_0) \geq d \text{ weakly} \tag{2.4}$$

is analogue and $u(z_0) = d$ if both (2.3) and (2.4) hold.

Let now $\{a_{ij}\}$, $i,j = 1,2,\dots,N$, be symmetric and such that

$$\lambda|\xi|^2 w(x) \leq \sum_{i,j=1}^{N} a_{ij}(x,t)\,\xi_i\xi_j \leq \Lambda|\xi|^2 w(x)$$

a.e. in Q, $\forall \xi \in \boldsymbol{R}^N$ and denote by P the parabolic operator

$$P = D_t - \sum_{i,j=1}^{N} D_{x_i}(a_{ij}D_{x_j})\,.$$

We denote again by a_{ij} the extension of the a_{ij} to $\boldsymbol{R}^{N+1}$ by $\delta_{ij}w(x)$ and by P the parabolic operator relative to the extension.

In the following G^z will be the Green function of the operator P in $\boldsymbol{R}^{N+1}$ with singularity at $z = (y,s)$. In [10] an upper and a lower bound for G^z (analogous to the one relative to the nondegenerate case) are proved, we have

$$c_1\left(\frac{1}{[h_x^{-1}(s-t)]^n} + \frac{1}{[h_y^{-1}(s-t)]^n}\right)\exp\left[-c_1\left(\frac{h_x(|x-y|)}{s-t}\right)^{1/(2\nu-1)}\right] \leq G^z(x,t) \leq$$

$$\leq c_2\left(\frac{1}{[h_x^{-1}(s-t)]^n} + \frac{1}{[h_y^{-1}(s-t)]^n}\right)\exp\left[-c_2\left(\frac{h_x(|x-y|)}{s-t}\right)^{1/(2\mu-1)}\right]$$

where c_1, c_2, ν, μ are positive constants, which depends only on N, λ, Λ, w, and $1 + N/2 \geq \mu \geq \nu > 1/2$ and w is assumed to be in R_ν (see [10] for the definition).

We define the regularized Green function G^z_ρ as the average of G^z on $Q(\rho; z)$; we observe that G^z_ρ is a weak solution in $\boldsymbol{R}^{N+1}$ of the equation

$$Pu = \frac{\chi_{Q(\rho;z)}}{|Q(\rho;z)|}$$

where $\chi_{Q(\rho;z)}$ is the characteristic function of the cylinder $Q(\rho; z)$ (we refer to the next section for the definition of weak solution). From the results in [10] we have that G^2_ρ converges to G^2 in $C_{\text{loc}}(\boldsymbol{R}^{N+1} - \{z\}) \cap W^1_{\text{loc}}(\boldsymbol{R}^{N+1} - \{z\}; w)$.

For an arbitrary set E such that $\text{cl}(E) \subseteq Q(r; z)$ we define the $\Gamma_{Q(r;z)}$ capacity by

$$\Gamma_{Q(r;z)}(E) = \inf\left\{ \operatorname*{esssup}_t \int_{B(r;z)} |u|^2\, dx + \int_{Q(r,z)} |D_x u|^2 w dx\, dt \right\}$$

where the infimum is taken over all functions of $V^2(Q(r; z), w)$ with $\text{supp}\,(u) \subseteq Q(r; z)$ and $E \subseteq \text{int}\,\{z; u(z) \geqslant 1\}$.

A function $v \in V^2(Q(r(1-\delta); z); w)$, $\delta > 0$, is $\Gamma_{Q(r;z)}$-quasi continuous (*i.e.* for every $\varepsilon > 0$ there is an open set $V \subseteq Q(r(1-\delta); z)$ such that u is continuous on $Q(r(1 - \delta); z) - V$ and $\Gamma_{(Q(r;z))}(V) \leqslant \varepsilon$) and if $u_m \to u$ in $V^2(Q(r(1-\delta); z); w)$ we have $u_m \to u$ $\Gamma_{Q(r;z)}$-quasi uniformly (the proof are analogous to the one given in [14] in the case $w = 1$ and $Q(r; z) = \boldsymbol{R}^{N+1}$).

Remark 1: Let $u \in W^1(Q \cap Q(r; z_0); w)$, $z_0 \in \partial Q$, and suppose $u(z) \leqslant 0$ weakly on $\partial Q \cap Q(r; z_0)$; we indicate by v the prolungate of u^+ by 0 to $Q(r; z_0)$. Let $D_t u = \text{div}\, g$, $g/w \in L^2(Q)$; by approximation by smooth functions we have easily

$$\int_{t'}^{t''} \int_{B(r;z_0)} g D_x(v\phi)\, dx\, dt = \frac{1}{2}\,\|v\phi(t'')\|^2_{L^2(B(r,x_0))} - \frac{1}{2}\,\|v\phi(t')\|^2_{L^2(B(r,x_0))}$$

for almost all $t', t'' \in (t_0 - h_{x_0}(r), t_0 - h_{x_0}(r))$, $t' \leqslant t''$, $\phi \in D(B(r; x_0))$ (for the imbedding of $W^1(Q; w)$ into $L^2(Q)$ see the Poincaré's inequality in section 4).

3. - Results

If $u \in V^2(Q; w)$ and

$$\int_Q \left\{ -u D_t \phi + \sum_{i,j=1}^N a_{ij} D_{x_j} u D_{x_i} \phi \right\} dx\, dt = 0 \tag{3.1}$$

for all $\phi \in C^\infty_0(Q)$ we say that u is a *weak solution* of the problem $Pu = 0$ in Q.

We observe that from (3.1) we have $D_t u \in Z(Q; w)$, then $u \in W^1(Q; w)$ and we can write

$$\langle D_t u, \phi \rangle + \sum_{i,j=1}^N \int_Q a_{ij} D_{x_j} u D_{x_i} \phi\, dx\, dt = 0 \tag{3.1'}$$

$v\phi \in H_0^{0,1}(Q)$ ($H_0^{0,1}(Q)$ = closure of $C_0(Q)$ in $H^{0,1}(Q)$) where $\langle\, ,\rangle$ is the duality between $H_0^{0,1}(Q)$ and its dual.

In the following for $z_0 \in \partial Q$ we indicate

$$\Delta_\theta(\rho) = \Gamma_{Q(2\rho;z_0)}(Q^C \cap Q_\theta - (\rho; z_0)),$$

where

$$Q_\theta(\rho; z_0) = \left(B\left(\frac{\rho}{2}; x_0\right) x (t_0 - (1-\theta)\, h_x(\rho), t_0 - 2\theta h_x(\rho)), \quad 0 < \theta < \frac{1}{4},\right.$$

and

$$\delta_\theta(\rho) = \left.\frac{\Delta_\theta(\rho)}{\Gamma_{Q(2\rho;z_0)}(Q(\rho; z_0))}\right).$$

A point $z_0 \in \partial Q$ is a *Wiener point* iff

(3.2) $$\int_r^1 \delta_\theta(\rho)\frac{d\rho}{\rho} \to +\infty \qquad \text{for } r \to +\infty$$

and for some θ.

Remark 2: The estimate $\Gamma - \underset{Q(2r;x)}{\text{cap}}\ Q(r;x) \approx r^N$ can be easily proved.

Denote $\omega_\theta(r) = \exp\left[-\int_r^1 \delta_\theta(\rho)\frac{d\rho}{\rho}\right]$; a point $z_0 \in \partial Q$ is a *logarithmic Wiener point* iff

$$\omega_\theta(r) \leqslant kr^\alpha,$$

$\alpha \in (0,1)$, for some θ.

Remark 3: If we denote by θ_0 a value of θ for which (3.2) or (3.3) hold, then these relations hold again for $\theta \in (0, \theta_0)$ being $\delta_{\theta_1}(r) \geqslant \delta_{\theta_2}(r)$, if $\theta_1 \leqslant \theta_2$.

Theorem 1: *Let u be a weak solution of the problem $Pu = 0$ in Q with $u(z_0) = d$ weakly, $z_0 \in \partial Q$. Suppose now $g \leqslant u \leqslant f$, $g(z_0) = d = f(z_0)$, where g and f are continuous on $Q(\overline{R}; z_0) \cap \partial Q$ and denote by $\Psi_f(r)$, $\Psi_g(r)$ the modulus of continuity of f, g.*

We have for $\theta \in (0, \theta_0)$, θ_0 suitable,

$$\underset{Q(r;z_0)\cap Q}{\text{osc}}\ u \leqslant K\omega_\theta(r)^{\beta(1-\alpha)} + \Psi_f(\omega_\theta(r)^\alpha), \qquad \forall \alpha, \beta \in (0,1)$$

where $K = K(\lambda, \Lambda, N, w)$ and $\omega_\theta(r)^\alpha < \theta \overline{R}$.

An easy consequence of the result in Theorem 1 is the following corollary:

Corollary 1: *Let the assumptions of* Theorem 1 *hold; if* z_0 *is a Wiener point, then* u *is continuous at* z_0*. Moreover if* z_0 *is a logarithmic Wiener point and* f, g *are Hölder continuous, then* u *is Hölder continuous at* z_0.

In the nondegenerate case the results in Theorem 1 have been proved in [2]; moreover the elliptic version of the boundary estimates in Theorem 1 have been proved in [1].

We use in the proof a method derived from the one used in [2]; an essential tool is a Poincaré's inequality for subsolutions of our problem which seems to be new.

Finally we observe hat the results given here can be also proved (under natural assumptions) in the nonlinear case with quadratic growth in the spatial gradient.

4. - A Poincaré's inequality

At first we will prove a Poincaré's type inequality involving only the spatial gradient for subsolution on $\boldsymbol{R}^N \times \boldsymbol{R}_+$ of a degenerate parabolic problem with weight w.

Let v be a non negative subsolution of our problem in the cylinder $Q(r; z_0)$ ($Pv \geqslant 0$ in $D'(Q(r; z_0))$) with Dv in $L^2(Q(r; z_0))$. Denote by $v_r(t)$ the weighted average of $v(\,.\,, t)$ defined by

$$v_r(t) = \frac{\int_{B(r;z_0)} v\chi_r \, dx}{\int_{B(r;z_0)} \chi_r \, dx},$$

where $\chi_r(x) = \chi\left(\frac{x - z_0}{r}\right)$ with $\chi \in C_0^\infty(B(1;0))$, $\chi = 1$ on $B\left(\frac{1}{2} r; 0\right)$ and $|D_x \chi| \leqslant 4$.

The following *spatial Poincaré's inequalities* can be easily proved (in the weighted case one can use the same methods as in [6]; see S. Chanillo, R. L. Wheeden, Am. J. Math., 107 (1985), 1191-1226 for (4.1))

$$\int_{B(r;x_0)} |g - g_r|^2 \, dx \leqslant Cr^2 \int_{B(r;x_0)} |D_x g|^2 w \, dx, \tag{4.1}$$

$$\int_{B(r;x_0)} |g - g_r|^2 w \, dx \leqslant Cr^2 \int_{B(r;x_0)} |D_x g|^2 w \, dx. \tag{4.2}$$

Lemma 1: *We have*

$$|(v_r(t) - v_r(s))^+| \leqslant \frac{C}{r^N} \int_{Q(r;x_0)} |D_x v| \, w \, dx \, d\eta \qquad (t \geqslant s). \tag{4.3}$$

PROOF: Using χ_r as test function, we have

$$(v_r(t) - v_r(s)) \leqslant \frac{C}{r^N} \int_{Q(r;z_0)} |D_x v|\,|D_x \chi_r|\, w dx\, d\eta \leqslant \frac{C}{r^{N+1}} \int_{Q(r;z_0)} |D_x v|\, w\, dx\, d\eta \leqslant$$

$$\leqslant C\left[\frac{1}{r^{2N+2}} w(B(r;x_0))\, h_{x_0}(r) \int_{Q(r;z_0)} |D_x v|^2 w\, dx\, d\eta\right]^{1/2}.$$

The result follows. ◆

Suppose that the following inequality holds for v:

$$(4.4)\qquad \Big|\|(v(t) - k)^+ \chi_r\|^2_{L^2(B(r;x_0))} - \|(v(s) - k)^+ \chi_r\|^2_{L^2(B(r;x_0))}\Big| \leqslant$$

$$\leqslant \frac{C}{\varepsilon} \int_t^s \int_{B(r;x_0)} (|D_x v|^2 + |D_x k|^2)\, w\, dx\, d\eta + \frac{C\varepsilon}{r^2} \int_t^s \int_{B(r;x_0)} |(v - k)^+ \chi_r|^2)\, w\, dx\, d\eta \qquad (s \geqslant t)$$

where k is a non negative function in $L^2(B(r;x_0)) \cap H_0^1(B(r;x_0);w)$. Choosing $k(x) = v(x,t)$ and $k(x) = v(s,x)$ we obtain

$$(4.5)\qquad \|(v(t) - v(s))\, \chi_r\|^2_{L^2(B(r;x_0))} \leqslant$$

$$\leqslant \frac{C_0}{\varepsilon} \int_t^s \int_{B(r;x_0)} (|D_x v|^2 + |D_x v(.\,,s)|^2 + |D_x v(.\,,t)^2)\, w\, dx\, d\eta +$$

$$+ \frac{C_1 \varepsilon}{r^2} \int_t^s \int_{B(r;x_0)} |(v - v_r(t))^+ \chi_r|^2 w\, dx\, d\eta + C_3\, \varepsilon (s - t)\, r^{-2} w(B(r;x_0))|(v_r(t) - v_r(s)|^2.$$

Let now

$$s \in \left(t_0 + \left(1 + \frac{\theta}{2}\right) h_{x_0}(r),\ t_0 + (1+\theta)\, h_{x_0}(r)\right) = I^\theta$$

and

$$t \in \left(t_0 - (1+\theta)\, h_{x_0}(r),\ t_0 - \left(1 + \frac{\theta}{2}\right) h_{x_0}(r)\right) = I_\theta$$

and denote by $\bar{v}(\underline{v})$ the average of $v_r(\eta)$ in the first (second) interval; from (4.5), the Lemma 1 and the spatial Poincaré's inequality we have

$$|v_r(t) - v_r(s)|^2 \leqslant \frac{C_4}{\varepsilon r^N} \int_{\tilde{Q}_\theta(r;z_0)} (|D_x v|^2 + |D_x v(.\,,s)|^2 + |D_x v(.\,,t)^2)\, w\, dx\, d\eta$$

where $\tilde{Q}_\theta(r;z_0) = B(r;x_0)\, x (t_0 - (1+\theta)\, h_{x_0}(r),\, t_0 + (1+\theta)\, h_{x_0}(r))$ on in t on I_θ and in s on

I^{θ} we obtain

$$(4.6) \qquad |\bar{v} - \underline{v}|^2 \leqslant C_5\, r^{-N} \int_{\tilde{Q}_{\theta}(r, z_0)} |D_x v|^2\, w\, dx\, d\eta\,. \qquad \blacklozenge$$

Lemma 2: *Let v the the average in time on*

$$\left(t_0 - (1+\theta)\, h_{x_0}(r),\ t_0 - \left(1 + \frac{\theta}{2}\, h_{x_0}(r)\right)\right)$$

of $v_r(t)$ and let (4.4) hold; we have

$$(4.7) \qquad \sup_{(t_0 - h_{x_0}(r),\, t_0 + h_{x_0}(r))} \|(v(t) - \underline{v})^+ \chi_r\|^2_{L^2(B(r; x_0))} \leqslant \frac{C}{\theta} \int_{\tilde{Q}_{\theta}(r, z_0)} |D_x v|^2\, w\, dx\, d\eta\,.$$

Proof: We apply (4.2) with $k = v_r(\tau)$ to the cylinder $\tilde{Q}_{\theta/2}(r, z_0)$ and we obtain

$$(4.8) \qquad \|(v(t) - v_r(\tau))^+ \chi_r\|^2_{L^2(B(r, x_0))} \leqslant C \int_{\tilde{Q}_{\theta}(r; z_0)} |D_x v|^2\, w\, dx\, d\eta + $$

$$+ \frac{C}{r^2} \int_{\tilde{Q}_{\theta/2}(r; z_0)} |(v - v_r(\tau))^+|^2\, w\, dx\, d\eta + \frac{C}{\theta h_{x_0}(r)} \int_{\tilde{Q}_{\theta/2}(r; z_0)} |(v - v_r(\tau))^+|^2\, dx\, d\eta$$

for t in $(t_0 - h_{x_0}(r),\ t_0 + h_{x_0}(r))$ and τ in $(t_0 - (1+\theta)\, h_{x_0}(r),\ t_0 - (1 + (\theta/2)\, h_{x_0}(r)))$.

From Lemma 1 we have

$$(4.9) \qquad (v_r(t) - v_r(s))^+ \leqslant \frac{C}{r^N} \int_{\tilde{Q}_{\theta}(r; z_0)} |D_x v|^2\, w\, dx\, d\eta$$

then

$$(4.10) \qquad \|(v(t) - \bar{v})^+ \chi_r\|^2_{L^2(B(r; x_0))} \leqslant C \int_{\tilde{Q}_{\theta}(r; z_0)} |D_x v|^2\, w\, dx\, d\eta + $$

$$+ C \int_{\tilde{Q}_{\theta}(r; z_0)} |v(t) - v_r(t)|^2 \left(\frac{1}{r^2}\, w + \frac{1}{\theta h_{x_0}(r)}\right) dx\, d\eta\,.$$

We apply the spatial Poincaré's inequality to the second term in the right hand side and we obtain the result. $\blacklozenge$

From (4.10) we obtain easily

LEMMA 2′: *Let the assumptions of Lemma 2 hold; we have*

$$\sup_{(t_0-h_{x_0}(r),\,t_0+h_{x_0}(r)}\|(v(t)-\bar v)^+\|^2_{L^2B(r/2;x_0)}\leq C\int_{\tilde Q_\theta(r;z_0)}|D_xv|^2\,w\,dx\,d\eta\,. \tag{4.11}$$

By the same methods of Lemma 2 we obtain also

LEMMA 2″: *We have*

$$\sup_{(t_0-h_{x_0}(r),\,t_0+h_{x_0}(r))}\|(v(t)-\bar v)^-\chi_r\|^2_{L^2(B(r;z_0))}\leq C\int_{\tilde Q_\theta(r;z_0)}|D_xv|^2\,w\,dx\,d\eta\,. \tag{4.12}$$

From Lemma 2′ and Lemma 2″ we have by standard methods

$$\bar v\leq\frac{C}{\Gamma_{Q(r;z_0)}(N_r)}\int_{\tilde Q_\theta(r;z_0)}|D_xv|^2\,w\,dx\,d\eta$$

where $N_r=\{(x,t)\in Q(r/2;z_0);\,v(x,t)=0\}$ then

PROPOSITION 1: *Let* (4.2) *hold; we have*

$$\int_{Q(r/2;z_0)}|v|^2\,dx\,d\eta\leq\frac{Cr^N h_{x_0}(r)}{\Gamma_{Q(r;z_0)}(N_r)}\int_{\tilde Q_\theta(r;z_0)}|D_xv|^2\,w\,dx\,d\eta\,. \tag{4.13}$$

From (4.7)(4.8) we have also easily

$$|v_r(t)-\bar v|^2\leq\frac{C}{r^N}\int_{\tilde Q_\theta(r;z_0)}|D_xv|^2\,w\,dx\,d\eta\,;$$

using the preciding methods we obtain also a weighted version of (4.13.):

PROPOSITION 2: *Let* (4.2) *hold; we have*

$$\int_{Q(r/2;z_0)}|v|^2\,w\,dx\,d\eta\leq\frac{Cr^{N+2}}{\Gamma_{\tilde Q_\theta(r;z_0)}(N_r)}\int_{\tilde Q_\theta(r;z_0)}|D_xv|^2\,w\,dx\,d\eta\,. \tag{4.14}$$

5. - A CACCIOPPOLI'S INEQUALITY

Let $z_0\in\partial Q$ and

$$k<(>)\,l=u(z_0)\ \text{weakly},$$

then there is $R_0 > 0$ depending on k, such that

$$(u-k)^-((u-k)^+) = 0 \quad \text{weakly in } Q(R_0; z_0) \cap \partial Q.$$

Proposition 3: *Let u be a weak solution of the problem $Pu = 0$ in Q; the following estimate holds*

$$\int_{t_0 - \theta h_{x_0}(R)}^{t_0} \int_{B(\theta^{1/2} R; x_0)} |D_x (u-k)^{\pm}|^2 G^{z_0} w \, dx \, dt + \sup_{Q(\theta^{1/2} R; z_0)} |(u-k)^{\pm}|^2 \leq$$

$$\leq C_1 \exp[-C_2(\theta)] \sup_{Q(R; z_0)} |(u-k)^{\pm}|^2 +$$

$$+ C_3(\theta) R^{-(N+2)} \int_{\bar{t} - (1-\theta) h_{x_0}(R)}^{\bar{t} - 2\theta h_{x_0}(R)} \int_{B(R/2; \bar{x})} |(u-k)^{\pm}|^2 w \, dx \, dt +$$

$$+ \frac{C_3(\theta)}{R^2 h_{x_0}(R)} \int_{\bar{t} - (1-\theta) h_{x_0}(R)}^{\bar{t} - 2\theta h_{x_0}(R)} \int_{B(R/2; \bar{x})} |(u-k)^{\pm}|^2 \, dx \, dt$$

where $C_2(\theta)$, $C_3(\theta) \to +\infty$ as $\theta \to 0$, $\theta \in (0, \theta_0)$ with θ_0 suitable.

Proof: We prove the result for $(u-k)^-$; the case $(u-k)^+$ is analogous. Consider as test function

$$\phi = (u-k)^- G_\rho^{\bar{z}} \tau^2 \eta^2 \qquad \bar{z} = (\bar{x}; \bar{t}) \in Q \cap Q(R; z_0)$$

$R \leq \dfrac{R_0}{2}$, where $\eta = \eta(x)$ is such that

$$\eta \in C^\infty(\mathbf{R}^N),$$

$$\eta = 1 \quad \text{for } x \in B\left(\frac{R}{8}; \bar{x}\right),$$

$$\eta = 0 \quad \text{for } x \notin B\left(\frac{R}{4}; \bar{x}\right),$$

$$0 \leq \eta \leq 1 \quad \text{for } x \in B\left(\frac{R}{4}; \bar{x}\right),$$

$$|D_x \eta| \leq \frac{16}{R},$$

and $\tau = \tau(t)$ is such that

$$\tau \in C^\infty(R^N),$$

$$\tau = 1 \quad \text{for } t \geq \bar{t} - 3\theta h_{x_0}(R),$$

$$\tau = 0 \quad \text{for } t \leq \bar{t} - (1-2\theta)\, h_{x_0}(R),$$

$$0 \leq \tau \leq 1 \quad \text{for } \bar{t} - 3\theta h_{x_0}(R) \leq t \leq \bar{t} - (1-2\theta)\, h_{x_0}(R),$$

$$|D_t \tau| \leq \frac{2}{h_{x_0}(R)}.$$

By standard methods we obtain

$$(5.1) \qquad \int_{Q(R;\bar{z})} |D_x(u-k)^-|^2 G_\rho^{\bar{z}} \tau^2 \eta^2 w\, dx\, dt + \frac{1}{|Q(\rho;\bar{z})|} \int_{Q(\rho;\bar{z})} |(u-k)^-|^2\, dx\, dt \leq$$

$$\leq C \int_{\bar{t}-(1-2\theta)h_{x_0}(R)}^{\bar{t}+\rho^2} \int_{B(R/4;\bar{x})} |D_x(u-k)^-|^2 |D_x \eta|^2 G_\rho^{\bar{z}} w\, dx\, dt +$$

$$+ \int_{\bar{t}-(1-2\theta)h_{x_0}(R)}^{\bar{t}+\rho^2} \int_{B(R/4;\bar{x})} |(u-k)^-|^2 \tau^2 \eta |D_x \eta|\, |D_x G_\rho^{\bar{z}}|\, w\, dx\, dt +$$

$$+ \int_{\bar{t}-(1-2\theta)h_{x_0}(R)}^{\bar{t}-3\theta h_{x_0}(R)} \int_{B(R/4;\bar{x})} |(u-k)^-|^2 |D_t \tau|\, G_\rho^{\bar{z}}\, dx\, dt.$$

Passing now to the limit as $\rho \to 0$ and taking into account that $G_\rho^{\bar{z}} \to G^{\bar{z}}$ in $C_{\text{loc}}(R^{N+1} - \{\bar{z}\}) \cap W^1_{\text{loc}}(R^{N+1} - \{\bar{z}\}; w)$; we have for almost all $\bar{z}$

$$(5.2) \qquad \int_{\bar{t}-3\theta h_{x_0}(R)}^{\bar{t}} \int_{B(R/8;\bar{x})} |D_x(u-k)^-|^2 G^{\bar{z}} w\, dx\, dt + |(u-k)^-|^2(\bar{z}) \leq$$

$$\leq C_1 R^{-2} \int_{\bar{t}-(1-2\theta)h_{x_0}(R)}^{\bar{t}} \int_{B(R/4;\bar{x}) - B(R/8;\bar{x})} |(u-k)^-|^2 G^{\bar{z}} w\, dx\, dt +$$

$$+\frac{C_2}{\theta h_{x_0}(R)}\int_{\bar t-(1-2\theta)h_{x_0}(R)}^{\bar t-3\theta h_{x_0}(R)}\int_{B(R/4;\bar x)}|(u-k)^-|^2 G^{\bar z}\,dx\,dt+$$

$$+\int_{\bar t-(1-2\theta)h_{x_0}(R)}^{\bar t}\int_{B(R/4;\bar x)}|(u-k)^-|^2\tau^2\eta|D_x\eta||D_xG^{\bar z}|\,w\,dx\,dt.$$

Consider now the last term in the right and side; we have:

$$\int_{\bar t-(1-2\theta)h_{x_0}(R)}^{\bar t}\int_{B(R/4;\bar x)}|(u-k)^-|^2\tau^2\eta|D_x\eta||D_xG^{\bar z}|\,w dx\,dt\leqslant$$

$$\leqslant\varepsilon R^{N/2}\int_{\bar t-(1-2\theta)h_{x_0}(R)}^{\bar t}\int_{B(R/4;\bar x)-B(R/8;\bar x)}|(u-k)^-|^2\tau^2\eta^2(G^{\bar z})^{-3/2}|DG^{\bar z}|^2 w\,dx\,dt+$$

$$+\frac{4}{\varepsilon}R^{(N/2-2)}\int_{\bar t-(1-2\theta)h_{x_0}(R)}^{\bar t}\int_{B(R/4;\bar x)-B(R/8;\bar x)}|(u-k)^-|^2(G^{\bar z})^{3/2}\,w\,dx\,dt,\qquad \varepsilon>0$$

From (5.2) and the Proposition in the Appendix we have

$$\text{(5.3)}\qquad \int_{\bar t-3\theta h_{x_0}(R)}^{\bar t}\int_{B(R/8;\bar x)}|D_x(u-k)^-|^2G^{\bar z}\,w\,dx\,dt+|(u-k)^-|^2(\bar z)\leqslant$$

$$\leqslant C_4R^{-2}\int_{\bar t-(1-2\theta)h_{x_0}(R)}^{\bar t}\int_{B(R/4;\bar x)-B(R/16;\bar x)}|(u-k)^-|^2\{G^{\bar z}+\varepsilon R^{-N/2}(G^{\bar z})^{1/2}+$$

$$+\varepsilon^{-1}R^{N/2}(G^{\bar z})^{3/2}+\varepsilon R^{-N}\}\,w\,dx\,dt+$$

$$+\frac{C_5}{\theta h_{x_0}(R)}\int_{\bar t-(1-2\theta)h_{x_0}(R)}^{\bar t-3\theta h_{x_0}(R)}\int_{B(R/4;\bar x)}|(u-k)^-|^2(G^{\bar z}+\varepsilon R^{-N/2}(G^{\bar z})^{1/2})\,dx\,dt\leqslant$$

$$\leqslant C_6\exp[-C_7(\theta)]\sup_{(\bar t,\bar t-3\theta h_{x_0}(R))XB(R/4;\bar x)}|(u-k)^-|^2+$$

$$+C_8(\theta)R^{-(N+2)}\int_{\bar t-(1-2\theta)h_{x_0}(R)}^{\bar t-3\theta h_{x_0}(R)}\int_{B(R/4;\bar x)}|(u-k)^-|^2 w\,dx\,dt+$$

$$+\frac{C_8(\theta)}{R^N h_{x_0}(R)}\int_{\bar t-(1-2\theta)h_{x_0}(R)}^{\bar t-3\theta h_{x_0}(R)}\int_{B(R/4;\bar x)}|(u-k)^-|^2\,dx\,dt.$$

where $C_7(\theta), C_8(\theta)\to+\infty$ as $\theta\to 0$.

Fixe now $\theta\in(0,\theta_0)$ with θ_0 suitable; taking the supremum for $\bar z\in Q(\theta^{1/2}R;z_0)$, we obtain the result. ◆

6. - The Wiener estimate

It is easy to prove that $(u-k)^\pm$ are subsolution and verify (4.2); then we can apply the Poincaré's inequalities of section 4 and we obtain

(6.1)
$$\int_{\bar t-(1-\theta)h_{x_0}(R)}^{\bar t-2\theta h_{x_0}(R)}\int_{B(R/2;\bar x)}|(u-k)^-|^2 w\,dx\,dt\leqslant$$
$$\leqslant\frac{CR^{N+2}}{\Delta_\theta(R)}\int_{\bar t-h_{x_0}(R)}^{\bar t-\theta h_{x_0}(R)}\int_{B(R;\bar x)}|D_x(u-k)^-|^2 w\,dx\,dt,$$

(6.2)
$$\int_{\bar t-(1-\theta)h_{x_0}(R)}^{\bar t-2\theta h_{x_0}(R)}\int_{B(R/2;\bar x)}|(u-k)^-|^2\,dx\,dt\leqslant$$
$$\leqslant\frac{CR^N h_{x_0}(R)}{\Delta_\theta(R)}\int_{\bar t-h_{x_0}(R)}^{\bar t-\theta h_{x_0}(R)}\int_{B(R;\bar x)}|D_x(u-k)^-|^2 w\,dx\,dt$$

where C depends on θ.

From the Caccioppoli's inequality, taking into account (6.1) (6.2) and the estimates on the Green function we obtain

(6.3)
$$\int_{t_0-\theta h_{x_0}(R)}^{t_0}\int_{B(\theta^{1/2}R;x_0)}|D_x(u-k)^\pm|^2 G^{z_0} w\,dx\,dt+\sup_{Q(\theta^{1/2}R;z_0)}|(u-k)^\pm|^2\leqslant$$
$$\leqslant C_1\exp[-C_2(\theta)]\sup_{Q(R;z_0)}|(u-k)^\pm|^2+$$
$$+\frac{1}{C_3(\theta)\,\delta_\theta(R)}\int_{t_0-h_{x_0}(R)}^{t_0-\theta h_{x_0}(R)}\int_{B(\theta^{1/2}R;x_0)}|D_x(u-k)^\pm|^2 G^{z_0} w\,dx\,dt$$

where $C_2(\theta)\to+\infty$, $C_3(\theta)\to 0$ as $\theta\to 0$.

Denote

$$\Phi^{\pm}(r)=\int_{t_0-h_{x_0}(r)}^{t_0}\int_{B(r;x_0)}|D_x(u-k)^{\pm}|^2 G^{z_0} w\,dx\,dt+\sup_{Q(r;z_0)}|(u-k)^{\pm}|^2;$$

choosing θ suitable and using the hole filling trick (see [2] for explicit computations) we obtain

$$\Phi^{\pm}(\theta^{1/2}R)\leq\frac{1}{1+C_4(\theta)\,\delta_\theta(R)}\,\Phi^{\pm}(R)\,. \tag{6.4}$$

From (6.4.) and the integration lemma in [13], we obtain

$$\Phi^{\pm}(r)\leq K\exp\left[-\beta\int_r^R\delta_\theta(\rho)\,\frac{d\rho}{\rho}\right]\Phi^{\pm}(R) \tag{6.5}$$

with K and β positive suitable constants, $R\leq\theta^{1/2}R_0$. ◆

7. - Proof of Theorem 1

Choose in (6.5) $R=\omega_\theta(r)^\alpha$, $\alpha\in(0,1)$. We observe that

$$\int_r^{\omega_\theta(r)^\alpha}\delta_\theta(\rho)\,\frac{d\rho}{\rho}=\int_r^1\delta_\theta(\rho)\,\frac{d\rho}{\rho}-\int_{\omega_\theta(r)^\alpha}^1\delta_\theta(\rho)\,\frac{d\rho}{\rho}\geq \tag{7.1}$$

$$\geq-\lg(\omega_\theta(r))+\lg(\omega_\theta(r)^\alpha)=\lg(\omega_\theta(r)^{\alpha-1})\,.$$

From (6.5) and (7.1) we obtain

$$|u(z)-u(z_0)^{\pm}|^2\leq K\omega_\theta(r)^{\beta(1-\alpha)}+\Psi_f(\omega_\theta(r)^\alpha)+\Psi_g(\omega_\theta(r)^\alpha) \tag{7.2}$$

where $z\in Q(r;z_0)$ and

$$\Psi_f(s)=\sup_{Q(s;z_0)\cap\partial Q}|f(z)-f(z_0)|;$$

we have so proved the result of Theorem 1.

The result in Corollary 1 is an easy consequence of (7.2). ◆

Appendix

Let v be a bounded positive subsolution relative to a weighted parabolic operator P as defined in 1 in $Q(R;\bar{z})$ and let be ω be a function in $C_0^\infty(B(R;\bar{x}))$ such that $\omega=\tilde{\omega}\eta$

where η is as in 5 and $\tilde{\omega} \in C_0^\infty(B(R;\bar{z}))$

$$\tilde{\omega} = 0 \text{ in } B\left(\frac{R}{16};\bar{x}\right) \text{ and for } x \notin B\left(\frac{R}{2};\bar{x}\right),$$

$$\tilde{\omega} = 1 \text{ in } B\left(\frac{R}{4};\bar{x}\right) - B\left(\frac{R}{8};\bar{x}\right),$$

$$|D_x \tilde{\omega}| \leqslant \frac{32}{R},$$

and τ be as in 5.

Proposition: *The following relation holds*

$$\int_{\bar{t}-h_{x_0}(R)}^{\bar{t}-\delta} \int_{B(R;\bar{x})} \omega^2 \tau^2 v^2 (G^{\bar{z}})^{-3/2} |DG^{\bar{z}}|^2 w\, dx\, dt \leqslant \frac{C_1}{h_{x_0}(R)} \int_{\bar{t}-h_{x_0}(R)}^{\bar{t}-\delta} \int_{B(R;\bar{x})} \omega^2 \tau^2 v^2 (G^{\bar{z}})^{1/2}\, dx\, dt +$$

$$+ C_2 R^{-2} \int_{\bar{t}-h_{x_0}(R)}^{\bar{t}-\delta} \int_{B(R/4;\bar{x}) - B(R/16;\bar{x})} \tau^2 v^2 (G^{\bar{z}})^{1/2} w\, dx\, dt +$$

$$+ C_3 R^{N/2} \int_{\bar{t}-h_{x_0}(R)}^{\bar{t}-\delta} \int_{B(R;\bar{x})} \omega^2 \tau^2 G^{\bar{z}} |Dv|^2 w\, dx\, dt +$$

$$+ C_4 R^{(N/2+2)} \int_{\bar{t}-h_{x_0}(R)}^{\bar{t}-\delta} \int_{B(R/4;\bar{x}) - B(R/16;\bar{x})} \tau^2 v^2 w\, dx\, dt + \|(G^{\bar{z}})^{1/2} \omega^2 v^2\|_{L^1(B(R,\bar{x}))} (\bar{t} - \delta).$$

Proof: From the definition of regularized Green function we have easily, for $\rho < R/16$,

$$\int_{\bar{t}-h_{x_0}(R)}^{\bar{t}-\delta} \left\langle -D_t G_\rho^{\bar{z}} - \sum_{i,j=1}^N D_{x_j} (a_{ij} D_{x_i} G_\rho^{\bar{z}}),\ \omega^2 \tau^2 (G_\tau^{\bar{z}})^{-1/2} v^2 \right\rangle dt = 0.$$

Being v^2 also a subsolution; we have

$$2 \int_{\bar{t}-h_{x_0}(R)}^{\bar{t}-\delta} \int_{B(R;\bar{x})} (G_\rho^{\bar{z}})^{1/2} \omega^2 \tau v^2 D_t \tau\, dx\, dt - \|(G_\rho^{\bar{z}})^{1/2} \omega^2 v^2\|_{L^1(B(R,\bar{x}))} (\bar{t} - \delta) +$$

$$- \frac{1}{2} \sum_{i,j=1}^N \int_{\bar{t}-h_{x_0}(R)}^{\bar{t}-\delta} \int_{B(R;\bar{x})} a_{ij} (G_\rho^{\bar{z}})^{3/2} \omega^2 \tau^2 v^2 D_{x_i} G_\rho^{\bar{z}} D_{x_j} G_\rho^{\bar{z}}\, dx\, dt +$$

$$+2\sum_{i,j=1}^{N}\int_{\bar t-h_{x_0}(R)}^{\bar t-\delta}\int_{B(R;\bar x)} a_{ij}(G_\rho^{\bar z})^{-1/2}\tau^2 v^2\,\omega D_{x_i}G_\rho^{\bar z}D_{x_j}\omega\,dx\,dt+$$

$$+4\sum_{i,j=1}^{N}\int_{\bar t-h_{x_0}(R)}^{\bar t-\delta}\int_{B(R;\bar x)} a_{ij}(G_\rho^{\bar z})^{-1/2}\tau^2 v\omega D_{x_i}\omega D_{x_j}v\,dx\,dt\geq 0$$

then

$$\int_{\bar t-h_{x_0}(R)}^{\bar t-\delta}\int_{B(R;\bar x)}\omega^2\tau^2v^2(G_\rho^{\bar z})^{-3/2}|DG_\rho^{\bar z}|^2w\,dx\,dt\leq\frac{C_1}{h_{x_0}(R)}\int_{\bar t-h_{x_0}(R)}^{\bar t-\delta}\int_{B(R;\bar x)}\omega^2\tau^2v^2(G_\rho^{\bar z})^{1/2}\,dx\,dt+$$

$$+C_2R^{-2}\int_{\bar t-h_{x_0}(R)}^{\bar t-\delta}\int_{B(R/4;\bar x)-B(R/16;\bar x)}\tau^2v^2(G_\rho^{\bar z})^{1/2}w\,dx\,dt+$$

$$+C_3R^{N/2}\int_{\bar t-h_{x_0}(R)}^{\bar t-\delta}\int_{B(R;\bar x)}\omega^2\tau^2G_\rho^{\bar z}|Dv|^2w\,dx\,dt+$$

$$+C_4R^{(N/2+2)}\int_{\bar t-h_{x_0}(R)}^{\bar t-\delta}\int_{B(R/4;\bar x)-B(R/16;\bar x)}\tau^2v^2w\,dx\,dt.$$

Passing to the limits as ρ and δ go to 0 we obtain the result. ◆

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