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On the Isoperimetric Deficit in Minkowskian Geometry (**)

SUMMARY. — In the set of all planar centrally symmetric convex bodies inscribed in a convex centrally symmetric annulus the figure of minimal isoperimetric deficit in Minkowskian geometry is described.

Sul deficit isoperimetrico nella geometria di Minkowski

SUNTO. — Si considera nel piano l'insieme di tutti i corpi convessi centralmente simmetrici inscritti in un dato anello convesso centralmente simmetrico e si determina in tale insieme la figura avente deficit isoperimetrico minimo nella geometria di Minkowski.

1. - INTRODUCTION

In 1920 the Danish mathematician T. Bonnesen obtained several results in the Euclidean plane on the isoperimetric problem and related questions. In particular he showed that for each convex body K the minimum circular annulus of K exists and is unique (see [3]). Moreover he proved the inequality

$$\frac{L^2}{4\pi} - A \geqslant \frac{\pi}{4}(R-r)^2$$

where L and A are respectively the perimeter and area of K and R, r are the two radii of the minimum annulus. From this there follows the isoperimetric inequality

$$L^2 - 4\pi A \geqslant 0$$

where equality holds for the circle.

It has been proved ([6]) that for a convex body K and for a smooth, strictly convex

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and centrally symmetric body C the following inequality holds

$$W(K,C)^2 - A(K)A(C) \geq \left(\frac{\sigma-\varrho}{2}\right)^2 A(C)^2 \tag{1}$$

where A denotes the area, $W(K,C)$ is the mixed area defined by

$$A(K+C) = A(K) + 2W(K,C) + A(C)$$

and σ and ϱ are the radii of the minimal convex annulus of K with respect to C (see [7]). Analogous results can be found in [1], [2] and [9].

The isoperimetric deficit

$$\delta K = W(K,C)^2 - A(K)A(C)$$

was investigated by Bonnesen for a circle C. He showed that in this case the minimum is attained for a figure composed of two parallel segments touching the inner circle and of four congruent circular arcs.

In this paper we obtain in Minkowskian geometry a result analogous to the one of Bonnesen in the set of all centrally symmetric convex bodies inscribed in a smooth strictly convex and centrally symmetric annulus.

We shall show that the minimum of the isoperimetric deficit is attained for a figure composed of two parallel line segments touching the inner homothetic image of C and of four arcs homothetic to arcs of C. Two of these arcs may be empty.

2. - Basic definitions and preliminary propositions

In the following we consider the plane equipped with a suitable Minkowski metric such that the boundary of C becomes the isoperimetrix of the plane. Then the Minkowski perimeter of a convex body K is

$$L(K) = 2W(K,C) \quad \text{(cf. [5, p. 310])}.$$

It follows that the isoperimetric deficit is

$$\delta K = \frac{L(K)^2}{4} - A(K)A(C).$$

We denote by $L(d)$ the Minkowski arc-length of the arc d contained in the boundary of K.

The abbreviations $\mathrm{bd}\,K$ and $\lambda F + x$ stand for the boundary and for the homothetic image of a convex body F. Given three distinct points a, b, c on $\mathrm{bd}\,F$ we denote by (a, c, b) the subarc of $\mathrm{bd}\,F$ containing c and with endpoints a, b. The convex hull of a set X is denoted by $\mathrm{conv}\,X$ and $[x, y]$ is a closed line segment with endpoints x and y.

If the arc d is contained in $\mathrm{bd}(\lambda C + x)$, λ is the C-radius of d.
Since for the Minkowski metric the triangle inequality holds, it follows that

$$\delta \operatorname{conv} X \leq \delta X .$$

In the following two propositions C is a smooth strictly convex and centrally symmetric body.

Proposition 2.1: Let F be a convex body such that $\mathrm{bd}\, F$ be inscribed in the annulus $(\sigma C, \varrho C)$. In each annulus $(\sigma' C, \varrho' C)$ with

$$\varrho \leq \varrho' < \sigma' \leq \sigma$$

it is possible to construct a convex body F' such that

$$\delta F' \leq \delta F .$$

Proof: We remark that δF is equal to the deficit of the curve $\mathrm{bd}\, F_x$ the outer exteriorly parallel to $\mathrm{bd}\, F$ at the distance x.
Indeed, from Steiner's formulas in Minkoskian geometry (cf. [5, p.310])

$$L(F_x) = L(F) + 2xA(C),$$

$$A(F_x) = A(F) + xL(F) + x^2 A(C),$$

we have that $\delta(F_x) = \delta F$.
The curve $\mathrm{bd}\, F_x$ is inscribed in the annulus $((\sigma + x)C, (\varrho + x)C)$. Therefore, it is possible, as in the circular case of the euclidean space, to find x by means of the following condition

$$\frac{\varrho'}{\varrho + x} = \frac{\sigma'}{\sigma + x} = \frac{\sigma' - \varrho'}{\sigma - \varrho} = \nu \quad (\nu \leq 1).$$

Hence

$$x = \frac{\varrho'(\sigma - \varrho) - \varrho(\sigma' - \varrho')}{(\sigma' - \varrho')} \quad (x \geq 0).$$

Now we consider the convex body $F' = \nu F_x$ which is homothetic to F_x with coefficient ν. From some properties of the areas and of the mixed areas (cf.[4]) we have that

$$\delta F' = \nu^2 \delta F_x$$

and the Proposition follows.

Proposition 2.2: Let $[a, b]$ be a line segment and γ a positive real number $(\gamma > L([a, b]))$. Moreover, let l_1 denote an arc with endpoints a and b such that $L(l_1) = \gamma$. For the figure K_1 such that

$$\operatorname{bd} K_1 = [a, b] \cup l_1$$

$A(K_1)$ is maximal if l_1 is an arc homothetic to an arc of $\operatorname{bd} C$.

Proof: Let λC denote an homothetic copy of C such that $\operatorname{bd}(\lambda C)$ contains an arc l_1 with endpoints a and b and $L(l_1) = \gamma$. Then we set

$$\operatorname{bd}(\lambda C) = l_1 \cup l_2$$

and we have that

$$L(\operatorname{bd}(\lambda C)) = \gamma + L(l_2).$$

Now we consider a set K such that $a, b \in \operatorname{bd} K$ and $\operatorname{bd} K = l \cup l_2$ where l is an arc of lenght γ and with endpoints a and b.

Then for the sets K_1 and K_2 such that

$$K_1 \cup K_2 = K,$$

$$\operatorname{bd} K_1 = [a, b] \cup l,$$

$$\operatorname{bd} K_2 = [a, b] \cup l_2,$$

it follows that

$$A(K_1) \leq \frac{[\gamma + L(l_2)]^2}{4A(C)} - A(K_2).$$

Here, equality holds if and only if K is homothetic to C (cf.[5, p. 310] or [8, p. 273]). Hence since l_2 is an arc of $\operatorname{bd}(\lambda C)$, the equality holds only if $K = \lambda C$.

Therefore, for the given points a and b and for the given γ, we have that the area of K_1 is maximal if l_1 is homothetic to an arc of $\operatorname{bd} C$.

3. - The theorem and its proof

Theorem 3.1: Let C be a smooth strictly convex and centrally symmetric (O-symmetric) body and $(\sigma C, \varrho C)$ a convex annulus. A convex body centrally symmetric (O-symmetric) such that $(\sigma C, \varrho C)$ be its minimal convex annulus and with the minimum isoperimetric deficit is a figure composed of two parallel line segments touching ϱC and four arcs which are homothetic to arcs of C and have all the same C-radius. Two of these arcs may be empty.

Proof: From (1) it follows that the isoperimetric deficit of the centrally symmetric convex bodies with boundary inscribed in the annulus $(\sigma C, \varrho C)$ attains an inferior limit δ_s. Blaschke's selection theorem yields the existence of a convex body Q centrally symmetric for which the extremum in question is attained. The aim of this theorem is to study the boundary of Q.

By denoting with $(\sigma C, \varrho C)$ the given annulus there are four points a, b, c, d on $\mathrm{bd}\, Q$ and ordered in this way, with $a, c \in \varrho C$ and $b, d \in \sigma C$ (see [7]).

Since Q and C are centrally symmetric it is possible to draw two parallel lines t and t' tangent to $\varrho C (a \in t,\ b \in t')$ which determine a subset G of σC which contains $\mathrm{bd}\, Q$ in its interior or as a part of its boundary. Therefore $\mathrm{bd}\, Q$ is divided in four parts by the lines a, c and b, d. It is possible that one of these parts contains, besides an arc l interior to G, either an arc of $\mathrm{bd}(\sigma C)$ or an arc of $\mathrm{bd}(\varrho C)$ or a part of t (or of t').

Assuming that l be not empty we show that l is either a line segment or an arc of $\mathrm{bd}(\lambda C + x)$.

Let p, q, r be three points of the interior of l and ordered in this way.

By assuming that the arc (p, q, r) of $\mathrm{bd}\, Q$ is not a line segment we consider, instead of (p, q, r) an arc (p, r) of $\mathrm{bd}(\lambda C + x)$ where $\lambda C + x$ contains the points p and r and is such that

$$L((p, r)) = L((p, q, r)).$$

So from Proposition 2.2, we have a new figure whose deficit is less than the one of the previous convex body. If the new figure is not convex we consider its convex hull. It is also possible to choose p and r close to q in such a way that the convex hull is contained in G.

Now we show that l cannot be a segment. Let $q = l \cap t$. We construct a convex body $\lambda C + x$ tangent to t. Let $p = \mathrm{bd}(\lambda C + x) \cap t$ and $[r_1, r_2] = \mathrm{bd}(\lambda C + x) \cap l$. We can determine $\lambda C + x$ in such a way that

$$L((r_1, r_2, p)) = L([r_1, q] \cup [q, p]).$$

Let F be the figure whose boundary contains the arc (r_1, r_2, p) of $\mathrm{bd}(\lambda C + x)$ instead of $[r_1, q] \cup [q, p]$ of $\mathrm{bd}\, Q$, then $\delta F \le \delta Q$. It is possible to choose r_1, r_2 and p close to q in such a way that $\mathrm{conv}\, F$ is a convex body with boundary inscribed in $(\sigma C, \varrho C)$ and with a deficit less than the one of Q.

Therefore l (if not empty) is an arc homothetic to an arc of C.

Now we prove that Q cannot contain arcs of $\mathrm{bd}(\sigma C)$. There are two cases:

i) one arc l is not empty (Then for symmetry reasons there are two nonempty arcs). As in the case studied above we construct a convex body $\lambda C + x$ tangent to l and we consider the points r, s of $\mathrm{bd}(\lambda C + x) \cap \mathrm{bd}(\sigma C)$, $q \in \mathrm{bd}(\sigma C) \cap l$ and $p \in \mathrm{bd}(\lambda C + x) \cap l$, belonging in this order to $\mathrm{bd}\, Q$. Moreover we choose $\lambda C + x$ such that the arc (r, s, p) of $\mathrm{bd}(\lambda C + x)$ and the arc (r, q, p) of $\mathrm{bd}\, Q$

have the same Minkowski arc-length. We can substitute the arc (r, q, p) of $\operatorname{bd} Q$ by the arc (r, s, p) of $\operatorname{bd}(\lambda C + x)$.

From Proposition 2.2, we have in this way a new figure D such that $\delta D \leqslant \leqslant \delta Q$.

Then we consider $\operatorname{conv} D$, since $\operatorname{conv} D$ is not contained in σC its minimal convex annulus is $(\sigma' C, \varrho C)$ with $\sigma' > \sigma$. By Proposition 2.1, we can construct a new convex body F of boundary inscribed in $(\sigma C, \varrho C)$ for which $\delta F \leqslant \delta Q$.

ii) each l is empty. Let q be a point of $\operatorname{bd}(\sigma C) \cap t$. We consider a convex body $\lambda C + x$ such that:

— it has t as tangent in a point p close to q,

— for the points r, s of $\operatorname{bd}(\lambda C + x) \cap \operatorname{bd}(\sigma C)$ and $p \in \operatorname{bd}(\lambda C + x) \cap t$

$$L((r, s, p)) = L((r, q) \cup [q, p]).$$

Therefore we can substitute the arc $(r, q) \cup [q, p]$ of $\operatorname{bd} Q$ by the arc (r, s, p) of $\operatorname{bd}(\lambda C + x)$. The new boundary is not contained in σC and is not convex, but from Proposition 2.1 it is possible to obtain a new convex body with deficit less than the previous and having $(\sigma C, \varrho C)$ as minimal convex annulus.

Now we show that $\operatorname{bd} Q$ cannot contain arcs of $\operatorname{bd}(\varrho C)$. As in the cases studied above it is possible to choose the points of the construction so close to each other that the new figure D is contained in G. Moreover, even if some parts of the new boundary are contained in ϱC and hence the minimal convex annulus of D is $(\sigma C, \varrho' C)$ with $\varrho' < \varrho$, there exists for Proposition 2.1 a new convex body with boundary inscribed in $(\sigma C, \varrho C)$ and with a deficit less than the previous one.

It remains to show that all the arcs l of Q have the same C-radius. To do this we suppose that the arcs $l_1 = (p, q)$ and $l_2 = (r, s)$ have different C-radius. Then we take the arc (q_1, q_2) on l_1 and the arc (s_1, s_2) on l_2 subtended by equal chords and we draw the arcs $l_1' = (p, q)$ with (s_1, s_2) instead of (q_1, q_2) and $l_2' = (r, s)$ with (q_1, q_2) instead of (s_1, s_2). The new figure has the same lenght of boundary, the same area of Q but l_1' and l_2' are not homothetic to arcs of C. This contradiction proves the statement.

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