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Local Regularity for Solutions to Schrödinger Equations Relative to Dirichlet Forms (**)

SUMMARY. — The Note concerns the study of Schrödinger equations relative to Dirichlet forms. The potential here belongs to the Kato class of measures. An Harnack inequality for non-negative local solutions is given, as well as an oscillation-energy estimate. From this last estimate it is possible to see that for general potentials belonging to the Kato class solutions are locally continuous but in general not locally hölderian.

Regolarità locale per soluzioni di equazioni di Schrödinger connesse con forme di Dirichlet

SUNTO. — Nella presente Nota si studiano equazioni di Schrödinger connesse con forme di Dirichlet. Il potenziale considerato appartiene alla classe delle misure di Kato. Si presentano: una diseguaglianza di Harnack per soluzioni locali non negative ed una stima locale dell'andamento di oscillazione-energia. In base a questa ultima stima si può vedere che per generici potenziali appartenenti alla classe di Kato le soluzioni sono localmente continue ma in genere non localmente hölderiane.

0. - INTRODUCTION AND RESULTS

Schrödinger operators have been studied by Chiarenza, Fabes and Garofalo [9], who obtained an Harnack inequality for a non negative solution of $Au = Vu$, where A is an elliptic second operator in divergence form and V belongs to the Kato class of potentials. This result extends a previous one proved by probabilistic methods by Ainzen-

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man and Simon [1] for $\Delta u = Vu$. In a recent paper Citti, Garofalo and Lanconelli [10] obtained an Harnack inequality in the case of the sum of squares of vectorial fields plus a potential belonging to the local Kato class K_n^{loc} (introduced by Kato in [16]), with $n \geq 3$; this result has been extended by Lu [18] to the weighted case. We now extend these results to Schrödinger operators associated to Dirichlet forms.

We consider a locally compact connected Hausdorff space: X, and a positive Radon measure m on X, with $\operatorname{supp} m = X$. We recall that a *Dirichlet form* on $H = L^2(X, m)$ is a closed, non-negative definite, symmetric bilinear form $a(u, v)$ defined on a dense linear subspace $D[a]$ of H, with the following *Markovian property*: if $u \in D[a]$, $v := 0 \vee u \wedge 1$, then $v \in D[a]$ and $a(v, v) \leq a(u, u)$.

We shall consider only Dirichlet forms of diffusion type, that is forms with the following *strong local property*: $a(u, v) = 0$ for every $u, v \in D[a]$ with v constant on a neighbourhood of $\operatorname{supp} u$. We shall also assume that the form a is regular in H, that is, there exists a subset C of $D[a] \cap C_0(X)$ which is both dense in $C_0(X)$ with the uniform norm and dense in $D[a]$ for the intrinsec norm $(a(u, u) + (u, u))^{1/2}$ ($C_0(X)$ denotes the space of continuous functions with compact support in X). C is called a *core* of a in H. For any regular Dirichlet form of diffusion type the following integral expression can be given:

$$a(u, v) = \int_X \mu(u, v)(dx) \tag{0.1}$$

for every $u, v \in D[a]$, where μ is a Radon-measure-valued nonnegative-definite bilinear form on $D[a]$, uniquely associated with a, called the *energy measure* of a.

By $D_0[a, A]$ we denote the closure of $D[a] \cap C_0(A)$ in $D[a]$ for the intrinsec norm $(a(u, u) + \|u\|^2_{L^2(X,m)})^{1/2}$; by $D_{loc}[a, A]$ the space of functions u that belong locally to the domain of the form a on a given open subset A of X, and simply by $D_{loc}[a]$ if $A = X$.

Regular Dirichlet forms of diffusion type satisfy some important properties that will be used in the following, such as the *Schwartz rule*, the *Leibnitz rule*, the *chain rule* and the *truncation lemma* ([14], [5]).

We suppose that our Dirichlet form a admits a *m-separating core*, that is, a core C such that:

(0.2) for every $x, y \in X$, $x \neq y$, $\exists \phi \in C$ with $\mu_a(\phi, \phi) \leq m$ on X, such that $\phi(x) \neq \phi(y)$.

Given our form a, we define the *distance function* $d = d_a : X \to [0, +\infty]$ by:

$$d(x, y) = \sup\{\phi(x) - \phi(y) : \phi \in C,\ \mu(\phi, \phi) \leq m \text{ on } X\}. \tag{0.3}$$

It can be verified that d is a distance; the balls given by the distance d associated with a are:

(0.4) $$B(x, r) = \{y \in X,\ d(x, y) < r\}, \qquad r > 0.$$

The notion of distance (0.3) allows us to formulate the following assumption:

Assumption I: The following two properties hold:

(i) The metric topology induced by the distance (0.3) on X is equivalent to the initial topology on X.

(ii) The measure m is doubling with respect to the balls (0.4), that is, there exists a constant $c_0 > 0$ such that:

$$0 < m(B(x, 2r)) \leq c_0 m(B(x, r)) < +\infty \qquad \text{for every } x \in X \text{ and } 0 \leq r \leq r_0.$$

Under this assumption the space X with the distance d acquires the structure of a *homogeneous space*, according to [11]. The next *Assumption II* establishes a good functional behaviour of the form:

Assumption II: Given a relatively compact open subset X_0 of X, there exists a constant $c_1 > 0$ and an integer $k \geq 1$, such that for every $x \in X_0$ and every $r > 0$ with $B(x, r) \subset X_0$ the following inequalities hold:

(j) $$\int_{B(x, r/k)} |u - \bar{u}|^2 m(dx) \leq c_1 r^2 \int_{B(x, r)} \mu(u, u)(dx)$$

for every $u \in D_{loc}[X_0]$, where

$$\bar{u} = \frac{1}{m(B(x, r/k))} \int_{B(x, r/k)} u m(dx).$$

As proved in [4], (see also [8] for connected results), the following Sobolev inequality

(jj) $$\left(\frac{1}{m(B(x, r))} \int_{B(x, r)} |u|^s m(dx)\right)^{1/s} \leq c_2 r \left(\frac{1}{m(B(x, r))} \int_{B(x, r)} \mu(u, u)(dx)\right)^{1/2}$$

with $\nu = \log_2 c_0$, $s = 2\nu/(\nu - 2)$ if $\nu > 2$, $s \in (2, +\infty)$ if $\nu \leq 2$, $u \in D_{loc}[X_0]$, $\operatorname{supp} u \subset B(x, r)$, is a consequence of (j) if $B(x, 2r) \neq X$. From (jj) it follows that:

$$\int_{B(x, r)} |u|^2 m(dx) \leq c_2 r^2 \int_{B(x, r)} \mu(u, u)(dx)$$

$\forall u \in D_{loc}[X_0]$ with $\operatorname{supp} u \subset B(x, r)$, $x \in X_0$; this implies that our bilinear form a is coercive on $D_0[a, B(x, r)]$ for the intrinsec norm.

We now suppose we are given a family of regular Dirichlet forms of diffusion type defined on a common domain $D \subset L^2(X, m)$, which are mutually *equivalent*, that is:

whenever we fix a form b in the family, there exist two constants $0 \leq \lambda \leq \Lambda$, depending on b, such that any other form a of the family is related to b by the condition:

$$\lambda b(u,u) \leq a(u,u) \leq \Lambda b(u,u) \tag{0.5}$$

for every $u \in D = D[a] = D[b]$. We remark that by a well known *domination principle* ([14]) this condition is equivalent to:

$$\lambda \mu_b(u,u) \leq \mu_a(u,u) \leq \Lambda \mu_b(u,u). \tag{0.6}$$

We suppose that there is a form in the family that admits a common separating core; clearly, if a set C is a core for a form in the family, it is also a core for any other form in the family.

It is easily checked that if Assumption I and II are satisfied by a given form of the family (0.5) with some constants r_0, c_0, c_1, k then they are also satisfied by any other form of the family, with possibly new constants r_0', c_0', c_1', k' depending on the initial r_0, c_0, c_1, k and on the ratio Λ / λ. This observation allows us to check Assumption I and II for an *arbitrary* form in the family (0.5).

If we consider a Dirichlet form of diffusion type a with energy measure μ and separating core C, that satisfies Assumption I, it is possible to prove the existence of *cut-off functions* between balls of different radius:

Proposition 0.1 ([5]): *Given $B(x, qr) \subset B(x, 2r) \subset\subset X$, $0 < r < r_0$, $q \in (0,1)$, there exists $\phi \in C$ with the properties: $\phi = 1$ on $B(x, qr)$, $\phi = 0$ on $X - B(x,r)$, $0 \leq \phi \leq 1$ and $\mu(\phi, \phi) \leq (10/((1-q)^2 r^2))\, m$ on X.* ■

Let X_0 be (here and in the following) an open compact set in X. We recall the definition of *Kato class* in X_0: $K(X_0)$ is the set of every Radon measure ν on X_0 such that:

$$\lim_{r \to 0} \sup_{x \in \bar{X}_0} \int_{X_0 \cap B(x,r)} \int_{d(x,y)}^{R} \frac{s^2}{m(B(x,s))} \frac{ds}{s} |\nu|(dy) = 0$$

where $R = 2\,\mathrm{diam}(X_0)$; $K_{loc}(X_0)$ = set of all Radon measures ν on X_0 such that $\nu \in K(X_0')$ for every open set $X_0' \subset\subset X_0$.

From the previous definitions we obtain ([6]) that if $\nu \in K(X_0)$ there exists an increasing function $\eta(r)$, $0 < r$, with $\lim_{r \to 0} \eta(r) = 0$ such that:

$$\sup_{x \in \bar{X}_0} \int_{X_0 \cap B(x,r)} \int_{d(x,y)}^{R} \frac{s^2}{m(B(x,s))} \frac{ds}{s} |\nu|(dy) \leq \eta(r).$$

Given $\nu \in K(X_0)$, we define

$$\|\nu\|_{K(X_0)} = \sup_{x \in \bar{X}_0} \int_{X_0} \int_{d(x,y)}^{R} \frac{s^2}{m(B(x,s))} \frac{ds}{s} |\nu|(dy).$$

It is easy to verify that $\|\ \|_{K(X_0)}$ is a norm on $K(X_0)$ and that

$$|v|(X_0) \leq cR^{-2}m(B_R)\|v\|_{K(X_0)}$$

with c structural constant, where B_R is an intrinsic ball of radius R and center at $x_0 \in X_0$, $R = 2 \operatorname{diam} X_0$ (see [6]).

Other results used in the following are: $K(X_0) \subset D'[a]$, where $D'[a]$ denotes the dual space of $D[a] \subset L^2(X_0, m)$; the space $C(\bar{X}_0)$ is dense in $K(X_0)$ (again, see [6]); the following embedding inequality is an extension to our case of analogous inequalities proved in [9], [10], [18]:

Proposition 0.2 ([3]): *Let $V \in K(X_0)$, $u \in D_{loc}(B_t)$, $\varepsilon > 0$ and $B_s \subseteq B_t \subsetneq B_{4t_0} \subseteq X_0$, where B_s, B_t, B_{4t_0} are concentric balls with fixed center x_0 and radius s, t, $4t_0$, and where $\eta(t_0) \leq \varepsilon^{1/2}/14$. Then:*

$$\int_{B_s} u^2|V|(dx) \leq \varepsilon \int_{B_t} \mu(u, u)(dx) + \frac{C_\varepsilon}{(t-s)^2} \int_{B_t - B_s} u^2 m(dx). \quad \blacksquare$$

Given a form a such that Assumptions I and II hold, we consider a solution u of

$$\int_{X_0} \mu(u, v)(dx) = -\int_{X_0} uvV(dx) \tag{0.7}$$

with $u \in D_{loc}[X_0]$, $V \in K_{loc}(X_0)$ and $\forall v \in D_0[X_0]$.

In Sections 1 and 2 we prove an Harnack inequality for such an u:

Theorem 2.1: *If u is a positive solution of (0.7), R_0 small enough and such that $B(x, MR_0) \subset X_0$, with $M = \max\{4k + 13, 33\}$ we have:*

$$\sup_{B(x,r)} u \leq c \inf_{B(x,r)} u$$

with $\bar{k}r \leq R_0$ for $\bar{k} \geq 1$, $\bar{k}$ depending on k, c depending on c_0, c_1 of Assumption I and II.

The proof is divided in two parts:

(1) a local L^∞ estimate of the solution;

(2) the proof of the Harnack inequality.

For the first part an essential tool is Proposition 0.2.

To prove (2) we consider the solution u as the sum of two functions obtained by solving separately two different problems similar to (1.7): one with the term Vu and zero boundary data and the other with $V = 0$ and boundary data u. The classical Harnack inequality ([5]) is then applied to the second problem and by combining this with the local L^∞ extimates the result follows.

In Section 3 we prove the following oscillation-energy estimate:

THEOREM 3.1: *Let u be a local solution in X_0 and $B(x_0, 16R_0) \subset X_0$. Let*

$$\psi(r) = \int_{B(x_0, r)} G^{x_0}_{B(x_0, 2q^{-1}r)} \mu(u, u)(dx) + \operatorname{osc}_{B(x_0, r)} u^2$$

where $G^{x_0}_{B(x_0, 2q^{-1}r)}$ is the Green function for the form a relative to the ball $B(x_0, 2q^{-1}r)$ and with singularity in x_0. We then have:

$$\psi(r) \leq c\left[\left(\frac{r}{R}\right)^{\beta} \psi(R) + \eta^*(R)\right]$$

for every $r \leq qR \leq q^2 R_0$ and $q \in (0, q_0]$, $q_0 = \min\{1/6, K^{-1}\}$, where $\beta \in (0, 1)$ and c are structural constants depending also on q, and where

$$\eta^*(r) = \overline{C^*} \cdot \frac{\|u\|_{L^2(B(x_0, R_0))}}{m(B(x_0, R_0))} \|V\|_{K(B(x_0, 16r))},$$

$\overline{C^*}$ structural constant depending again also on q.

We will see that a consequence of Theorem 3.1 is the following:

COROLLARY 3.1: *Let u be as in Theorem 3.1, then u is locally continuous in X_0; moreover, if V is such that $\|V\|_{K(B_R)} \to 0$ as a power of R, as $R \to 0$, we obtain the local Hölder continuity of u.*

Our results can be applied to classes of operators of the form $Au - Vu$, where A is a partial differential operator relative to a Dirichlet form. As examples we consider:

a) A uniformly elliptic operator in divergence form with measurable coefficients:

$$A = \sum_{i,j=1}^{n} D_{x_i}(\alpha^{ij}(x) D_{x_j})$$

where $\alpha = (\alpha^{ij})$ is a symmetric matrix of measurable functions on R^n such that:

$$\lambda|\xi|^2 \leq \alpha(x)\xi \cdot \xi \leq \Lambda|\xi|^2 \quad \text{a.e. in } R^n$$

for every $\xi \in R^n$, for some given constants $0 < \lambda \leq \Lambda$. We consider the form:

$$a(u, v) = \int_{R^n} \sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i}(x) \frac{\partial v}{\partial x_j}(x) \alpha^{i,j}(x)\, dx, \qquad u, v \in C_0^1(R^n).$$

The form relative to A is obtained by the closure of a in $L^2(R^n)$. Here $X = R^n$, m is the usual Lebesgue measure and the distance is the usual euclidean one. Theorem 2.1 was obtained for this class of operators in [9].

b) A is an operator of uniform Hormander type

$$A = \sum_{i,j=1}^{m} X_i^* (a^{ij}(x) X_j), \quad x \in R^n$$

with X_i, $i = 1, \ldots, m$, m smooth vector fields in R^n satisfying Hormander condition and $\alpha = (\alpha^{ij})$ symmetric $m \times m$ matrix of measurable functions on R^n such that:

$$\lambda |\xi|^2 \leq \alpha(x) \xi \cdot \xi \leq \Lambda |\xi|^2,$$

for every $\xi \in R^n$, a.e. on R^n, for some given constants $0 < \lambda \leq \Lambda$. We consider the form:

$$a(u, v) = \int_{R^n} \sum_{i,j=1}^{m} \alpha^{ij}(x) X_i u X_j v \, dx, \quad u, v \in C_0^\infty(R^n).$$

The domain of the form is obtained by the closure of a in $L^2(R^n)$. Here $m(dx) = dx$ and the distance induced on R^n by the operator according to our definition is equal to the one defined in [19], hölderian with respect to the euclidean one. For such operators, if (α^{ij}) is the identity, the Harnack inequality was first found in [10].

c) A is as in case *b*) except that $\alpha = (\alpha^{ij})$ is a symmetric $m \times m$ matrix of measurable functions on R^n such that:

$$\lambda w(x) |\xi|^2 \leq \alpha(x) \xi \cdot \xi \leq \Lambda w(x) |\xi|^2$$

with $w(x) \in A^2(R^n, d)$, where d is as in case *b*) and $w \in A^2$ if

$$\sup_{B \subset R^n} \frac{1}{|B|} \int_B w(x) \, dx \cdot \frac{1}{|B|} \int_B w^{-1}(x) \, dx \leq c_w$$

with c_w independent of the metric balls $B \subset R^n$.

We now consider the form:

$$a(u, v) = \int_{R^n} \sum_{i,j=1}^{m} \alpha^{ij}(x) X_i u X_j v \, dx, \quad u, v \in C_0^\infty(R^n).$$

The domain of the form is obtained by the closure of a in $L^2(R^n, w\,dx)$. In this case, the measure is $m(dx) = w(x)\,dx$ and the metric is again as in *b*). The duplication property holds because w is A_2 with respect to the spheres of the distance d, the scaled Poincaré and Sobolev-Poincaré inequalities were proved in [17] and the Harnack inequality for $Au - Vu$ was proved in [18].

d) We consider now a selfadjoint subelliptic form b on $C_0^1(R^n)$:

$$b(u, v) = \int_{R^n} \sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i}(x) \frac{\partial v}{\partial x_j}(x) \beta^{ij}(x) \, dx$$

where $\beta = (\beta^{ij})$, $i, j = 1, \ldots, n$ is a symmetric matrix of smooth functions on R^n such that:

$$0 \leq \beta(x)\,\xi\cdot\xi \quad \text{in } R^n,\ \forall \xi \in R^n$$

and where b satisfies:

$$c\|u\|^2_{H^\varepsilon} \leq b(u, u) + \|u\|^2_{L^2} \qquad \forall u \in C_0^1(R^n) \text{ and for some } \varepsilon \in (0, 1).$$

We take now the family of the operators and relative forms whose expression is as in case *a*), except that now $\alpha = (\alpha^{ij})$ is a symmetric matrix of mesurable function on R^n satisfying the uniform subelliptic condition:

$$\lambda\beta(x)\,\xi\cdot\xi \leq \alpha(x)\,\xi\cdot\xi \leq \Lambda\beta(x)\,\xi\cdot\xi$$

a.e. in R^n, $\forall \xi \in R^n$.

The measure is in this case the Lebesque measure, the distance related to b by our definition is equal to the one considered in [12] and in [13] and the scaled Poincaré inequalities can be easily obtained from [15]. Our results seem to be new in this uniform subelliptic setting.

1. - L^∞ ESTIMATES

In this section we prove a L^∞-estimate for the solution of the problem:

$$a(u, v) + \int_{B_R} uvV(dx) = \int_{B_R} fvm(dx) \tag{1.1}$$

with $B_R = B(x_0, R)$ intrinsec ball of radius R contained in X_0, and $\forall u, v \in D_0[B_R]$, $V \in K(B_R)$; $f \in L^p(B_R)$.

We recall that the existence of the solution, for R small enough, comes from Proposition 0.2.

We can now prove:

THEOREM 1.1: *Let u be a solution of* (1.1) *with $f \in L^p(B_R, m)$, $p > (1/2)\nu\nu$, $R \leq R_0$, $R_0 > 0$ suitable, $B_{20R_0} \subset X_0$. Then:*

$$\sup_{B_R} |u| \leq cR^2 m(B_R)^{-1/p}\|f\|_{L^p(B_R, m)}$$

with c structural constant and R small enough.

PROOF: Let $G_\varrho^{x,b}$ be the regularized Green function for the operator $a(u, v)$, relative to a ball $B_R = B(x_0, R)$ and to a ball $B(x, \varrho)$ contained in B_R. We can approximate V in Ω with a sequence $\{V_b\} \in C(\Omega)$ converging to V in $D'[a, \Omega]$, such that for every $\varepsilon > 0$ there exist $r_\varepsilon > 0$ independent on b and $b_\varepsilon > 0$ for which, taking $b > b_\varepsilon$ and $r < r_\varepsilon$

it results:

$$\sup_{x \in \bar\Omega} \int_{\Omega \cap B(x,s)} \int_{d(x,y)}^{\bar R} \frac{s^2}{m(B(x,s))} \frac{ds}{s} V_h m(dy) \leq \varepsilon$$

with $\bar R = 2\,\mathrm{diam}\,(\Omega)$ (see [6]). Given u_h solution of

$$a(u_h, v) + \int_{B_R} u_h v V_h m(dx) = \int_{B_R} fv m(dx)$$

the existence at least for R small enough of u_h is a consequence of Proposition 0.2; it results:

$$\fint_{B(x,\varrho)} u_h(y) = \int_{B_R} G_\varrho^{a,x} fm(dx) - \int_{B_R} G_\varrho^{a,x} u_h V_h m(dx).$$

From the $L^{p'}$ estimate of $G_\varrho^{x,a}$ given in [5], Lemma 4.2, we obtain:

$$\fint_{B(x,\varrho)} u_h(y) \leq \frac{cR^2}{m(B_R)^{1/p}} \|f\|_{L^p} + \int_{B_R} G_\varrho^{a,x} u_h V_h m(dx).$$

Taking the limit as $\varrho \to 0$, $\varepsilon = \varepsilon_0$ suitable, $h > h_{\varepsilon_0}$, $R < r_{\varepsilon_0}$ and considering the estimate of $G^{a,x}$ given in [5] Theorem 1.3, we obtain:

$$\|u_h\|_{L^\infty(B_R)} \leq \frac{cR^2}{m(B_R)^{1/p}} \|f\|_{L^p} + 1/2 \|u_h\|_{L^\infty(B_R)}$$

we get the result by taking the limit for $h \to +\infty$; after extraction of subsequences u_h weakly converges to u in $D[a, B_R]$ and V_h strongly converges to V in $D'[a, B_R]$; $u_h v$ weakly converges to uv in $D'[a, B_R]$, in fact: we know that $u_h v$ weakly converges to some w in $D'[a, B_R]$; as u_h weakly converges to u in $D[a, B_R]$, we can find a finite linear combination of vectors $\overline{u_n} = \sum_{i=n}^{n+m} \alpha_i u_i$ with $\sum_{i=n}^{n+m} \alpha_i = 1$, $\alpha_i \geq 0$, $m \geq 0$, such that $\overline{u_n}$ strongly converges to u in $D[a, B_R]$ (see Mazur theorem). So $\overline{u_n}$ punctually a.e. converges to u in $D[a, B_R]$. We have $\overline{u_n}$ weakly converges to some w in $D[a, B_R]$, in fact $\forall \varepsilon > 0$, for $n > n_\varepsilon$:

$$\left| \sum_{i=n}^{n+m} \langle \alpha_i u_i v - w, \psi \rangle \right| =$$

$$= \left| \sum_{i=n}^{n+m} \alpha_i \langle u_i v - w, \psi \rangle \right| \leq \sum_{i=n}^{n+m} \alpha_i |\langle u_i v - w, \psi \rangle| \leq \sum_{i=n}^{n+m} \alpha_i \varepsilon \leq \varepsilon.$$

As $\overline{u_n}$ is bounded in $L^\infty(B_R) \cap H^1(B_R)$, $\overline{u_n} \to u$ in $D[a, B_R]$, $v \in L^\infty(B_R) \cap H^1(B_R)$, taking into account that $|\alpha(v, v)|$ for $v \in D[a, B_R]$ does not charge sets of 0-capacity, $\overline{u_n} v$ strongly converges to u in $D[a, B_R]$; so $\overline{u_n} v$ punctually a.e. converges to uv. We have then that $\overline{u_n} v$ weakly converges to w in $D[a, B_R]$ and punctually a.e. converges to

uv and, as the two limits must be equal, we have $w = uv$ as required. So, as V_h strongly converges to V in $D'[a, B_R]$ and $u_n v$ weakly converges to uv in $D'[a, B_R]$, and as V, being a Kato measure, belongs to $D'[a, B_R]$, we obtain:

$$\int_{B_R} V_h u_h v m(dx) \to (V, uv)_{D', D} = \int_{B_R} uv V(dx).$$

We have the same result for a general $v \in D[a, B_R]$ as $uv_n \in L^2(\Omega, V)$ for $v_n \in \in D[a, B_R] \cap C(B_R)$, and strongly converges to uv in $L^2(\Omega, V)$. ■

We define now the regularized Green function for the complete form $a + V$, relative to a ball $B(x, R) \subset X_0$, $R \leq R_0$ and to a ball $B(y, \varrho) \subset\subset B(x, R)$, both intrinsec balls for a, as the solution of the problem:

$$a(G^y_{\varrho, B(x,R)}, v) + \int_{B(x,R)} G^y_{\varrho, B(x,R)} v V(dx) = \fint_{B(y,\varrho)} v m(dx),$$

$$\forall v \in D_0[a, B(x, R)], \qquad G^y_{\varrho, B(x,R)} \in D_0[a, B(x, R)],$$

again the existence of the solution is a consequence of Proposition 0.2.

LEMMA 1.1: *We have* $G^y_{\varrho, B(x,R)} \in L^\infty(B(x, R), m)$ *and, for* $R \leq R_0$, R_0 *small enough*:

$$\left(\frac{1}{m(B(x,R))} \int_{B(x,R)} (G^y_{\varrho, B(x,R)})^{p'} m(dx)\right)^{1/p'} \leq c \frac{R^2}{m(B(x,R))} \tag{1.2}$$

$\forall y \in B(x, R - \varrho)$, *where* c *is a structural constant. Then*:

$$\int_{B(x,R)} G^y_{\varrho, B(x,R)} m(dx) \leq cR^2 \tag{1.3}$$

where $p' = p/(p-1)$, p *as in Theorem* 1.1.

PROOF: From Theorem 1.1 it follows, for R small enough:

$$a(G^y_{\varrho, B(x,R)}, u) + \int_{B(x,R)} G^y_{\varrho, B(x,R)} u V(dx) = \fint_{B(y,\varrho)} u m(dx) =$$

$$= a(u, G^y_{\varrho, B(x,R)}) + \int_{B(x,R)} u G^y_{\varrho, B(x,R)} V(dx) =$$

$$= \int_{B(x,R)} f G^y_{\varrho, B(x,R)} m(dx) \leq cR^2 \|f\|_{L^p(B(x,R), m)} m(B(x, R))^{-1/p}.$$

Then:

$$\left(\int_{B(x,R)} (G^{p}_{Q,B(x,R)})^{p'} m(dx) \right)^{1/p'} \leq cR^2 m(B(x,R))^{-1/p} .$$

By Schwartz inequality we obtain (1.2) and (1.3). ■

2. - Proof of Harnack inequality

We first prove a *Caccioppoli type* inequality:

Proposition 2.1: *Let u be a local solution in X_0 of the problem:*

$$a(u,v) = -\int uvV(dx) \tag{2.1}$$

with $u \in D_{loc}[X_0]$, $V \in K_{loc}(X_0)$, $\forall v \in D_0[X_0]$.

Let $B_s \subset B_t \subset B_{2t} \subset\subset X_0$ be concentric balls, $0 < s < t \leq 1$. Then there exists a constant $b \geq 1$ such that:

$$\int_{B_s} \mu(u,u)(dx) \leq \frac{b}{(t-s)^2} \int_{B_t - B_s} u^2 m(dx) .$$

Proof: We choose as test function $u_k \phi^2$, where ϕ is the cut-off function of B_s relative to B_t; u_k is the truncation of u to $\pm k$. The result follows as in [5], taking into account Proposition 0.2. ■

We recall now a Real Analysis lemma (for the proof, see a.e. [5], Lemma 5.2).

Lemma 2.1: *Let $u \in L^{\infty}(B(x,t), m)$ and assume there exists positive constants C, L such that, for every s, t with $1/2 \leq s < t \leq 1$ we have:*

$$\sup_{B(x,sr)} |u| \leq \frac{C}{(t-s)^L} \left(\fint_{B(x,tr)} |u|^{2d} m(dx) \right)^{1/2d}, \quad d > 0 . \tag{2.2}$$

Then for every $p > 0$ there exists a constant c_p, which depends on p, C, L, d and on the constant c_0 in Assumption I (ii) *but does not depend on u, such that:*

$$\sup_{B(x,r/2)} |u| \leq c_p \left(\fint_{B(x,r)} |u|^p m(dx) \right)^{1/p} . \quad ■$$

By the some method of Lemma 2.1 it is possible to obtain (for the proof see again a.e. [5], Proposition 5.3):

Proposition 2.2: *Let u be a local solution of* (2.1). *Then for every $r > 0$, with $B(x, 2r) \subset\subset X_0$, we have:*

$$\left(\fint_{B(x,r/2)} |u|^2 m(dx) \right)^{1/2} \leq c \fint_{B(x,r)} |u| m(dx)$$

where c is a structural constant. ∎

We now prove a boundedness result for a local solution of (2.1):

Proposition 2.3: *Let u be a local solution of* (2.1) *in $B(x, 4r) \subset\subset X_0$; then for every $p > 0$ we have:*

$$\sup_{B(x,r/2)} |u| \leq c_p \left(\fint_{B(x,r)} |u|^p m(dx) \right)^{1/p}$$

where c_p is a structural constant depending on p.

Proof: By Lemma 2.1 it is enough to prove that for a suitable $\beta > 0$ and for every s, t with $1/2 \leq s < t \leq 1$ we have:

$$\sup_{B(x,r/2)} |u| \leq \frac{c}{(t-s)^\beta} \left(\fint_{B(x,sr)} |u|^2 m(dx) \right)^{1/2}.$$

Let ϕ be the cut-off function of $B(x, (s+\varepsilon)r)$ with respect to $B(x, (t-\varepsilon)r)$ where $\varepsilon = (t-s)/4$. It results then by Proposition 1.1:

$$\mu(\phi, \phi) \leq \frac{10}{(t-s)^2 r^2} m.$$

We consider the regularised Green function $G_\varrho^y = G_{\varrho, B(x,4r)}^y$, $y \in B(x, sr)$ for $A + V$, that is the solution of:

$$a(G_\varrho^y, v) + \int_{B(x,4r)} G_\varrho^y v V(dx) = \fint_{B(y,\varrho)} vm(dx).$$

We take $v = u_k \phi$, where u_k is the truncate of u. We obtain then:

$$\fint_{B(y,\varrho)} u_k \phi m(dx) = \int_{B(x,4r)} (\phi \mu(u_k, G_\varrho^y)(dx) + u_k \mu(\phi, G_\varrho^y)(dx)) + \int_{B(x,4r)} G_\varrho^y u_k \phi V(dx) =$$

$$= \int_{B(x,4r)} (\mu(u_k, \phi G_\varrho^y)(dx) - G_\varrho^y \mu(u_k, \phi)(dx) + u_k \mu(\phi, G_\varrho^y)(dx)) +$$

$$+\int_{B(x,4r)} G_\varrho^y u_k \phi V(dx) \leq \frac{c}{(t-s)r}\left(\int_{B(x,(t-\varepsilon)r)-B(x,(s+\varepsilon)r)} (G_\varrho^y)^2\right)^{1/2}\cdot$$

$$\cdot\left(\int_{B(x,(t-\varepsilon)r)-B(x,(s+\varepsilon)r)} \mu(u_k, u_k)\right)^{1/2} + \frac{c}{(t-s)r}\left(\int_{B(x,tr)} |u_k|^2 m(dx)\right)^{1/2}\cdot$$

$$\cdot\left(\int_{B(x,(t-\varepsilon)r)-B(x,(s+\varepsilon)r)} \mu(G_\varrho^y, G_\varrho^y)(dx)\right)^{1/2}.$$

Passing to the limit as $k \to +\infty$ we obtain:

$$\fint_{B(y,\varrho)} u\phi m(dx) \leq \frac{c}{(t-s)r}\left(\int_{B(x,(t-\varepsilon)r)-B(x,(s+\varepsilon)r)} (G_\varrho^y)^2 m(dx)\right)^{1/2}\cdot$$

$$\cdot\left(\int_{B(x,(t-\varepsilon)r)-B(x,(s+\varepsilon)r)} \mu(u,u)(dx)\right)^{1/2} + \frac{c}{(t-s)r}\left(\int_{B(x,tr)} |u|^2 m(dx)\right)^{1/2}\cdot$$

$$\cdot\left(\int_{B(x,(t-\varepsilon)r)-B(x,(s+\varepsilon)r)} \mu(G_\varrho^y, G_\varrho^y)(dx)\right)^{1/2}.$$

If u is a local solution, the same relation holds for $-u$, then in general we can write:

$$\left|\fint_{B(y,\varrho)} u\phi m(dx)\right| \leq \frac{c}{(t-s)r}\left(\int_{B(x,(t-\varepsilon)r)-B(x,(s+\varepsilon)r)} (G_\varrho^y)^2 m(dx)\right)^{1/2}\cdot$$

$$\cdot\left(\int_{B(x,(t-\varepsilon)r)-B(x,(s+\varepsilon)r)} \mu(u,u)(dx)\right)^{1/2} + \frac{c}{(t-s)r}\left(\int_{B(x,tr)} |u|^2 m(dx)\right)^{1/2}\cdot$$

$$\cdot\left(\int_{B(x,(t-\varepsilon)r)-B(x,(s+\varepsilon)r)} \mu(G_\varrho^y, G_\varrho^y)(dx)\right)^{1/2}.$$

From the above inequality the result follows as in [5], using also the estimate in Lemma 1.1. ■

We have:

PROPOSITION 2.4: *If u satisfies (2.1), $u \in D_{loc}[X_0] \cap L^\infty(X_0, m)$, $u \geq 0$ a.e. in X_0, and if $B(x, 4kr) \subset X_0$, with r small enough, then there exists a constant c, depending only on*

c_0, c_1 *and* k, *such that, for every* $\varepsilon > 0$:

$$\fint_{B_r} \left| \log(u+\varepsilon) - \frac{1}{m(B_R)} \int_{B_r} \log(u+\varepsilon) \right|^2 m(dx) \leqslant c$$

that is, $\log(u+\varepsilon)$ *is* BMO *on* B_r.

Proof: It we call $u_\varepsilon = u + \varepsilon$, $\log u_\varepsilon$, u_ε^{-1} belong to $D_{loc}[X_0] \cap L^\infty(X_0, m)$. If we consider $\phi$the cut-off function of $B(x, r)$ in $B(x, 2r)$, we have:

$$\int_{B(x,2r)} \phi^2 u(\log u_\varepsilon, \log u_\varepsilon)(dx) = - \int_{B(x,2r)} \phi^2 \mu(u_\varepsilon, u_\varepsilon^{-1})(dx) =$$

$$= - \int_{B(x,2r)} \mu(u_\varepsilon, \phi^2 u_\varepsilon^{-1})(dx) + \int_{B(x,2r)} u_\varepsilon^{-1} \mu(u_\varepsilon \phi^2)(dx) =$$

$$= \int_{B(x,2r)} u_\varepsilon \phi^2 u_\varepsilon^{-1} V(dx) + \int_{B(x,2r)} u_\varepsilon^{-1} \mu(u_\varepsilon \phi^2)(dx) \leqslant$$

$$\leqslant \int_{B(x,2r)} u_\varepsilon \phi^2 u_\varepsilon^{-1} V(dx) + 2 \int_{B(x,2r)} \phi u_\varepsilon^{-1} \mu(u_\varepsilon \phi)(dx).$$

As $V \in K^{loc}(X_0)$, we have ([9], pag. 420, Lemma 2.2; [6]):

$$\int_{B(x,2r)} |V|(dx) \leqslant \frac{\bar{c}m(B(x,r))}{r^2} \|V\|_{K(B(x,2r))}.$$

We have also:

$$2 \int_{B(x,2r)} \phi u_\varepsilon^{-1} \mu(u_\varepsilon, \phi)(dx) \leqslant 2 \left(\int_{B(x,2r)} \phi^2 u_\varepsilon^{-2} \mu(u_\varepsilon, u_\varepsilon)(dx) \right)^{1/2} \cdot$$

$$\cdot \left(\int_{B(x,2r)} \mu(\phi, \phi)(dx) \right)^{1/2} \leqslant \frac{1}{2} \int_{B(x,2r)} \phi^2 \mu(\log u_\varepsilon, \log u_\varepsilon)(dx) + 2 \int_{B(x,2r)} \mu(\phi, \phi)(dx).$$

By Proposition 0.1 and by the doubling property of m we have:

$$\int_{B(x,2r)} \phi^2 \mu(\log u_\varepsilon, \log u_\varepsilon)(dx) \leqslant \frac{20 c_0 m(B(x,r))}{r^2}.$$

As $\phi = 1$ on $B(x, r)$, we obtain:

$$\int_{B(x,r)} \mu(\log u_\varepsilon, \log u_\varepsilon)(dx) \leqslant \frac{cm(B(x,r))}{r^2}.$$

By Poincaré inequality (j) of Assumption II, it results:

$$\int_{B(x,r)} \left| \log u_\varepsilon - \frac{1}{m(B(x,r))} \int_{B(x,r)} \log u_\varepsilon \right|^2 m(dx) \leq c_1 r^2 \int_{B(x,kr)} \mu(\log u_\varepsilon, \log u_\varepsilon)(dx)$$

from which the result comes, by the duplication property of m. ■

It is known, see a.e. [5], that from Proposition 2.4 we have;

PROPOSITION 2.5: *If u satisfies (2.1), $u \geq 0$, m-a.e. in X_0 and if $B(x_0, (4k+12)R) \subset X_0$, $x \in B(x_0, R)$, $0 < r < R$, $u \neq 0$ m-a.e. con $B(x,r)$, then:*

$$⨍_{B(x,r)} u^\gamma m(dx) ⨍_{B(x,r)} u^{-\gamma} m(dx) \leq A$$

where $\gamma \in (0,1)$ is a suitable constant depending only on c_0, c_1, k, $A \geq 1$ is a constant depending only on c_0.

This means that u^γ is in the A_2-*Muckenhoupt's class*; then it can be easily proved that:

COROLLARY 2.1: *If u satisfies (2.1), $u \geq 0$, m-a.e. on X_0 and $B(x_0, (4k+12)R) \subset X_0$, $x \in B(x_0, r)$, $0 < r \leq R$, then:*

$$\int_{B(x,r)} u^\gamma m(dx) \leq c_0' \int_{B(x,r/2)} u^\gamma m(dx)$$

where $c_0' = Ac_0^2$ and γ, A are the constants in Proposition 2.5.

We come now to the proof of Theorem 2.1: let us consider the ball $B(x, R_0)$ such that $B(x, MR_0)$ is contained in X_0, with $M = \max\{4k+13, 33\}$ and let us consider in $B(x, R_0)$ the problem:

$$a(u,v) = -\int_{B(x,R_0)} uvV(dx)$$

with $u \in D_{\mathrm{loc}}[B(x,R_0)]$, $V \in K_{\mathrm{loc}}(X_0)$, $\forall v \in D_0[B(x,R_0)]$ we can regard u in $B(x,R_0)$ as $u = h + w$ with h, w satisfying:

$$\begin{cases} a(h,v) = 0, \\ h - u \in D_0[B(x,R_0)], \end{cases}$$

$$\begin{cases} a(w,v) = -\displaystyle\int_{B(x,R_0)} uvV(dx), \\ w \in D_0[B(x,R_0)], \end{cases}$$

it results ([5], Theorem 1.1): $\sup_{B(x,r)} h \leq c \inf_{B(x,r)} h$, with $B(x,r) \subset B(x,R_0)$, $\bar{k}r \leq R_0$, with $\bar{k} \geq 1$, $\bar{k}$ depending on k, c depending on c_0, r_0, c_1 Assumptions I and II.

By Proposition 2.3, Corollary 2.1 and by the estimates of the Green function G^a for the form a given in [5], Theorem 1.3 we have then:

$$\sup_{B(x,r)} u \leq c \inf_{B(x,r)} u + (c+1) \sup_{y \in B(x,r)} \left| \int_{B(x,R_0)} G^{a,y}_{B(x,R_0)} u V(dz) \right| \leq$$

$$\leq c \inf_{B(x,r)} u + (c+1) \|u\|_{L^\infty(B(x,R_0))} \cdot \sup_{y \in B(x,r)} \int_{B(y,2R_0)} G^{a,y}_{B(y,2R_0)} |V|(dz) \leq$$

$$\leq c \inf_{B(x,r)} u + (c+1) \left(\int_{B(x,2R_0)} |u|^\gamma m(dx) \right)^{1/\gamma} \cdot$$

$$\cdot \sup_{y \in B(x,r)} \int_{B(y,2R_0)} G^{a,y}_{B(y,32R_0)} |V|(dz) \leq$$

$$\leq c \inf_{B(x,r)} u + c(c+1) \left(\int_{B(x,r)} |u|^\gamma m(dx) \right)^{1/\gamma} \cdot$$

$$\cdot \sup_{y \in B(x,r)} \int_{B(y,2R_0)} \int_{d(z,y)}^{32R_0} \frac{s^2}{m(B(y,s))} \frac{ds}{s} |V|(dz) \leq$$

$$\leq c \inf_{B(x,r)} u + c(c+1) \sup_{B(x,r)} u \cdot$$

$$\cdot \sup_{y \in B(x,16R_0)} \int_{B(y,2R_0)} \int_{d(z,y)}^{32R_0} \frac{s^2}{m(B(y,s))} \frac{ds}{s} |V|(dz) \leq$$

$$\leq c \inf_{B(x,r)} u + c(c+1) \sup_{B(x,r)} u \cdot c(R_0).$$

Then if R_0 is small enough, $c(R_0) < 1/(2c(c+1))$ and the result follows. ∎

3. - Energy decay

We prove a weighted Caccioppoli's inequality:

Proposition 3.1: *Let v be a local solution in X_0, i.e.*

$$a(v,w) = -\int vwV(dx),$$

$\forall w \in D_0[X_0]$, with $v \in D_{loc}[X_0]$, $v \in K_{loc}[X_0]$.

If r is such that $B(x, 16r) \subset X_0$, we have:

$$\int_{B(x_0, qr)} G^{a,x_0} \mu(v, v)(dx) + \frac{1}{2} \sup_{B(x_0, qr)} v^2 \leq c \frac{1}{m(B(x_0, r))} \int_{B(x_0, r) - \overline{B(x_0, qr)}} v^2 m(dx) + \eta(r)$$

with $q \in (0, q_0]$ for some $q_0 < 1$, c structural constant depending on q_0, G^{a,x_0} Green function for a with singularity at x_0 with respect to $B(x_0, 2r)$, and with $\eta(r) = c' \sup_{B(x_0, r)} v^2 \|V\|_{K(B(x_0, 16r))}$, c' structural constant depending again also on q.

Proof: We take $z \in B(x_0, r)$ with $B(z, sr) \subset B(z, tr) \subset B(x_0, r)$, $s < t < 1$ and we denote by ϕ the cut-off function of $B(z, sr)$ with respect to $B(z, tr)$. We choose as test function $\phi^2 v G_\varrho^{a,z}$ where $G_\varrho^{a,z}$ denotes the regularised Green function for a relative to z and to the ball $B(x, 2r)$. Proceeding as in [5], Proposition 7.1, we obtain:

$$\int_{B(z, tr)} \phi^2 G_\varrho^{a,z} \mu(v, v)(dx) + \frac{1}{2} \int_{B(z, tr)} \mu(v^2 \phi^2 G_\varrho^{a,z})(dx) \leq$$

$$\leq \frac{c\delta}{(t-s)^2 r^2} \int_{B(z, tr) - B(z, s^* r)} v^2 G_\varrho^{a,z} m(dx) - \int_{B(z, tr)} v^2 \phi^2 G_\varrho^{a,z} V(dx)$$

where $s^* = s^2 / t$ and δ is a constant depending only on s / t. By the definition of G_ϱ^z and since $\phi = 1$ on $B(z, sr)$ we obtain:

$$\int_{B(z, sr)} \phi^2 G_\varrho^{a,z} \mu(v, v)(dx) + \frac{1}{2} \fint_{B(z, \varrho)} v^2 m(dx) \leq$$

$$\leq \frac{c\delta'}{(t-s)^2 r^2} \int_{B(z, tr) - B(z, s^* r)} v^2 G_\varrho^{a,z} m(dx) - \int_{B(z, tr)} v^2 \phi^2 G_\varrho^{a,z} V(dx) \leq$$

$$\leq \frac{c\delta'}{(t-s)^2 r^2} \int_{B(z, tr) - B(z, s^* r)} v^2 G_\varrho^{a,z} m(dx) + \int_{B(z, tr)} v^2 G_\varrho^{a,z} |V|(dx).$$

Passing to the limit as $\varrho \to 0$ and taking the supremum for $z \in B(x_0, qr)$, by choosing $q \in (0, 1/3)$, $s = \sqrt{2q(1-q)}$, $t = 1 - q$, we obtain:

$$\int_{B(x_0, qr/2)} G^{a,x_0} \mu(v, v)(dx) + \sup_{B(x_0, qr)} v^2 \leq$$

$$\leq \frac{c_q}{m(B(x_0, r))} \int_{B(x_0, r) - B(x_0, qr/2)} v^2 m(dx) + \sup_{z \in B(x_0, qr)} \int_{B(z, tr)} v^2 G^{a,z} |V|(dx) \leq$$

$$\leq \frac{c_q}{m(B(x_0, r))} \int_{B(x_0, r) - B(x_0, qr/2)} v^2 m(dx) + c' \int_{B(x_0, r)} v^2 G^{a,x_0} |V|(dx)$$

from which the result follows. ■

By Proposition 3.1 we have also proved:

$$\int_{B(x_0,R)} G^{a,x_0}_{B(x_0,q^{-1}R)}\mu(v,v)(dx) \le \frac{1}{m(B(x_0,R_0))}\|v\|^2_{L^2(B(x_0,R_0))} + \eta(R_0) \tag{3.1}$$

with $q^2 R \le q^3 R_0$; $B(x_0, 20R_0) \subset X_0$. ■

We now prove Theorem 3.1: we take as test function $w = (u-k)G^{a,z}_{\varrho}\phi$, where $G^{a,z}_{\varrho}$ is the regularised Green function for a relative to z and to the ball $B(z,tr)$, ϕ is the capacitary potential of $B(z,sr)$ with respect to $B(z,tr)$ (see a.e. [5], Section 6 for the definition) and k is a constant that will be specified later, $z \in B(x_0, qr)$, $s < t < 1$, q to be fixed. We have

$$\int_{B(z,tr)} \mu(u,(u-k)\phi G^{a,z}_{\varrho})(dx) = \int_{B(z,tr)} \phi G^{a,z}_{\varrho}\mu(u,u)(dx) +$$

$$+\frac{1}{2}\int_{B(z,tr)} \mu((u-k)^2\phi, G^{a,z}_{\varrho})(dx) - \frac{1}{2}\int_{B(z,tr)} (u-k)^2\mu(\phi, G^{a,z}_{\varrho})(dx) +$$

$$+\int_{B(z,tr)} G^{a,z}_{\varrho}(u-k)\mu(u,\phi)(dx) = -\int_{B(z,tr)} u(u-k)G^{a,z}_{\varrho}\phi V(dx).$$

From the definition of $G^{a,z}_{\varrho}$ we obtain:

$$\int_{B(z,tr)} \phi G^{a,z}_{\varrho}\mu(u,u)(dx) + \frac{1}{2}\fint_{B(z,\varrho)} (u-k)^2 m(dx) \le \int_{B(z,tr)} (u-k)^2\mu(\phi, G^{a,z}_{\varrho})(dx) -$$

$$-\int_{B(z,tr)} G^{a,z}_{\varrho}(u-k)\mu(u,\phi)(dx) + -\int_{B(z,tr)} u(u-k)G^{a,z}_{\varrho}\phi V(dx) =$$

$$= \int_{B(z,tr)} (u-k)^2\mu(\phi, G^{a,z}_{\varrho})(dx) - \int_{B(z,tr)} G^{a,z}_{\varrho}(u-k)\mu(u,\phi)(dx) +$$

$$-\int_{B(z,tr)} (u-k)^2 G^{a,z}_{\varrho}\phi V(dx) - \int_{B(z,tr)} kG^{a,z}_{\varrho}\phi(u-k)V(dx) \le$$

$$= \int_{B(z,tr)} (u-k)^2\mu(\phi, G^{a,z}_{\varrho})(dx) - \int_{B(z,tr)} G^{a,z}_{\varrho}(u-k)\mu(u,\phi)(dx) +$$

$$+2\int_{B(z,tr)} (u-k)^2 G^{a,z}_{\varrho}\phi|V|(dx) + \int_{B(z,tr)} k^2 G^{a,z}_{\varrho}\phi|V|(dx).$$

Proceeding now as in [5], Theorem 7.3, we obtain:

$$\int_{B(z,\sigma r)} \phi G_\varrho^{a,z}\mu(u,u)(dx) + \frac{1}{2} \fint_{B(z,\varrho)} (u-k)^2 m(dx) \leq$$

$$\leq \left(\frac{1}{2} + c\eta\right) \sup_{B(z,\sigma r)} (u-k)^2 + \frac{1}{\eta} \int_{B(z,\sigma r) - B(z-\sigma r)} G_\varrho^{a,z}\mu(u-u)(dx) +$$

$$+2 \int_{B(z,\sigma r)} (u-k)^2 G_\varrho^{a,z}\phi|V|(dx) + \int_{B(z,\sigma r)} k^2 G_\varrho^{a,z}\phi|V|(dx)$$

for an arbitrary $\eta > 0$ and with c structural constant.

We now take the supremum for $z \in B(x_0, qr)$, with $q \in (0, q_0]$, $q_0 = \min\{1/6, k^{-1}\}$ and, if $\gamma = 2c\eta$, we find:

$$\sup_{B(x_0,qr)} (u-k)^2 \leq (1+\gamma) \sup_{B(x_0,r)} (u-k)^2 + \frac{C}{\gamma} \int_{B(x_0,r) - B(x_0,qr)} G^{a,x_0}_{B(x_0,2r)}\mu(u,u)(dx) +$$

$$+4C \int_{B(x_0,r)} (u-k)^2 G^{a,x_0}_{B(x_0,2r)}|V|(dx) + 2C \int_{B(x_0,r)} k^2 G^{a,x_0}_{B(x_0,2r)}|V|(dx)$$

with C structural constant depending on q.

By Proposition 3.1 we have:

$$\int_{B(x_0,qr)} G^{a,x_0}_{B(x_0,2r)}\mu(u,u)(dx) \leq c \sup_{B(x_0,r)} (u-k)^2 + C' \int_{B(x_0,r)} (u-k)^2 G^{a,x_0}_{B(x_0,2r)}|V|(dx).$$

And so:

$$\int_{B(x_0,qr)} G^{a,x_0}_{B(x_0,2r)}\mu(u,u)(dx) + \sup_{B(x_0,qr)} (u-k)^2 \leq$$

$$\leq (c+\gamma) \sup_{B(x_0,r)} (u-k)^2 + \frac{C}{\gamma} \int_{B(x_0,r) - B(x_0,qr)} G^{a,x_0}_{B(x_0,2r)}\mu(u,u)(dx) +$$

$$+ \overline{C'} \int_{B(x_0,r)} (u-k)^2 G^{a,x_0}_{B(x_0,2r)}|V|(dx) + 2C \int_{B(x_0,r)} k^2 G^{a,x_0}_{B(x_0,2r)}|V|(dx).$$

By hole filling, after multiplication by γ, we obtain:

$$(C+\gamma) \int_{B(x_0,qr)} G^{a,x_0}_{B(x_0,2r)}\mu(u,u)(dx) + \gamma \sup_{B(x_0,qr)} (u-k)^2 \leq$$

$$\leq \gamma(c+\gamma) \sup_{B(x_0,r)} (u-k)^2 + C \int_{B(x_0,r)} G^{a,x_0}_{B(x_0,2r)}\mu(u,u)(dx) +$$

$$+ \overline{C''}\gamma \int_{B(x_0,r)} (u-k)^2 G^{a,x_0}_{B(x_0,2r)}|V|(dx) + 2C\gamma \int_{B(x_0,r)} k^2 G^{a,x_0}_{B(x_0,2r)}|V|(dx).$$

Choosing $K = \bar{u}$ on $B(x_0, qr)$, where $\bar{u} = 1 / (m(B(x_0, qr))) \int_{B(x_0, qr)} u\, dx$, by Poincaré inequality (j) and the doubling properties of m, we find:

$$(C + \gamma) \int_{B(x_0, qr)} G^{\varepsilon, x_0}_{B(x_0, 2r)} \mu(u, u)(dx) + (C' + \gamma) \sup_{B(x_0, qr)} |u - \bar{u}|^2 \leq$$

$$\leq \gamma(c + \gamma) \sup_{B(x_0, r)} |u - \bar{u}|^2 + C \int_{B(x_0, r)} G^{\varepsilon, x_0}_{B(x_0, 2q^{-1} r)} \mu(u, u)(dx) +$$

$$+ \overline{C^*} \sup_{B(x_0, r)} [(u - \bar{u})^2 + \bar{u}^2] \int_{B(x_0, r)} G^{\varepsilon, x_0}_{B(x_0, 2r)} |V|(dx).$$

If we call

$$\psi^*(r) = \int_{B(x_0, r)} G^{\varepsilon, x_0}_{B(x_0, 2q^{-1} r)} \mu(u, u) + \frac{c' + \gamma}{4C + \gamma} (\mathrm{osc}_{B(x_0, r)} u)^2$$

the result comes by standard methods, by considering also that $(\bar{u})^2 \leq (\sup_{B(x_0, r)} |u|)^2$ and by using Proposition 2.3. ■

We can now prove Corollary 3.1: if u is a local solution in X_0 and $B(x_0, 16R_0) \subset X_0$, then:

$$(\mathrm{osc}_{B(x_0, r)} u)^2 \leq c \left[\left(\frac{r}{R} \right)^{\beta} (\mathrm{osc}_{B(x_0, R)} u)^2 + \eta^{**}(r, R) \right]$$

with r, R, β as in Theorem 3.1, c constant of Theorem 3.1 and with $\eta^{**}(r, R) = c\eta(R) + \eta^*(R)$, c constant of Proposition 3.1. This means that u is locally continuous on X_0: moreover, if V is such that $\|V\|_{K(B_R)} \to 0$ as a power of R as $R \to 0$, we obtain the local holder continuity of u.

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