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Modified Routh Algorithms Applied to Aircraft Stability (**)

SUMMARY. — Routh criterion allows to decide about the asymptotic stability of an equilibrium state, but no features of the transient motion can be assessed a priori. In this paper we propose suitable modifications to the Routh algorithm so that prescribed damping properties can be assigned to any system converging to equilibrium. In particular, these new algorithms are applied to the stick-fixed dynamic of an aircraft to effect critical times and damping ratios of the longitudinal and lateral-directional dynamics.

Algoritmi di Routh modificati con applicazione alla stabilità degli aerei

SUNTO. — Il criterio di Routh consente di determinare condizioni per la stabilità asintotica, ma non di influire sulle caratteristiche del moto transitorio. Nella pesente nota si propongono delle modifiche all'algoritmo di Routh sicché le principali caratteristiche del moto smorzato possano essere assegnate a priori. Gli algoritmi proposti sono applicati alla dinamica a controlli bloccati dei velivoli per influenzare i tempi critici e le frequenze della dinamica longitudinale e laterale.

1. - INTRODUCTION

Stability is an important requirement for any system in engineering.

Of most interest, for pratical purposes, is *asymptotic stability* , that is, if the system is perturbed from an equilibrium state, then it is required to converge again to equilibrium, after some (hopfully short) time.

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By Liapunov theory [1], such stability is verified if the linearized system about its equilibrium ($x = 0$) state, $\dot{x} = Ax$, is asymptotically stable.

Of course this will be the case if the solutions of the characteristic equations (i.e. the eigenvalues of A)

$$\det(A - \lambda I) = \lambda^n + p_1 \lambda_{n-1} + \ldots + p_n = 0 \tag{1}$$

have all negative real part.

However, if the degrees of freedom of the system are more than one, there is no general rule to solve equation (1) explicitly.

In 1876 Routh published his celebrated algorithm which allows to decide about stability of the solutions of a linear equation without actually solve it.

Namely, if the coefficients p_i in (1) satisfy certain inequalities [2] (see Sect. 2) then all the solutions λ_i will be in the plane $\text{Re} < 0$, and the asymptotic stability of the equilibrium state is ensured.

Sometimes we need to know something more about the transient motion, to asses the mechanical properties of the system: in particular critical time and damping ratio.

Also, it may be of interest to have the possibility to establish prescribed damping properties to the system by a suitable choice of the physical parameters, as it is done in the Routh algorithm for stability.

In this paper we propose new algorithms by which we can assign a priori the properties of the transient motion.

Namely, we prove in Proposition 2 that, if the coefficients p_i of (1) verify a set of inequalities, then the solutions λ_i are to the left of a stright line through x_1 and parallel to the imaginary axis (see Fig. 1).

That is, the λ_i's will have real part less than a prescribed value $x_1 = -c$ (with $c > 0$) so that the critical time of the system will be less than $1/c$.

In Proposition 3 we obtain conditions such that the solutions are inside a cone of amplitude α (see Fig. 2).

Therefore the damping ratio ζ will be within the values $(\cos\alpha, 1)$.

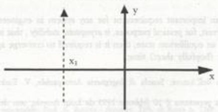

Figure 1.

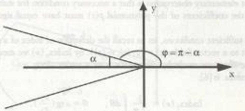

Figure 2.

The sets of inequalities to be satisfied are found in Propositions 2 and 3 by «modified Routh algorithms».

Although such algorithms seem to be a natural generalization of the Routh test, we were not able to find them in the literature, see [3], [4], [5], [6] and the many references therein. We believe such algorithms rather useful, and we applied them to a classical subject: the stick-fixed dynamical stability of an aircraft.

It is well kwown that the longitudinal and the lateral dynamic under infinitesimal perturbations of an equilibrium flight are described by biquadratic linear equations on λ's .

Generally, the Routh test is performed to ensure asymptotic stability, and the flight qualities are described approximating the biquadratic equation by the product of two quadratics.

For longitudinal stability, the two quadratics correspond to the phugoid and short period modes, and in the lateral case they give rise to spiral, roll and Dutch roll modes [7], [8].

Applying the algorithms of Proposition 2, we force the critical times of the longitudinal system to be lower than a definite time interval , such as 20 sec.

Proposition 3 is applied to the lateral system to assign prescribed damping ratio and critical time, thus having an influence on the frequences.

The paper proceeds as follows: in Section 2 the Routh algorithm is recalled in view of the proof of Propositions 2 and 3.

Section 3 is devoted to such proofs and the new algorithms are presented.

The last Section concerns with two applications in dynamical stability of an aircraft.

2. - The routh algorithm.

To make our exposition self-contained, we review here the basic ideas behind the Routh algorithm (for further details, see [2], [3], [6]).

The first elementary observation is that a necessary condition for stability is the following: all the coefficients of the polynomial $p(z)$ must have equal sign, for instance positive.

To find sufficient conditions, let us recall the definition of index of a closed curve γ with respect to a vector field $v(z) = (v_1(z), v_2(z))$: by $\text{Index}_\gamma(v)$ we mean the number of complete rotations made by the vector v as z varies along γ.

The formula is [6]:

(2) $$\text{Index}_\gamma(v) = \frac{1}{2\pi}\int_\gamma d\theta\,, \qquad \theta = a\,\text{tg}\left(\frac{v_2}{v_1}\right).$$

As an example, consider the complex field $p(z) = z^n$, represented on the plane as

(3) $$\begin{cases} v_1 = \text{Re}\,p = \varrho^n \cos n\phi\,, \\ v_2 = \text{Im}\,p = \varrho^n \sin n\phi\,, \end{cases}$$

so that $\text{tg}\,\theta = \text{tg}\,n\phi$.

If we take γ to be the unit circle, then, as z moves along γ, ϕ goes from 0 to 2π and θ performs n complete rotations.

In the example, the curve γ contains the origin which is the solution of $z^n = 0$; more generally, it is not difficult to show that given any polynomial with real coefficients

(4) $$p(z) = p_0 z^n + p_1 z^{n-1} + \ldots + p_{n-1} z + p_n$$

and any closed curve γ containing all its zeroes, then $\text{Index}_\gamma(p) = n$ [6].

Therefore, the solutions of (4) will be stable if the index with respect to the field $(\text{Re}\,p, \text{Im}\,p)$ of a curve γ «containing the left half plane $Re < 0$» is equal to n.

As such curve γ, we can consider the semi-circle of Fig. 3 as its radius R goes to infinity.

Then the integral in (2) is the sum of the integral along the curve C_R and the integral

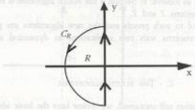

Figure 3.

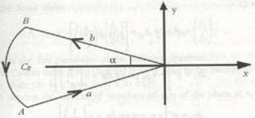

Figure 4.

along the imaginary axis followed from $-\infty$ to $+\infty$; for stability we must have:

(5) $$\text{Index}_y(p) = \frac{1}{2\pi}\int_{C_R} d\theta + \frac{1}{2\pi}\int_{-i\infty}^{+i\infty} d\theta = n .$$

To compute the first integral, note that if C_R is the curve of Fig. 4, we have, as $R \to \infty$:

(6) $$\frac{1}{2\pi}\int_{C_R} d\theta = \frac{1}{2\pi} 2n\alpha .$$

For the reader convenience, we reproduce here a proof of (6):

Proposition 1: Given any polynomial of degree n, the integral in (6), C_R being the curve of Fig. 4, is equal to $n\alpha/\pi$ as the radius R goes to infinity.

Proof: *Let us write the polynomial in polar cordinates* $z = \varrho e^{i\phi}$:

$$v_1 = \operatorname{Re} p = \varrho^n \cos n\phi + \varrho^{n-1} p_1 \cos(n-1)\phi + \dots + p_n ,$$

$$v_2 = \operatorname{Im} p = \varrho^n \sin n\phi + \varrho^{n-1} p_1 \sin(n-1)\phi + \dots + p_{n-1} \sin\phi .$$

By definition of $\theta = a\,\mathrm{tg}\left(\frac{v_2}{v_1}\right)$, *it follows:*

$$d\theta = \frac{v_1 dv_2 - v_2 dv_1}{v_1^2 + v_2^2}$$

where: $(k \geq 1)$

$$v_1^2 + v_2^2 = \varrho^{2n}\left[p_0 + o\left(\frac{1}{\varrho^k}\right)\right]$$

$$dv_1 = \varrho^n\left[-n\,p_0 \sin n\phi\, d\phi + o\left(\frac{1}{\varrho^k}\right)\right],$$

$$dv_2 = \varrho^n\left[n\,p_0 \cos n\phi\, d\phi + o\left(\frac{1}{\varrho^k}\right)\right].$$

Then $d\theta = n\,d\phi + o(1/\varrho^k)$ *tends, for* $R \to \infty$, *to* $n\,d\phi$, *and along the arch of amplitude* 2α *we get:*

$$\int_{C_R} d\theta = n \int_{\pi-\alpha}^{\pi+\alpha} d\phi = 2n\alpha$$

and this ends the Proposition. ■

Now, according to (5) and Prop. 1, the zeroes of (4) will be stable (i.e. on the plane $\mathrm{Re} < 0$) if

$$\frac{1}{2\pi}\int_{-\infty i}^{+\infty i} d\theta = \frac{n}{2}\,.$$

That is θ must perform $n/2$ turns as z goes along the imaginary axis.

This is equivalent to say that $\mathrm{tg}\,\theta = \mathrm{Im}\,p / \mathrm{Re}\,p$ must be subjected to n jumps from $+\infty$ to $-\infty$ (see for instance Fig. 9 where $\mathrm{tg}\,\theta$ has 4 jumps as θ rounds two times).

The stability problem is then reduced to show that $\mathrm{tg}\,\theta = (\mathrm{Im}\,p/\mathrm{Re}\,p)(it)$ undergoes n jumps from $+\infty$ to $-\infty$ as t goes from $-\infty$ to $+\infty$ (or, equivalently, that $\mathrm{ctg}\,(\mathrm{Re}\,p/\mathrm{Im}\,p)(it)$ is subjected to n jumps from $-\infty$ to $+\infty$).

Actually, it is necessary an explicit way to compute the so called «Cauchy index» of a rational function $R(x)$:

(7) $I_a^b\, R(x) = \{$number of jumps from $-\infty$ to $+\infty\}$ −

−{number of jumps from $+\infty$ to $-\infty\}$ as x goes from a to b $(a < b)$.

Routh proposed to use the *Sturm chain* to compute the Cauchy index.

The computation is as follows: given any rational function $R = (f_2/f_1)$ it is possible to define the *Sturm chain*:

(8) $$f_1, f_2, f_3 = -\mathrm{Rem}\left(\frac{f_1}{f_2}\right), \quad f_4 = -\mathrm{Rem}\left(\frac{f_2}{f_3}\right), \ldots$$

where Rem is the first remainder of the indicated quotients.

Then the Cauchy index $I_a^b R(x)$ defined in (7) is equal to the number of variations of sign in the chain (8) as x goes from a to b [3].

If $V(x)$ denotes the number of variations of sign of the chain at x, we have in formulae:

(9) $$I_a^b R(x) = V(a) - V(b).$$

EXAMPLE *a*): Consider the rational function $R = f_2 / f_1$ with $f_2 = p_1 x$ and $f_1 = p_0 x^2 + p_2$.

Then

$$f_1(x) = q(x) f_2(x) - f_3(x)$$

where $f_3 = -p_2$, $q(x) = (p_0 / p_1) x$, and the Sturm chain is:

$$f_1(x), \quad f_2(x), \quad f_3(x).$$

To show identity (9), suppose that f_1, f_2, f_3 have signs in $x = a$:

$$f_1(a), f_2(a), f_3(a) \to (+), (-), (+)$$

then the number of variations of sign in the chain at $x = a$ is: $V(a) = 2$.

As x goes from a to b, f_2 can not vanish, since, if $f_2(x_0) = 0$, then $f_1(x_0) = -f_3(x_0)$, that is f_1 and f_3 must be of opposite sign.

A variation of sign may happen only if f_1 goes through a zero x_0, so we have:

$$(+), (-), (+) \to (-), (-), (+),$$

$$V(a) = 2 \to V(x_0 + \varepsilon) = 1.$$

Note that a variation of sign in the chain is lost: $V(a) - V(x_0 + \varepsilon) = 1$, and correspondingly $R = f_2 / f_1$ has a jump at x_0 from $-\infty$ to $+\infty$.

So, from definition (7),

$$I_a^{x_0+\varepsilon} R = 1 = V(a) - V(x_0 + \varepsilon)$$

that is (9) holds true.

Viceversa, if in $x = a$ we have signs:

$$(-), (-), (+), \qquad V(a) = 1$$

then f_2 can have a zero, but that would not change the number of variations of sign.

Again, this number will cange only if f_1 goes through a zero x_0, and we have in $x_0 + \varepsilon$:

$$(-),\ (-),\ (+),\qquad V(x_0+\varepsilon) = 2,$$

i.e. a variation of sign in the chain is gained.

Correspondingly $R = f_2 / f_1$ has a jump from $+\infty$ to $-\infty$.

Again we get

$$I_a^{x_0+\varepsilon} R = -1 = V(a) - V(x_0+\varepsilon)$$

as in formula (9). ■

Now, we have to compute the Cauchy index

$$I_{-\infty}^{+\infty} \frac{\operatorname{Im} p}{\operatorname{Re} p}$$

(or $I_{-\infty}^{+\infty}(\operatorname{Re} p / \operatorname{Im} p)$, if the degree of $\operatorname{Re} p$ is bigger than the degree of $\operatorname{Im} p$) using Sturm chains.

For, we have on the imaginary axis $p(it) = \operatorname{Re} p + i \operatorname{Im} p$, where for even n:

$$\operatorname{Re} p = (-1)^{n/2}(p_0 t^n - p_2 t^{n-2} + p_4 t^{n-4} + \ldots),$$

$$\operatorname{Im} p = (-1)^{(n-2)/2}(p_1 t^{n-1} - p_3 t^{n-3} + p_5 t^{n-5} + \ldots)$$

and for odd n:

$$\operatorname{Re} p = (-1)^{(n-1)/2}(p_1 t^{n-1} - p_3 t^{n-3} + \ldots),$$

$$\operatorname{Im} p = (-1)^{(n-1)/2}(p_0 t^n - p_2 t^{n-2} + \ldots).$$

Then from (5) and (7) it follows that

$$2\frac{1}{2\pi}\int_{-\infty i}^{+\infty i} d\theta = I_{-\infty}^{+\infty} \frac{\operatorname{Re}}{\operatorname{Im}} p \ \text{ if } n \text{ is odd}$$

$$= -I_{-\infty}^{+\infty} \frac{\operatorname{Im}}{\operatorname{Re}} p \ \text{ if } n \text{ is even}.$$

Both the cases lead to the computation of the Cauchy index [3]:

$$I_{-\infty}^{+\infty} \frac{p_1 t^{n-1} - p_3 t^{n-3} + p_5 t^{n-5} + \ldots}{p_0 t^n - p_2 t^{n-2} + p_4 t^{n-4} + \ldots} = V(-\infty) - V(+\infty)$$

where $V(x)$ is the number of variations of sign in the Sturm chain:

$$(10)\qquad \begin{cases} f_1 = p_0 t^n - p_2 t^{n-2} + p_4 t^{n-4} + \dots, & f_2 = p_1 t^{n-1} - p_3 t^{n-3} + p_5 t^{n-5} + \dots \\ f_3 = -\operatorname{Re} m\left(\dfrac{f_1}{f_2}\right), & f_4 = -\operatorname{Re} m\left(\dfrac{f_2}{f_3}\right), \text{ etc.} \end{cases}$$

Computing the above remainders, one soon realizes that the coefficients of the polynomials f_1, f_2, f_3, etc. in the chain can be obtained by the <u>*Routhian*</u>:

$$(11)\qquad \begin{cases} p_0 \quad p_2 \quad p_4 \quad \cdots \\ p_1 \quad p_3 \quad p_5 \quad \cdots \\ c_1 \quad c_2 \quad c_3 \quad \cdots \\ d_1 \quad d_2 \quad d_3 \quad \cdots \\ \text{etc., where} \\ c_1 = \dfrac{p_1 p_2 - p_0 p_3}{p_1}, \quad c_2 = \dfrac{p_1 p_4 - p_0 p_5}{p_1}, \text{ etc.} \\ d_1 = \dfrac{c_1 p_3 - p_1 c_2}{c_1}, \quad d_2 = \dfrac{c_1 p_5 - p_1 c_3}{c_1}, \text{ etc.} \end{cases}$$

Namely, the c_i's are the coefficients of f_3, d_i's those of f_4, etc.

For stability it is required:

$$(12)\quad 2\left(\frac{1}{2\pi}\int_\gamma d\theta\right) = I_{-\infty}^{+\infty}\, \frac{p_1 t^{n-1} - p_3 t^{n-3} + p_5 t^{n-5} + \dots}{p_0 t^n - p_2 t^{n-2} + p_4 t^{n-4} + \dots} = V(-\infty) - V(\infty) = n\,.$$

Since the entries of the first column of the Routhian are the leading terms of the f_i's, $V(\infty)$ is equal to the number of variations of sign in the first column

$$V(\infty) = [p_0, p_1, c_1, d_1, \dots]$$

and

$$V(-\infty) = [p_0, -p_1, c_1 - d_1, \dots].$$

Then

$$(13)\qquad V(-\infty) = n - V(\infty)$$

By (12) and (13) it follows $V(\infty) = 0$, i.e.:

ROUTH CRITERION: The solutions of (4) have all negative real part if and only if p_i have equal sign and the first column of the Routhian (11) have all its entries with equal sign.

3. - Modified Routhians

Now we look for conditions such that the solutions of (4) have all real part less than a given negative number $x_1 = -c$ $(c > 0)$ i.e. they stay on the half plane to the left of the line $x = x_1$ (see Fig. 1).

We get these conditions just applying the Routh criterion to the polynomial $p(z_1)$ obtained by translation:

$$z = z_1 + x_1$$

Namely

$$p(z) = p_0(z_1 + x_1)^n + p_1(z_1 + x_1)^{n-1} + \dots + p_n \tag{14}$$

has zeroes in the z_1 plane:

$$\lambda_1' = \lambda_1 - x_1\,, \quad \lambda_2' = \lambda_2 - x_1, \dots, \lambda_n' = \lambda_n - x_1\,,$$

λ_i being zeroes in the z plane.

Then if $p(z_1)$ satisfies the Routh condition, it follows that λ_i' are all to the left of 0_1 (see Fig. 5), so the λ_i are all to the left of the line $x = x_1 = -c$.

All we have to do is just to apply the Routh criterion to $p(z_1)$, which is obtained expanding the monomials in (14) by the Newton formula:

$$(a+b)^k = a^k + \binom{k}{1} a^{k-1} b + \binom{k}{2} a^{k-2} b^2 + \dots + b^k, \quad \text{where } \binom{k}{h} = \frac{k!}{h!(k-h)!}\,.$$

Then we have:

Proposition 2: All the solutions of

$$p_0 z^n + p_1 z^{n-1} + \dots p_n$$

will have real part less than $x_1 = -c$ $(c > 0)$ if and only if p_i, a_i, b_i have the same sign,

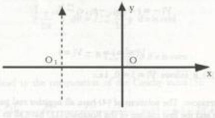

Figure 5.

and the same it is for the first column of the Routhian

$$(15)\quad \begin{cases} a_1 \; a_2 \; a_3 \dots \\ b_1 \; b_2 \; b_3 \dots \\ c_1 \; c_2 \; c_3 \dots \\ \vdots \\ \text{where} \\ a_1 = p_0 \,, \quad a_2 = \sum_{k=0}^{2} p_k \binom{n-k}{2-k} x_1^{2-k} \,, \quad a_3 = \sum_{k=0}^{4} p_k \binom{n-k}{4-k} x_1^{4-k} \,, \text{ etc.} \\ b_1 = \sum_{k=0}^{1} p_k \binom{n-k}{1-k} x_1^{1-k} \,, \quad b_2 = \sum_{k=0}^{3} p_k \binom{n-k}{3-k} x_1^{3-k} \,, \text{ etc.} \\ c_1 = \dfrac{b_1 a_2 - a_1 b_2}{b_1} \,, \quad c_2 = \dfrac{b_1 a_3 - a_1 b_3}{b_1} \,, \text{ etc.} \end{cases}$$

EXAMPLE (1): $z^2 + p_1 z + p_2 = 0.$

The first column of the Routhian (15) is given by:

$$a_1 = 1 \,, \qquad b_1 = p_1 + 2x_1 \,, \qquad c_1 = x_1^2 + p_1 x_1 + p_2 \,.$$

The inequalities $b_1 > 0, c_1 > 0$ imply that the two parameters p_1 and p_2 must be taken in the shaded region bounded by the two straight lines $b_1 = 0$, $c_1 = 0$ of Fig. 6.

From the formula of the solution of a second order equation, the reader may easily verify that conditions $b_1 > 0$, $c_1 > 0$ are just those needed to have $\mathrm{Re}(\lambda) < x_1$. ■

EXAMPLE (2): $z^3 + p_1 z^2 + p_2 z + p_3 = 0.$

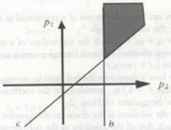

Figure 6.

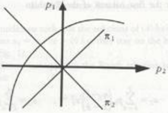

Figure 7.

Proposition (2) gives the conditions:

$$(a) = (3x_1 + 2P_1)x_1 + P_2,$$

$$(b) = p_1 + 3x_1,$$

$$(c) = (p_1 + 3x_1)(2x_1p_1 + p_2 + 3x_1^2) - (x_1^3 + p_1x_1^2 + p_2x_1 + p_3),$$

$$(d) = p_1x_1^2 + p_2x_1 + p_3 + x_1^3,$$

all positive.

The region on the space (p_1, p_2, p_3) of the coefficients verifying the previous inequalities is bounded by the planes $(a) = 0$, $(b) = 0$, $(d) = 0$ and the inner part of the paraboloid $(c) = 0$. The latter intersects the axes in the points $(-4c_1^2, 0, 0)$, $(0, -4c_1^2, 0)$, $(0, 0, -8c_1^3)$ and it is oriented along the principal planes

$$\pi_1: (x_1 + \sqrt{x_1^2 + 1})p_1 + p_2 = 0, \quad \pi_2: (x_1 - \sqrt{x_1^2 + 1})p_1 + p_2 = 0$$

as shown in Fig. 7. ■

By Proposition 2 we can locate the zeroes of a polynomial to the left of an arbitrary stright line parallel to the imaginary axis.

If such zeroes $\lambda_k = x_k + iy_k$ represent the eigenvalues of a matrix in a differential system, then the motion is obtained by linear (complex conjugate) combinations of the functions $e^{\lambda_k t} = e^{x_k t}(\cos y_k t + i \sin y_k t)$.

Therefore, since we have a control on the real parts x_k, we can extablish a priori and at our will the critical time $t^* = 1/x_k$, provided that the coefficients of the characteristic equation satisfy the inequalities of Prop. 2.

Another control on the solutions of the characteristic equation can be the following: find conditions on p_i such that all the solutions of the polynomial are inside a cone of angle 2α (with $R \to \infty$) (see Fig. 4).

If all the solutions are inside the cone, then we must have

$$\frac{1}{2\pi}\int_\gamma d\theta = n \tag{16}$$

where γ is the boundary of the cone: $\gamma = (b) \cup C_{R\to\infty} \cup (a)$.

Before finding conditions such that (16) holds true, let us consider an easy example.

EXAMPLE b): Take the polynomial

$$p(z) = z^2 + z + \frac{1}{2}$$

having zeroes $\lambda_{1,2} = -1/2 \pm i/2$ with polar coordinates: $\varrho_{1,2} = 1/\sqrt{2}$, $\phi_1 = (3/4)\pi$, $\phi_2 = (5/4)\pi$. The cone of amplitude $\alpha = \pi/3$ contains the zeroes of p, then we must have:

$$\int_\gamma d\theta = 4\pi$$

That is θ will make two complete rotations as z goes along the boundary γ of the cone $\alpha = \pi/3$.

Namely, the line (b) in Fig. 4 has parametric equation:

$$\begin{cases} x = s\cos(\pi - \alpha), \\ y = s\sin(\pi - \alpha), \\ s \in [0, \infty), \end{cases} \tag{17}$$

and $p(z)$ evaluated along (b) is:

$$\operatorname{Re} p(b) = -\frac{s^2}{2} - \frac{s}{2} + \frac{1}{2},$$

$$\operatorname{Im} p(b) = -s^2\frac{\sqrt{3}}{2} + s\frac{\sqrt{3}}{2}.$$

By direct analysis of the function $\operatorname{tg}\theta = (\operatorname{Im} p/\operatorname{Re} p)(b)$ we get the diagram of Fig. 8, as $s \in [0, \infty)$.

We can get the same result using the Sturm chain:

$$f_1 = \operatorname{Re} p, \quad f_2 = \operatorname{Im} p, \quad f_3 = s - \frac{1}{2}, \quad f_4 = s\frac{\sqrt{3}}{4}, \quad f_5 = \frac{1}{2},$$

which implies $V_{(b)}(0) - V_{(b)}(\infty) = 2 - 3 = -1$, i.e. a variation of sign is gained, thus showing that f_2/f_1 has a jump from $+\infty$ to $-\infty$.

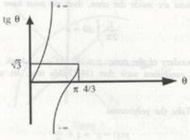

Figure 8.

To compute the variation along C_R, we use Prop. 1, which states that θ is rotated by an angle equal to $2n\alpha = (4/3)\pi$, then $\theta(A) = (8/3)\pi$.

Finally, the line (a) has parametric equation:

$$\begin{cases} x = s\cos(\alpha), \\ y = s\sin(\alpha), \\ s \in (-\infty, 0], \end{cases} \tag{18}$$

so that

$$\operatorname{Re} p(a) = -\frac{s^2}{2} + \frac{s}{2} + \frac{1}{2},$$

$$\operatorname{Im} p(a) = s^2\frac{\sqrt{3}}{2} + s\frac{\sqrt{3}}{2}.$$

The Sturm chain gives: $V_{(a)}(-\infty) - V_{(a)}(0) = -1$, then $(\operatorname{Im}/\operatorname{Re})(a)$ has a jump from $+\infty$ to $-\infty$. The final diagram is in Fig. 9.

Therefore, either by direct calculation of the variation of θ (as we did along C_R) or by inspecting the behaviour of its tangent (as in (a), (b)), we can conclude that θ rotates by 4π as z moves along γ.

Summarizing we computed the index by the formula:

$$2\operatorname{Index}_\gamma(p) = \Gamma + V_{(b)}(\infty) - V_{(a)}(-\infty)$$

(since $V_{(b)}(0) = V_{(a)}(0)$), where Γ is equal to the number of jumps of $\operatorname{tg}\theta$ along the arch C_R connecting the points B and A of Fig. 4.

In the general case it is not difficult to calculate such Γ, since θ rotates by $2n\alpha$ along

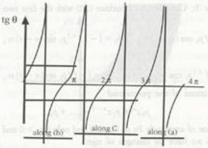

Figure 9

C_R, so we have the number of jumps of the tangent knowing the value of θ in B:

$$\theta(B) = \lim_{t \to \infty} \frac{\operatorname{Im} p}{\operatorname{Re} p}(b).$$

However, let us note that the conditions of greatest interest are for damping ratio $\zeta = \cos\alpha$ big enough , that is α small.

Then we choose α to be so small that, on the arch C_R, $\operatorname{tg}\theta$ does not undergo to any jump from $+\infty$ to $-\infty$.

In such hypothesis, the condition for the solutions of a polynomial (4) to be inside the cone of Fig. 2 is:

(19) $$2\,\operatorname{Index}_\gamma(p) = V_{(b)}(\infty) - V_{(a)}(-\infty) = 2n\,.$$

To obtain the variations of sign of (19), let us note that the leading terms of the needed Sturm chains can be obtained by the «modified Routh scheme»

(20) $$\begin{cases} a_0 \quad a_1 \quad a_2 \; \cdots & \qquad a_1 \quad a_2 \; \cdots \\ b_0 \quad b_1 \quad b_2 \; \cdots & \qquad b_1 \quad b_2 \; \cdots \\ c_0 \quad c_1 \quad c_2 \; \cdots & \qquad c_1 \quad c_2 \; \cdots \\ d_0 \quad d_1 \quad d_2 \; \cdots & \qquad d_1 \quad d_2 \; \cdots \\ \text{with} & \\ c_1 = \dfrac{a_0 b_1 - b_0 a_1}{b_0}, \quad c_1 = \dfrac{a_0 b_2 - b_0 a_2}{b_0}, \text{ etc.} & \\ d_1 = \dfrac{b_0 c_1 - c_0 b_1}{c_0}, \quad d_2 = \dfrac{b_0 c_2 - c_0 b_2}{c_0}, \text{ etc.} & \end{cases}$$

and the first two rows are given according to the

Proposition 3: Consider the Routhian (20) with the first two rows given by, for even n:

$$a_i = (-1)^i p_i \cos(n-i)\alpha\,, \qquad b_i = (-1)^{i+1} p_i \sin(n-i)\alpha\,, \qquad i = 0, n\,,$$

for odd n:

$$a_i = (-1)^{i+1} p_i \cos(n-i)\alpha\,, \qquad b_i = (-1)^i p_i \sin(n-i)\alpha\,, \qquad i = 0, n\,,$$

then all the solutions of the polynomial

$$p_0 z^n + p_1 z^{n-1} + \ldots + p_n$$

are inside the cone of amplitude 2α of Fig. 2 if and only if $p_i > 0$ and in the first column of the Routhian we have $2n$ changes of sign.

Proof: *In the hypothesis* $\Gamma = 0$ *we have condition* (19), *then we have to evaluate* $\operatorname{Re} p$, $\operatorname{Im} p$ *along* (a) *and* (b).

For $(b) = \{s \in [0, \infty), \phi = \pi - \alpha\}$ *we have:*

$$\operatorname{Re} p(b) = p_0 \varrho^n \cos n(\pi - \alpha) + p_{n-1} \varrho^{n-1} \cos(n-1)\alpha + p_1 \varrho \cos(\pi - \alpha) + p_n\,,$$

$$\operatorname{Im} p(b) = p_0 \varrho^n \sin n(\pi - \alpha) + p_{n-1} \varrho^{n-1} \sin(n-1)\alpha + p_1 \varrho \sin(\pi - \alpha)\,.$$

Then $V_{(b)}(\infty) = [a_0 b_0 c_0 d_0, \ldots]$ *with* [*] *the first column of the Routhian as defined in the statement of the proposition.*

To have $V_{(a)}(-\infty)$ *we need to evaluate* $\operatorname{Re} p$, $\operatorname{Im} p$ *along* $(a) = \{s \in (-\infty, 0], \phi = \alpha\}$.

Then we can construct the Routhian according to (20) *with* $a_i = p_i \cos(n-i)\alpha$, $b_i = p_i \sin(n-i)\alpha$ $i = 0, n$.

Moreover we must take the variations at $-\infty$, *so the leading terms* $[a_0, b_0, c_0, etc.]$ *must be taken according to the following rules:*

$$(21) \qquad \begin{cases} [a_0, b_0, c_0, d_0, \ldots] \to [a_0, b_0, -c_0, -d_0, \ldots] \text{ if } n \text{ is even}\,, \\ a_0, b_0, c_0, d_0, \ldots \to [-a_0, -b_0, c_0, d_0, \ldots] \text{ if } n \text{ is odd}\,, \end{cases}$$

From the relations found, it follows $V_{(b)}(\infty) = 2n - V_{(a)}(-\infty)$.

Therefore $V_{(b)}(\infty) = 2n$ *and* $V_{(a)}(-\infty) = 0$.

Equality $V_{(b)}(\infty) = 2n$ *is just the statement of the Proposition.*

Moreover, equality $V_{(a)}(-\infty) = 0$ *implies:*

Proposition 4: The solutions λ_i are inside a cone of amplitude 2α if and only if $p_i > 0$, and taking the first column of the Routhian (20) with

i) $a_i = p_i \cos(n-i)\alpha$, $b_i = p_i \sin(n-i)\alpha$, $i = 0, n$

ii) signs taken according to (21),

we have all the entries with equal sign.

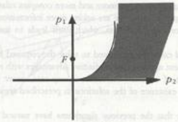

Figure 10.

EXAMPLE 3: $z^2 + p_1 z + p_2 = 0$.

In the hypothesis $\Gamma = 0$, the trigonometric functions in the routhian are all positive, then for the signs to be alternate, we just need

$$p_2 \sin 2\alpha - \frac{p_1^2}{2} \operatorname{tg} \alpha < 0 .$$

It follows that p_1, p_2 must be taken to the right of the branch of the parabola with focus $F = (0, 2\cos^2\alpha)$ of Fig. 10. ■

EXAMPLE 4: $z^3 + p_1 z^2 + p_2 z + p_3 = 0$.

According to Proposition 3, the following column must have alternate signs:

$$(22) \qquad \begin{cases} -\cos 3\alpha , \\ \sin 3\alpha , \\ -p_1 \dfrac{\sin \alpha}{\sin 3\alpha} , \\ d_0 = p_2 \sin 2\alpha \sin 3\alpha - p_1 \sin \alpha \sin 2\alpha , \\ e_0 = -p_1^2 p_2 \sin^3 \alpha + p_1 p_3 \sin \alpha \sin^2 3\alpha - p_2^2 \sin^2 2\alpha \sin 3\alpha + p_1 p_2 \sin \alpha \sin^2 2\alpha , \\ f_0 = d_0 p_3 - e_0 \left(\dfrac{p_1 p_2 \sin^2 \alpha - p_3 \sin^2 3\alpha}{e_0 \sin 3\alpha} \right), \\ -p_3 . \end{cases}$$

Therefore the required coefficients are bounded by the algebraic surfaces $e_0 = 0, f_0 = 0$ and the plane $d_0 = 0$.

In general, it is possible to set the algebraic surfaces in normal form, so that the inequalities become easier and the required region can be visualized, as we did in Example 2. ■

Of course, for increasing n we get more and more complex relations. On the other hand this had to be expected since we are asking more informations to the algorithm than we do with the ordinary Routhian, which itself leads to intricate relations for large n.

Therefore in several cases we are forced to solve the required inequalities numerically. Even so, the present approach has definitive advantages with respect to any other numerical manipolation of equation (1),which, in any case, never provides with *sufficient conditions* for the existence of the solutions in prescribed regions of the complex plane.

Finally, let us note that the previous algorithms have natural generalizations:

PROPOSITION 5: If the number of variations of sign in the algorithm of Proposition 2 is $2k$, then only $n-k$ zeroes of $p(z)$ are to the left of the straight line $x = x_1$.

PROPOSITION 6: If the number of variations of sign in the algorithm of Proposition 3 is $2k$, then only $n-k$ zeroes of $p(z)$ are inside the cone of amplitude 2α.

Both Propositions 5 and 6 can be proved in the same lines of Prop. 2,3 with the additional observation that if γ contains only k zeroes, then $1/2\pi \int_\gamma d\theta = k$.

The combined use of Prop. 2, Prop. 3 (or Prop. 4) and Prop. 5, 6 allows us to «box» the solutions within arbitrary regions of the plane bounded by sections of cones or strips.

4. - AIRCRAFT STABILITY

The equation of the motion of an aircraft, regarded as rigid body are in a body fixed reference frame:

$$\frac{dv}{dt} + \omega \wedge v = X_A + X_T + X_g ,$$

$$I\frac{d\omega}{dt} + \omega \wedge I\omega = L_A + L_T ,$$

where X_A, X_T, X_g are the aereodynamic, thrust and gravity forces, L_A, L_T are the momenta (no gravity gradient L_g appairs since the gravity field is supposed to be constant).

The first equation gives the motion of the baricenter, having velocity (u, v, w), while the second gives the motion about the baricenter, which can be represented—for

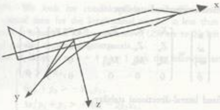

Figure 11.

small deviations from an equilibrium state—by the angles:

ϕ: *small angle around the x axis = roll angle*,

θ: *small angle around the y axis = pitch angle*,

ψ: *small angle around the z axis = yaw angle*,

see Fig. 11.

The xz plane is the plane of symmetry of the aircraft, and if we have initial conditions in the plane of symmetry:

$$\omega_1(0) = \omega_2(0) = \omega_3(0) = v(0) = \phi(0) = \psi(0) = 0$$

and take the «stability reference system» of Fig. 12, then we have the equilibrium state $(u_0, 0)$ (straight unperturbed flight) and the motion under small perturbation from such equilibrium is described by [7]:

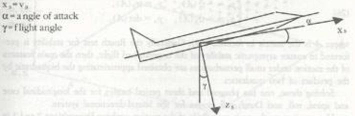

Figure 12.

(A) Stick-fixed longitudinal stability:

(23)
$$\begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{\omega}_2 \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} X_u & X_w & X_{\omega_2} & -g\cos\gamma_0 \\ Z_u & Z_w & u_0 & -g\sin\gamma_0 \\ M_u & M_w & M_{\omega_2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ \omega_2 \\ \theta \end{bmatrix}.$$

(B) Stick-fixed lateral-directional stability:

(24)
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -I_{xz}/I_x & 0 \\ 0 & -I_{xz}/I_z & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{\omega}_1 \\ \dot{\omega}_3 \\ \dot{\phi} \end{bmatrix} =$$
$$= \begin{bmatrix} Y_v & Y_{\omega_1} & Y_{\omega_3} - u_0 & g\cos\gamma_0 \\ L_v & L_{\omega_1} & L_{\omega_3} - u_0 & 0 \\ N_v & N_{\omega_1} & N_{\omega_3} - u_0 & 0 \\ 0 & 1 & \mathrm{tg}\,\gamma_0 & 0 \end{bmatrix} \begin{bmatrix} v \\ \omega_1 \\ \omega_3 \\ \phi \end{bmatrix}$$

where the components of the forces X are denoted by (X, Y, Z) and those of the momenta L by (L, M, N).

The physical meaning of the partial derivatives (stability derivatives) in equations (23), (24) is described in [8].

The characteristic equations for (23) and (24) are biquadratic.

For instance (23) gives:

(25)
$$\lambda^4 + p_1\lambda^3 + p_2\lambda^2 + p_3\lambda + p_4$$

with

(26)
$$\begin{cases} p_1 = -\mathrm{tr}(A), & p_2 = \mathrm{tr}_2(A), \\ p_3 = -\mathrm{tr}_3(A), & p_4 = \det(A), \end{cases}$$

where A is the matrix in formula (23). Generally the Routh test for stability is performed to ensure asymptotic stability of the unperturbed flight, then the main features of the motion under small perturbations are obtained approximating the biquadratic by the product of two quadratics.

Solving those, one has phugoid and short period modes for the longitudinal case and spiral, roll, and Dutch roll modes for the lateral-directional system.

Here we impose the main properties of the motion applying Propositions 2 and 3 in two examples.

EXAMPLE (5): We look for conditions on the coefficients of equation (25) such that the critical time for the longitudinal motion is less than 20 seconds.

This is equivalent to say that all the solutions of (25) are to the left of the stright line $x = -0.05$ and parallel to the imaginary axis.

Specialization of Prop. 2 to $n = 4$ $p_0 = 1$ give the following set of inequalities

$$(27)\quad \begin{cases} p_1 > -4x_1\,, \\ 6x_1^2 + p_2 > -3x_1 p_1\,, \\ 3x_1^2 p_1 + p_3 > -4x_1^3 - 2p_2 x_1\,, \\ p_1(p_2 + 15x_1^2) > -3p_1^2 x_1 + p_3 - 20x_1^3 - 2p_2 x_1\,, \\ d_1 > 0\,, \\ x_1^4 + p_2 x_1^2 + p_4 > -p_1 x_1^3 - p_3 x_1\,. \end{cases}$$

A way to approach these inequalities is to choose p_1 according to the first of (27), p_2 and p_3 according to the second-fourth, p_4 as in the last and test $d_1 > 0$.

In general, we can take a grid of points in the 4-dimensional parameter space and check the inequalities numerically. For instance it is possible to visualize families of surfaces parametrized by fixed p_1.

Figure 13 shows the allowed region for the parameters p_2 (varying from 6.5 to 7.5 with step 0.1), p_3 (varying from 0.6 to 1.5 with step 0.1), p_4 (from 0.05 to 0.1 with step 0.005) while p_1 is fixed at 2.55.

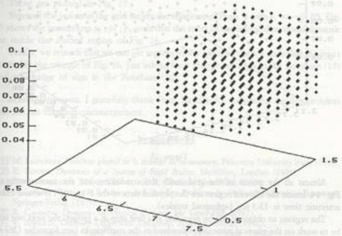

Figure 13.

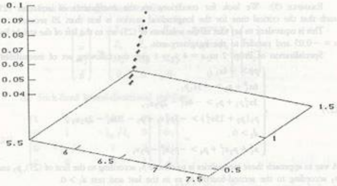

Figure 14.

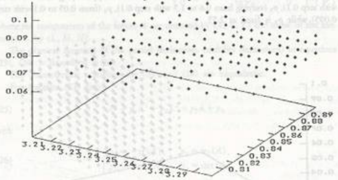

Figure 15.

Almost all the points of the grid verify the conditions, but the narrow strip of Fig. 14. Inside the allowed region are the physical data of the F-16 (AFTI) whose characteristic time is 13.3 sec («phugoid mode»).

The regions so obtained can be regarded as a first step in a proget: the next one will be to work on the physical parameters involved in the coefficients (see formulae (26) in such a way to get the required range of values for the p_i's.

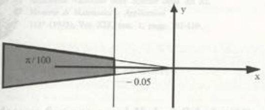

Figure 16.

Example (6): The second application concerns the lateral-directional dynamics: we find conditions such that the critical time is less than 20 sec (the critical time of the spiral mode) and the damping ratio is close to the critical value (almost no Dutch-Roll motion).

We have to superinpose to the previous inequalities the ones coming from Prop. 3 for $n = 4$, $p_0 = 1$.

Again a numerical analysis is helpful.

A grid of 1000 points is considered, where p_2 varies between 3.2 and 3.3, p_3 is in [0.8, 0.9] and p_4 in [0.06, 0.1] (p_1 is fixed at 4); 158 of the grid points satisfy the inequalities of Prop. 3 with $\alpha = \pi / 100$.

These are plotted in Fig. 15.

Since all the points in the grid satisfy the relations of Prop. 2 with $x_1 = -0.05$, Fig. 15 shows the parameters p_2, p_3, p_4 such that the roots of the lateral-directional dynamic are inside the shaded region of Fig. 16.

Finally, we remark that we can get regions in the space of the p_i's such that the roots are inside the triangle of Fig. 16, just asking for n changes of signs in the Routhian (15) and no change of sign in the Routhian (20).

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REFERENCES

[1] M. Liapunov: *Problème général de la stabilité du mouvement*, Princeton University Press (1949).
[2] E. Routh: *Dynamics of a System of Rigid Bodies*, McMillan, London (1905).
[3] F. Gantmacher: *Matrix Theory*, vol. 2, Chelsea, New York (1959).
[4] L. Cesari: *Asymptotic Behaviour and Stability Problems in Ordinary Differential Equations*, Springer-Verlag (1959).
[5] R. Cannon: *Dynamics of Physical Systems*, Mc Graw-Hill (1967).
[6] V. Arnold: *Ordinary Differential Equations*, Springer, Berlin (1992).
[7] L. Mangiacasale: *Flight Mechanics*, Levrotto e Bella, Torino (1992).
[8] B. Etkin: *Dynamics of Flight*, Wiley, New York (1965).