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Microfunctions Along Submanifolds with Constant Levi Rank (**)

SUMMARY. — Let M be a real analytic hypersurface of a complex manifold X, and let S be an analytic submanifold of M. If the Levi form L_M of M has constant rank, the conormal bundle $\Lambda = T^*_M X$ admits a local partial complexification $\tilde{\Lambda} \subset T^* X$ which is regular involutive. This result, already stated in [Z], is recovered here by a much simpler argument. Assume moreover that L_M is semi-definite and that $\Sigma = S \times_M T^*_M X$ is regular involutive in the real symplectic space Λ. We prove here that S is generic in X, and that the bicharacteristic flow of $\tilde{\Lambda}$ issued from Σ is locally the conormal bundle to a hypersurface $W \subset X$. We then consider a complex of «microfunctions at the boundary» naturally associated to this geometrical setting. Concerning this complex, we get vanishing theorems for its cohomology groups, and we prove that it satisfies the principle of analytic continuation along the integral leaves of $\tilde{\Lambda}$. Notice that the above results extend those obtained in [D'A-Z 2] under the assumption of maximal rank for L_M.

Microfunzioni lungo sottovarietà a rango di Levi costante

RIASSUNTO. — Sia M una sottovarietà analitica reale di una varietà complessa X, e sia S una sottovarietà analitica di M. Nell'ipotesi che la forma di Levi L_M di M abbia rango costante, il fibrato conormale $\Lambda = T^*_M X$ ammette una complessificazione parziale locale $\tilde{\Lambda} \subset T^* X$ regolare involutiva. Questo risultato, già enunciato in [Z], è qui riottenuto con una dimostrazione molto più semplice. Si assuma inoltre che L_M sia semi-definita e che $\Sigma = S \times_M T^*_M X$ sia regolare involutiva nello spazio simplettico reale Λ. In tali ipotesi si mostra che S è generica in X, e che il flusso bicaratteristico di $\tilde{\Lambda}$ sortito da Σ è localmente il fibrato conormale ad un'ipersuperficie $W \subset X$. Si considera quindi un complesso di «microfunzioni al bordo» naturalmente associato a questa situazione geometrica. Riguardo ad esso, si mostrano teoremi di annullamento per i suoi gruppi di coomologia, e si prova che soddisfa il principio del prolungamento analitico lungo le foglie integrali di $\tilde{\Lambda}$. Si noti che i risultati citati estendono quelli ottenuti in [D'A-Z 2] nell'ipotesi di rango massimo per L_M.

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1. - SOME RESULTS ON SYMPLECTIC GEOMETRY

Let X be a complex manifold of dimension n. Denote by TX the tangent bundle to X, by T^*X the cotangent bundle, and let $\pi: T^*X \to X$ be the projection. Denote by $X^{\mathbb{R}}$ the real underlying manifold to X. We will identify $T^*(X^{\mathbb{R}})$ and $(T^*X)^{\mathbb{R}}$ by $df = \partial f + \bar{\partial} f \mapsto \partial f$, for any real function f. Let ω be the canonical one-form, $\sigma = d\omega$ be the canonical two-form on T^*X, and decompose it as $\sigma = \sigma^{\mathbb{R}} + i\sigma^I$, for $\sigma^{\mathbb{R}}$ and σ^I real symplectic two-forms. The forms σ and σ^I induce Hamiltonian isomorphisms that we denote by H and H^I respectively.

Let M be a real analytic submanifold of $X^{\mathbb{R}}$, and denote by $T^*_M X$ the conormal bundle to M in X. For $p \in \dot{T}^*X = T^*X \setminus T^*_X X$, $z = \pi(p)$, we set:

$$T^{\mathbb{C}}_z M = T_z M \cap iT_z M,$$

$$\lambda_M(p) = T_p T^*_M X,$$

$$\lambda_0(p) = T_p \pi^{-1}\pi(p),$$

$$\nu(p) = CH\omega(p),$$

$$\delta_M(p) = \lambda_M(p) \cap i\lambda_M(p),$$

$$d_M(p) = \dim_{\mathbb{C}}(\delta_M(p)),$$

$$\gamma_M(p) = \dim_{\mathbb{C}}(\delta_M(p) \cap \lambda_0(p)).$$

If no confusion may arise, we shall drop down the indices p or M in the above notations.

Recall that $T^*_M X$ is said to be I-symplectic when $\mathrm{NS}(\sigma^I|_{\lambda_M(p)}) = \{0\}$, where NS denotes the null-space. It is immediate to see that $\mathrm{NS}(\sigma^I|_{\lambda_M(p)}) = \delta_M(p)$.

Fix $p_0 \in \dot{T}^*_M X$, set $z_0 = \pi(p_0)$, and let $\phi_1 = 0, \ldots, \phi_l = 0$ be a set of independent equations for M near z_0. The vectors $w_i = \partial\phi_i(z_0)$ constitute a basis for $(T^*_M X)_{z_0}$. Consider the morphism

$$\psi: \mathbb{R}^l \times M \to T^*_M X, \qquad ((t_i), z) \mapsto \left(z; \sum_{i=1}^{l} t_i \partial\phi_i(z)\right).$$

Let ϕ be a real function such that $\phi|_M \equiv 0$, $(z_0; \partial\phi(z_0)) = p_0$. The Levi form of M at p_0 is defined by:

$$L_M(p_0) = \partial\bar{\partial}\phi|_{T^{\mathbb{C}}_{z_0}M}. \tag{1.1}$$

This is a good definition in the sense that if $\tilde{\phi}$ is any other function with $\tilde{\phi}|_M \equiv 0$ and $\partial\tilde{\phi}(z_0) = p_0$, the corresponding form $\partial\bar{\partial}\tilde{\phi}|_{T^{\mathbb{C}}_{z_0}M}$ has the same signature and rank as (1.1).

Let $(t_{0i}) \in \mathbb{R}^l$ be such that $\partial\phi(z_0) = \sum_{i=1}^{l} t_{0i} w_i$. Consider the tangent morphism to ψ

at $((t_0), z_0)$:

$$\text{(1.2)}\quad \psi_*: \mathbb{R}^l \times T_{z_0} M \to \lambda_M(p_0), \qquad ((t_i), v) \mapsto \left(v; \sum_{i=1}^{l} t_i w_i + \partial\langle \partial\phi, v\rangle + \partial\langle \overline{\partial}\phi, \overline{v}\rangle\right).$$

We remark that the pull back of $\sigma|_{\lambda_M}$ by ψ_* is described as

$$\text{(1.3)}\quad \psi_*^*(\sigma|_{\lambda_M})(((t_i), v), ((\overline{t}_i), \overline{v})) = (dt \wedge \partial\phi)(((t_i), v) \wedge ((\overline{t}_i), \overline{v})) + \partial\overline{\partial}\phi(\overline{v} \wedge \overline{v}).$$

Lemma 1.1: *The following relation holds*:

$$\dim \mathrm{NS}(L_M(p_0)) = d_M(p_0) - \gamma_M(p_0).$$

Proof: Let $((t_i), v)$ be such that in (1.3) we get 0 for all $((\overline{t}_i), \overline{v})$. Then

$$\begin{cases} \partial\overline{\partial}\phi(\overline{v} \wedge \overline{v}) + \sum_{i=1}^{l} \langle t_i w_i, \overline{v}\rangle = 0 & \forall \overline{v} \in T_{z_0} M, \\ \sum_{i=1}^{l} \langle \overline{t}_i w_i, v\rangle = 0 & \forall (\overline{t}_i) \in \mathbb{R}^l, \end{cases}$$

(here, to obtain the first condition we put $(\overline{t}_i) = 0$ in (1.3), and to obtain the second we put $\overline{v} = 0$). By the second equality we get $v \in T^{\mathbb{C}}_{z_0} M$; by the first we have

$$\mathfrak{Im}\left(\langle \partial(\overline{\partial}\phi, \overline{v}), \overline{v}\rangle + \sum_{i=1}^{l} \langle t_i w_i, \overline{v}\rangle\right) = 0 \qquad \forall \overline{v} \in T_{z_0} M,$$

which is equivalent to

$$\partial(\overline{\partial}\phi, \overline{v}) + \sum_{i=1}^{l} t_i w_i \in i(T_M^* X)_{z_0},$$

or else to

$$\begin{cases} \partial(\overline{\partial}\phi, \overline{v}) \in (T_M^* X)_{z_0} + i(T_M^* X)_{z_0}, \\ \sum_{i=1}^{l} t_i w_i \in \mathfrak{Re}(-\partial(\overline{\partial}\phi, \overline{v})) + ((T_M^* X)_{z_0} \cap i(T_M^* X)_{z_0}) \end{cases}$$

(where $\mathfrak{Re}$ is the real part, well-defined modulo $(T_M^* X)_{z_0} \cap i(T_M^* X)_{z_0}$). Summarizing up,

$$\dim(\mathrm{NS}(\sigma|_{\lambda_M(p_0)})) = \dim\{v \in T^{\mathbb{C}}_{z_0} M: \partial(\overline{\partial}\phi, \overline{v}) \in (T_M^* X)_{z_0} + i(T_M^* X)_{z_0}\} + \gamma_M(p_0) = \\ = \dim \mathrm{NS}(L_M(p_0)) + \gamma_M(p_0).$$

Since $\mathrm{NS}(\sigma|_{\lambda_M(p_0)}) = \delta_M(p_0)$, we get the conclusion. ■

For $p \in \dot{T}^*_M X$, consider the numbers $s_M^{+,-,0} = s_M^{+,-,0}(p)$ given by:

$$\begin{cases} s_M^0 = d_M - \gamma_M, \\ s_M^+ + s_M^- + s_M^0 = n - \operatorname{codim}_{TX} T^{\mathbb{C}} M, \\ s_M^+ - s_M^- = \frac{1}{2}\tau(\lambda_M, i\lambda_M, \lambda_0), \end{cases}$$

where τ is the Maslov index in the sense of [K-S]. In [D'A-Z 1] it is proved that $s_M^{+,-,0}$ are equal to the numbers of respectively positive, negative, and null eigenvalues for the Levi form L_M. The latter are moreover equal to the corresponding numbers of eigenvalues for $\sigma(v, v^c)|_{\lambda_M \cap \lambda_0}$, where v^c is the conjugate with respect to λ_M in the quotient $\lambda_M + i\lambda_M / \lambda_M \cap i\lambda_M$. (Note that these results in case of $\operatorname{codim}_X M = 1$ were obtained in [S1] and [K-S].)

In the following we will write $\Lambda = \dot{T}^*_M X$. Let Σ be a conic submanifold of Λ, and recall that $\Sigma \subset \Lambda$ is called *regular* if $\omega^1|_\Sigma$ is everywhere different from zero.

Proposition 1.2 (cf. [Z]): *Let $p_0 \in \Sigma$, and assume that for a conic neighborhood V of p_0 in T^*X we have:*

$$\begin{cases} d_M(p) = d \qquad \forall p \in \Lambda \cap V, \\ \dim\,(\nu(p_0) \cap \lambda_M(p_0)) = 1, \\ T_p\Sigma \supset \delta_M(p) \qquad \forall p \in \Sigma \cap V, \\ \Sigma \text{ is regular in } V, \\ T_p\Sigma / \delta(p) \text{ is involutive in } \delta(p)^\perp / \delta(p) \ \forall p \in V \cap \Sigma. \end{cases}$$

Then, there exist complex manifolds X', X'' of dimension $n-d$ and d respectively, real analytic submanifolds $S' \subset M' \subset X'$ with $\operatorname{codim}_{X'} M' = 1$, $\operatorname{codim}_{M'} S' = \operatorname{codim}_\Lambda \Sigma$, and a germ of symplectic transformation at p_0

$$\chi: T^*X \xrightarrow{\sim} T^*(X' \times X''),$$

such that, identifying X'' with $T^*_{X''}X''$, the following isomorphisms hold locally at p_0:

(i) $\chi(\Lambda) = \Lambda' \times X''$, $\quad \Lambda' = T^*_{M'}X'$ *I-symplectic*;

(ii) $\chi(\Sigma) = \Sigma' \times X''$, $\quad \Sigma' = S' \times_\Sigma M' T^*_{M'}X'$ *regular involutive in* $T^*_{M'}X'$.

Proof: (i) We give here a new and simpler proof with respect to [Z].

Performing a first complex contact transformation, it is not restrictive to assume that Λ is the conormal bundle to a hypersurface M such that $s_M^-(p_0) = 0$ (cf. [K-S]). By Lemma 1.1, the form L_M has constant rank (in particular, M is the boundary of a pseudoconvex domain). By [R], [F] we may find an analytic foliation of M by the (complex) integral leaves of $NS(L_M)$. By a result of [B-F] (concerning complex curves, but easily extended to higher codimensional complex submanifolds), this foliation can be lifted to a foliation of Λ by the complex integral leaves of $\lambda_M \cap i\lambda_M$. This is a real analytic submersion $\Lambda \to \Lambda'$ with complex fibers. Moreover, using [F, th. 2.14] we may find a complex analytic manifold X'' of dimension d and a germ of real analytic isomorphism at p_0:

$$f: \Lambda \xrightarrow{\sim} \Lambda' \times X'', \tag{1.4}$$

with the submersion mentioned above corresponding to the first projection. Note that, for $p = (p', z'')$, $T_p f^{-1}(p' \times X'') = \delta(p)$, and hence σ^I induces a non-degenerate real symplectic form on Λ'. A complexification $(\Lambda')^{\mathbb{C}}$ of Λ' is then endowed with a non-degenerate complex symplectic two-form, and we may locally identify it with the cotangent bundle to a complex manifold X'. Finally, since $\dim_{\mathbb{R}}(T\Lambda' \cap \nu^{\delta}) = 1$, by applying a contact transformation in T^*X' it is not restrictive to assume $\Lambda' = T^*_{M'}X'$, for a hypersurface M' of X' which is the boundary of a pseudoconvex domain.

(ii) Due to (i), and the identifications $T\Sigma^{\delta} = T\Sigma'$, $T\Lambda' = \delta^{\perp}/\delta$, we have

$$\chi(\Sigma) = \Sigma' \times X''$$

with Σ' regular involutive in $\Lambda' = T^*_{M'}X'$. Finally, since $\Sigma' \subset T^*_{M'}X'$ and M' is a hypersurface, we may find $S' \subset M'$ with $\operatorname{codim}_{M'} S' = \operatorname{codim}_{\Lambda} \Sigma$ such that $\Sigma' = S' \times_{M'} T^*_{M'}X'$. ■

2. - Genericity and involutivity

Let M be an analytic hypersurface of X, let $S \subset M$ be a submanifold, and set $\Lambda = \dot{T}^*_M X$, $\Sigma = S \times_M \dot{T}^*_M X$. Assuming that L_M has constant rank, we shall discuss here the link between «genericity» of S and «regular involutivity» of Σ in the real symplectic space (Λ, σ^I). (These results were obtained in [D'A-Z 2] under the hypothesis of L_M being non-degenerate.)

By the results from the previous section, keeping the above notations it is not restrictive to assume (after a contact transformation) that $\Lambda = \Lambda' \times X''$ and $\Sigma = \Sigma' \times X''$ at $p_0 \in \Sigma$. For $z_0 = \pi(p_0) \in S$, write $z_0 = (z_0', z_0'')$ with $z_0' \in S'$ and $z_0'' \in X''$. Let $\phi = 0$ be a local equation for M' in X' at z_0' (hence an equation of $M \simeq M' \times X''$ in X at z_0), and set $p_0' = (z_0'; \partial\phi(z_0'))\in \Sigma'$, so that $p_0 = (p_0', z_0'')$.

We consider the diagram

$$\begin{array}{ccc} T_{z_0} M & \xrightarrow{\psi_*} & \lambda_M(p_0) \\ \downarrow & & \downarrow \\ T_{z_0'} M' & \xrightarrow{\psi_*} & \lambda_{M'}(p_0') \end{array}$$

where the horizontal arrows are induced from (1.2) with $(t_i) = 0$ and the vertical arrows are the projections $v = (v', v'') \mapsto v'$. One has

$$\psi^* \sigma(v, w) = \langle \partial \langle \bar\partial \phi, \bar v \rangle, w \rangle - \langle \partial \langle \bar\partial \phi, \bar w \rangle, v \rangle =$$

$$= \langle \partial \langle \bar\partial \phi, \bar v' \rangle, w' \rangle - \langle \partial \langle \bar\partial \phi, \overline{w'} \rangle, v' \rangle = \psi^* \sigma((0, v'), (0, w')).$$

In [D'A-Z 2] it is proved that $\forall \theta' \in T^*_{z_0} M'$

$$H^I(\pi^* \theta')|_{\psi_* T^{\mathbb{C}}_{z_0} M'} = \psi_*(v'),$$

where v' is the unique solution of

$$\partial \langle \bar\partial \phi, \bar v' \rangle = i\theta' |_{T^{\mathbb{C}}_{z_0} M'}$$

in the identification $T^{\mathbb{C}}_{z_0} M' \xrightarrow{\sim} (T^{\mathbb{C}}_{z_0} M')^*$, given by $v \mapsto \langle v, \cdot \rangle$.

From now on we will consider the following geometrical situation. Let S and M be submanifolds of X, $S \subset M$, $\operatorname{codim}_X M = 1$. Let $\Sigma = S \times_M \dot{T}^*_M X$ and consider it as a submanifold of $\Lambda = \dot{T}^*_M X$, endowed with the (degenerate) two-form σ^I. Let p_0 be a point of Σ, V a conic neighborhood of p_0 and set $z_0 = \pi(p_0)$.

Recall that S is called generic at z_0 if $(T^*_S X)_{z_0} \cap i(T^*_S X)_{z_0} = \{0\}$.

Proposition 2.1: (*a*) *Assume that*

(i) $d_M(p) = d \qquad \forall p \in \Lambda \cap V,$

(ii) $T_p \Sigma \supset \delta_M(p) \qquad \forall p \in \Sigma \cap V.$

Then

$$\Sigma \textit{ is regular at } p_0 \iff \dim(\nu(p_0) \cap \lambda_S(p_0)) = 1.$$

(*b*) *Let us also assume that L_M is semi-definite (i.e., M is the boundary of a (weakly) pseudoconvex or pseudoconcave domain). Then*

$$\begin{cases} \Sigma \textit{ is regular at } p_0 \\ T_p \Sigma^\delta \textit{ is involutive in } \delta(p)^\perp / \delta(p) \qquad \forall p \in \Sigma \cap V \end{cases}$$

implies

$$S \text{ is generic at } z_0.$$

(*Here* $\bullet^{\delta} = ((\bullet \cap \delta^{\perp}) + \delta)/\delta$.)

PROOF: Let us choose coordinates such that $\Lambda = \dot{T}^*_M X' \times X''$ and $\Sigma = (S' \times_{M'} \dot{T}^*_{M'} X') \times X''$. Then:

$$T\Sigma^{\delta} \simeq T(S' \times_{M'} T^*_{M'} X'),$$

and

$$(\lambda_S \cap \lambda_0)^{\delta} \simeq \lambda_{S'} \cap \lambda_0.$$

The conclusion follows by the results of [D'A-Z 2]. ■

Let us denote by $\tilde{\Sigma}$ the bicharacteristic flow of $\bar{\Lambda}$ issued from Σ. Note that $\tilde{\Sigma}$ is the union of the partial complexifications of the integral leaves of $T\Sigma^{\perp}$.

THEOREM 2.2: *Assume that*

(i) $d_M(p) = d \ \ \forall p \in \Lambda \cap V$;

(ii) $T_p\Sigma \supset \delta_M(p) \ \ \forall p \in \Sigma \cap V$;

(iii) Σ *is regular and* $T_p\Sigma^{\delta}$ *is involutive in* $\delta(p)^{\perp}/\delta(p)$ $\forall p \in \Sigma \cap V$.

Then

$$\dim_{\mathbb{R}}(T_p\tilde{\Sigma} \cap \lambda_0(p)) = 1 + d_S(p) - d_M(p), \qquad \forall p \in \Sigma \cap V.$$

PROOF: (Cf. [D'A-Z 2] for a different proof in the case $d_M(p) \equiv 0$.)

Let $\Lambda = \dot{T}^*_{M'} X' \times X''$, $\Sigma = (S' \times_{M'} \dot{T}^*_{M'} X') \times X''$. Since $T\tilde{\Sigma} \cap \lambda_0 \simeq T\tilde{\Sigma}' \cap \lambda_0$, $\dim(\lambda_S \cap i\lambda_S) = \dim(\lambda_M \cap i\lambda_M) + \dim(\lambda_{S'} \cap i\lambda_{S'})$, then it is not restrictive to suppose $d_M = 0$ from the beginning.

For $p \in \Sigma$, one has:

$$T\tilde{\Sigma} \simeq T\Sigma + iH^I(T^*_\Sigma T^*_M X) \simeq$$

$$\{(u + iv; s + \partial\langle\partial\phi, u + iv\rangle + \partial\langle\bar{\partial}\phi, \overline{u - iv}\rangle) : u \in TS,$$

$$s \in (T^*_M X)_{z_0},\ v \in T^{\mathbb{C}}M \cap TS,\ \partial\langle\bar{\partial}\phi, \bar{v}\rangle|_{T^{\mathbb{C}}M} \in iT^*_S X|_{T^{\mathbb{C}}M}\}$$

where the first equality is due to $T\Sigma^{\perp} \subset T\Sigma$. Thus $T\tilde{\Sigma} \cap \lambda_0$ is described by

$$\text{(2.1)} \qquad \begin{cases} u = -iv \in T^{\mathbb{C}}S, \\ \partial\langle\bar{\partial}\phi, \overline{u - iv}\rangle|_{T^{\mathbb{C}}M} \in T^*_S X|_{T^{\mathbb{C}}M}. \end{cases}$$

Let us set $Z = \{v \in T^{\mathbb{C}}_{z_0} S;\ \partial\langle\bar{\partial}\phi, \bar{v}\rangle \in (T^*_S X)_{z_0}|_{T^{\mathbb{C}}M}\}$. Then $\dim(T\tilde{\Sigma} \cap \lambda_0) = 1 +$

$+ \dim Z$. One has:

$$Z + iZ = \{v \in T^{\mathbb{C}}_{z_0} S;\ \partial\langle\bar\partial\phi, \bar v\rangle \in ((T^*_S X)_{z_0} + i(T^*_S X)_{z_0})|_{T^{\mathbb{C}}M}\} = \mathrm{NS}(L_M),$$

$$Z \cap iZ = \{v \in T^{\mathbb{C}}_{z_0} S;\ \partial\langle\bar\partial\phi, \bar v\rangle \in ((T^*_S X)_{z_0} \cap i(T^*_S X)_{z_0})|_{T^{\mathbb{C}}M}\} =$$

$$= \{v \in T^{\mathbb{C}}_{z_0} S;\ \partial\langle\bar\partial\phi, \bar v\rangle \in (T^*_S X)_{z_0} \cap i(T^*_S X)_{z_0}\},$$

where the last equality follows from the implication:

$$\dim(\lambda_S \cap \nu) = 1 \quad \Rightarrow \quad T^*_S X \cap iT^*_S X \xrightarrow{\sim} T^*_S X|_{T^{\mathbb{C}}M} \cap iT^*_S X|_{T^{\mathbb{C}}M}.$$

We then have:

$$\dim_{\mathbb{R}} Z = \dim_{\mathbb{C}}(Z + iZ) + \dim_{\mathbb{C}}(Z \cap iZ) =$$

$$= \dim NS(L_S) + \dim(\lambda_S \cap i\lambda_S \cap \lambda_0) = (\lambda_S \cap i\lambda_S). \quad \blacksquare$$

Corollary 2.3: *Assume that*

(i) $d_M(p) = d \qquad \forall p \in \Lambda \cap V$;

(ii) $T_p\Sigma \supset \delta_M(p) \qquad \forall p \in \Sigma \cap V$;

(iii) Σ *is regular and* $T_p\Sigma^\delta$ *is involutive in* $\delta(p)^\perp/\delta(p)$ $\forall p \in \Sigma \cap V$;

(iv) L_M *is semi-definite.*

Then $\tilde\Sigma = T^*_W X$ for a hypersurface $W \subset X$.

Proof: By Proposition 2.1, S is generic (i.e., $\gamma_S = 0$) and $\mathrm{NS}(L_S) = \mathrm{NS}(L_M)$ in Σ. It follows that $\delta_S(p) = \delta_M(p)$ $\forall p \in \Sigma$. In particular, by Theorem 2.2, $\dim(T\tilde\Sigma \cap \lambda_0) = 1$ and thus we may find a hypersurface $W = W' \times X''$ such that

$$\tilde\Sigma = T^*_W X = T^*_{W'} X' \times X''. \quad \blacksquare$$

3. - Microlocalization along submanifolds

Let S, M be submanifolds of a complex manifold X of dimension n with $S \subset M$, $\mathrm{codim}_X M = 1$, $\mathrm{codim}_M S = r$. Let $\Sigma = S \times_M \dot T^*_M X$, $\Lambda = \dot T^*_M X$, fix $p_0 \in \Sigma$ and denote by V a conic neighborhood of p_0. Let us assume

$$(3.1) \qquad \begin{cases} d_M(p) = d & \forall p \in \Lambda \cap V, \\ T_p\Sigma \supset \delta_M(p) & \forall p \in \Sigma \cap V, \\ \Sigma \text{ regular in } V, \\ T_p\Sigma^{\delta(p)} \text{ involutive in } \delta(p)^\perp/\delta(p) & \forall p \in \Sigma \cap V. \end{cases}$$

We recall from the preceding section that if $L_M \geqslant 0$ (or $L_M \leqslant 0$), then there exists a germ of contact transformation χ at p_0 such that $\chi(\tilde{\Sigma}) = T^*_W X \simeq T^*_{W'} X \times X''$ with $\operatorname{codim}_X W = 1$. In this case, we shall consider the numbers $s_W^{+,-,0}$. For $p \in \Sigma$, we have the following equalities in $T_p T^* X$:

$$(T_p\tilde{\Sigma} + iT_p\tilde{\Sigma}) \cap \lambda_0 + (\lambda_0 \cap \lambda_S)^{\mathbb{C}} = (T_p\Sigma + iT_p\Sigma) \cap \lambda_0 + (\lambda_0 \cap \lambda_S)^{\mathbb{C}} =$$
$$= (\lambda_S + i\lambda_S) \cap \lambda_0,$$

where the first equality holds since $T_p\tilde{\Sigma} = T_p\Sigma + iT_p\Sigma^{\perp} \subset T_p\Sigma + iT_p\Sigma$, and the second follows from the arguments between formulas (2.7) and (2.8) in [D'A-Z1]. It follows that:

$$\sigma(v, v^{c_{T\tilde{\Sigma}}})|_{(T\tilde{\Sigma} + iT\tilde{\Sigma}) \cap \lambda_0} \sim \sigma(v, v^{c_{T\Sigma}})|_{(T\Sigma + iT\Sigma) \cap \lambda_0} \sim \sigma(v, v^{c_{\lambda_S}})|_{(\lambda_S + i\lambda_S) \cap \lambda_0}$$

where the symbol $\sim$ means that the above forms have the same signature and rank. In particular, if $L_M \geqslant 0$ or $L_M \leqslant 0$ one has $L_W \sim L_S$ in Σ and thus $s_W^{\pm} = s_S^{\pm}$. One may also immediately see that $s_W^0 = r + d_M$. It follows that $s_W^{\mathbb{C}} = s_S^0 + \operatorname{codim}_{T^{\mathbb{C}}W} T^{\mathbb{C}} S = s_S^0 + \operatorname{codim}_M S$.

Consider the microlocalization of the sheaf $\mathcal{O}_X$ of the holomorphic functions in X along W (cf. [K-S]):

$$\mu_W(\mathcal{O}_X) = \mu\hom(\mathbb{Z}_W, \mathcal{O}_X).$$

Choosing a quantization of χ by a kernel, we define the complex of microfunctions along $\tilde{\Sigma}$ by:

$$\mathcal{C}_{\tilde{\Sigma}} = \chi^{-1}\mu_W(\mathcal{O}_X)[1]. \tag{3.2}$$

This is a good definition (i.e. independent of the choice of χ and its quantization) by the results of [K-S, ch.11]. Let:

$$\mathcal{C}_A = \mu_M(\mathcal{O}_X)[1 + s_M^-].$$

Proposition 3.1: Let us assume (3.1). Then:

(i) *$\mathcal{C}_{\tilde{\Sigma}}$ and $\mathcal{C}_A$ are concentrated in degree zero;*

(ii) *there exists a natural «restriction» morphism*

$$\mathcal{C}_{\tilde{\Sigma}} \to \mathcal{C}_A, \tag{3.3}$$

(iii) *the third term $\mathcal{C}_{A \setminus \tilde{\Sigma}}$ of a distinguished triangle associated to (3.3) is concentrated in degree zero;*

(iv) *$\mathcal{C}_{\tilde{\Sigma}}$ satisfies the principle of the analytic continuation along the integral leaves of $T\tilde{\Sigma} \cap iT\tilde{\Sigma}$.*

Proof (Cf. [S2]): Let $p_0 = (p_0', z_0'') \in \Sigma' \times X''$. By a contact transformation χ' at p_0' (cf. [K-S]) we may interchange Λ' (resp. Σ') with the conormal bundle to a strictly (resp. weakly) pseudoconvex domain U (resp. U') with $U' \supset U$. Setting $\tilde{U} = U \times X''$, $\tilde{U}' = U' \times X''$, and choosing a quantization of χ', we get:

$$(1) \qquad (H^j \mathcal{C}_\Lambda)_p \simeq \begin{cases} \varinjlim_B \dfrac{\mathcal{O}_X(\tilde{U} \cap B)}{\mathcal{O}_X(B)} & j = 0 \\ 0 & j \neq 0, \end{cases} \qquad \text{for } p \in \Lambda,$$

$$(2) \qquad (H^j \mathcal{C}_\Sigma)_p \simeq \begin{cases} \varinjlim_B \dfrac{\mathcal{O}_X(\tilde{U}' \cap B)}{\mathcal{O}_X(B)} & j = 0 \\ 0 & j \neq 0, \end{cases} \qquad \text{for } p \in \Sigma,$$

where B ranges through the family of open neighborhoods of $(\pi(\chi'(p_0')), z_0'')$. (i) follows. The morphism in (ii) is induced by the restriction $\mathcal{O}_X(U') \to \mathcal{O}_X(U)$. (iii) follows. The propagation of (iv) (for microfunctions with holomorphic parameters) can be proved by the aid of Bochner's theorem. ■

Example 3.2: Let $X = \mathbf{C}^n$, and let $\chi: T^*X \to T^*X$ be a germ of contact transformation with $\psi(T^*_{\mathbf{R}^n}X) = T^*_M X$, where $\operatorname{codim}_X M = 1$ and M is the boundary of a strictly pseudoconvex domain. Let N be an analytic hypersurface of $\mathbf{R}^n$, and write $\Sigma = \chi(N \times_{\mathbf{R}^n} T^*_{\mathbf{R}^n} X)$. If $\mathbf{R}^n \setminus N = \Omega^+ \cup \Omega^-$, denoting by $\mathcal{C}_{\Omega^\pm|X} = \mu_{\Omega^\pm}(\mathcal{O}_X)[n]$ the complexes of [S1], one has

$$\chi_*(\mathcal{C}_{\Omega^+|X} \oplus \mathcal{C}_{\Omega^-|X}) = \mathcal{C}_{\Lambda \setminus \Sigma}.$$

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