Rendiconti
Accademia Nazionale delle Scienze detta dei XL
Memorie di Matematica e Applicazioni
113° (1995), Vol. XIX, fasc. 1, pagg. 273-283

NICOLA CUFARO PETRONI (*)

On the Series of Fermi Random Variables (**)

Summary. — In the computer simulations of random acoustic signals to be recognized by neural networks the amplitudes of the partial vibrations are extracted from a Fermi distribution. This kind of distributions, well known in the physical literature, seems not to have received much attention in the specifically probabilistic one. In this paper a few propositions are proved which are useful, for example, to discuss the convergence of series of random variables with distributions of the Fermi type.

Sulle serie di variabili aleatorie di Fermi

Sunto. — Nella simulazione su computer di segnali acustici aleatori, prodotti per il riconoscimento da reti neurali, le ampiezze delle vibrazioni parziali vengono estratte sulla base di una distribuzione di Fermi. Questo tipo di distribuzioni, ben noto nella letteratura di carattere fisico, non sembra aver ricevuto molta attenzione in quella più specificamente probabilistica. In questo lavoro vengono dimostrate alcune proposizioni utili, ad esempio, per discutere la convergenza di serie di variabili aleatorie con densità del tipo di Fermi.

1. - Introduction

In a few recent papers about the detection of a pitch in acoustic signals by neural networks (see [3] and [4]) it has been pointed out the necessity of producing random signals in order to simulate the realistic acoustic environment to be used in the training of the network. These signals, be they either tones (with a pitch) or noises (without a pitch), are simulated by giving their discretized Fourier transform. For example, to characterize the set of all the periodic signals we can consider their representation by means of a trigonometric Fourier series. Of course, by considering all the possible sequences $\{r_n, \theta_n\}_{n \in N}$ of coefficients (with $r_n \geq 0$ and $0 \leq \theta_n \leq 2\pi$) such that the

(*) Indirizzo dell'Autore: Dipartimento di Fisica dell'Università di Bari e G.N.C.B.-C.N.R. Unità di Bari; via Amendola 173, 70126 BARI (Italy): E-mail cufaro@bari.infn.it.
(**) Memoria presentata il 28 settembre 1995 da Giorgio Letta, uno dei XL.

trigonometric series

$$\sum_{n=0}^{\infty} r_n \cos(2\pi n\nu t + \theta_n)$$

converges (in a suitable sense) we will recover all the set of the *realistic* periodic signals. The simplest sufficient condition on the coefficients which guarantees that the trigonometric series converges is

$$\sum_{n=0}^{\infty} r_n < +\infty .$$

On the other hand, since following the Ohm's law (see for example [2], pag. 114) in a first approximation only the amplitudes and not the phases are relevant for the pitch perception, these conditions seem to embody the essential physical requirements for a useful simulation. If now we want to pick up at random a periodic signal we can consider the stochastic process $\xi(t)$ given by the *random trigonometric series*

$$\xi(t) = \sum_{n=0}^{\infty} \xi_n \cos(2\pi n\nu t + \zeta_n) \tag{1}$$

by substituting respectively the numbers r_n, θ_n with the sequence of independent random variables ξ_n, ζ_n ($\xi_n \geq 0$; $0 \leq \zeta_n < 2\pi$; $\forall n \in N$). To every sample of the random sequence $\{\xi_n, \zeta_n\}_{n \in N}$, if the corresponding series converges, it is associated a periodic signal given by the sum of the trigonometric series. Of course our problem will be now that of the characterization of the random variables ξ_n, ζ_n in such a way that the series (1) be convergent (P-*a.s.*).

From the previous remarks it is clear that a sufficient condition for the convergence (P-*a.s.*) of (1) is the convergence (P-*a.s.*) of $\sum_{n=0}^{\infty} \xi_n$, and a sufficient condition for that is the convergence of the series

$$\sum_{n=0}^{\infty} E\xi_n .$$

An interesting example can be given by assuming that each ξ_n admits a *Fermi density function*. We recall that a Fermi density is a probability density function of the form

$$f(x) = \begin{cases} \dfrac{\lambda}{\ln(1+e^{\lambda a})} \dfrac{1}{1+e^{\lambda(x-a)}}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0, \end{cases}$$

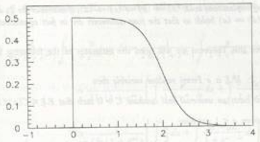

Fig. 1.

with cumulative distribution function

$$F(x) = \begin{cases} 1 - \dfrac{\ln(1 + e^{\lambda a} e^{-\lambda x})}{\ln(1 + e^{\lambda a})}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

where $a > 0$, $\lambda > 0$ (see Fig. 1 for an example of the Fermi density with $a = 2$ and $\lambda = 15$). Since these distributions, well known to the physicists (see for example [7]), seem to have received little attention in the probabilitstic literature, we will prove now a few proposition useful to study the convergence of series of Fermi random variables.

2. - Series of Fermi random variables

First of all it would be very interesting to find precise conditions on the parameters of our Fermi random variables which entail the convergence of the series (1). These conditions are be spelled out in the following Theorem:

Theorem 1: *Let $\{\xi_n\}_{n\in N}$ be a sequence of Fermi random variables with parameters a_n and λ_n; among the four statements*

(a) $\sum_{n\in N} E(\xi_n \wedge 1) < +\infty$,

(b) $\sum_{n\in N} (a_n + \lambda_n^{-1}) < +\infty$,

(c) $\sum_{n\in N} E\xi_n < +\infty$,

(d) $\sum_{n\in N} \xi_n < +\infty \quad P\text{-a.s.}$, *the following implications hold:* $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$; *if moreover the* ξ_n *are independent, also* $(d) \Rightarrow (a)$ *holds so that the four statements are in fact equivalent.*

To prove this Theorem we will need the estimates of the following Lemma:

LEMMA 2: *If* ξ *is a Fermi random variable then*

(I) *it exists an universal real constant* $C > 0$ *such that* $E\xi \leq C(a \vee \lambda^{-1})$;

(II) *when* $a < 1$ *we have*

$$E(\xi \wedge 1) \geq \frac{1}{\lambda \ln(1 + e^{\lambda a})} \left(\frac{\lambda^2 a^2}{4} + \int_0^{\lambda(1-a)} \frac{t}{1 + e^t}\, dt \right).$$

PROOF: We remark first of all that $E\xi = \lambda^{-1} g(\lambda a)$ where g is the following function defined on $[0, +\infty[$:

$$g(x) = \frac{1}{\ln(1 + e^x)} \int_0^{\infty} \frac{t}{1 + e^{t-x}}\, dt .$$

When $x \to +\infty$ we have for this function

$$g(x) \sim \frac{1}{x} \int_0^{\infty} \frac{t}{1 + e^{t-x}}\, dt = \int_{-\infty}^{x} \frac{1}{1 + e^{-u}}\, du - \frac{1}{x} \int_{-\infty}^{x} \frac{u}{1 + e^{-u}}\, du ,$$

so that from the relations

$$\int_{-\infty}^{x} \frac{1}{1 + e^{-u}}\, du \sim x , \qquad \frac{1}{x} \int_{-\infty}^{x} \frac{u}{1 + e^{-u}}\, du \sim \frac{1}{x} \left(\frac{x^2}{2} \right) = \frac{x}{2}$$

we immediately have $g(x) \sim x - (1/2)x = (1/2)x$. Hence we can say that it exists a real number $b > 0$ (which of course is independent from the parameters of our Fermi random variable) such that $g(x) \leq x$ for $x > b$. If then m is the maximum of $g(x)$ on the interval $[0, b]$, we have

$$E\xi = \begin{cases} m\lambda^{-1} & \text{if } a\lambda \leq b ; \\ \lambda^{-1}(a\lambda) = a & \text{otherwise} . \end{cases}$$

As a consequence, our universal constant is $C = m \vee 1$ and the statement (I) is proved.

The proof of the statement (II) is in the following relations which hold for $a < 1$:

$$E(\xi \wedge 1) \geqslant \frac{\lambda}{\ln(1+e^{a\lambda})} \int_0^1 \frac{x}{1+e^{\lambda(x-a)}}\,dx = \frac{1}{\lambda \ln(1+e^{a\lambda})} \int_0^{\lambda} \frac{u}{1+e^{u-\lambda a}}\,du =$$

$$= \frac{1}{\lambda \ln(1+e^{a\lambda})} \left(\int_0^{a\lambda} \frac{u}{1+e^{u-\lambda a}}\,du + \int_{a\lambda}^{\lambda} \frac{u}{1+e^{u-\lambda a}}\,du \right) \geqslant$$

$$\geqslant \frac{1}{\lambda \ln(1+e^{a\lambda})} \left(\int_0^{a\lambda} \frac{u}{2}\,du + \int_{a\lambda}^{\lambda} \frac{u-a\lambda}{1+e^{u-\lambda a}}\,du \right) =$$

$$= \frac{1}{\lambda \ln(1+e^{\lambda a})} \left(\frac{\lambda^2 a^2}{4} + \int_0^{\lambda(1-a)} \frac{t}{1+e^t}\,dt \right).$$

This completes the proof of the Lemma ■

Proof of Theorem 1: We observe that the implication $(b) \Rightarrow (c)$ follows from (I) of Lemma 2, and that $(c) \Rightarrow (d)$ is trivial. Moreover the validity of $(d) \Rightarrow (a)$ in the case of independence is a well known result (see for example [6] Ch. III, § 4) so that we are reduced to prove just the implication $(a) \Rightarrow (b)$.

If (a) holds, then $E(\xi_n \wedge 1) \xrightarrow{n} 0$ and this also implies that $\xi_n \xrightarrow{P} 0$, namely that

$$(2) \qquad P(\xi_n > \varepsilon) = \frac{\ln(1+e^{\lambda_n(a_n-\varepsilon)})}{\ln(1+e^{\lambda_n a_n})} \xrightarrow{n} 0, \qquad \forall \varepsilon > 0 .$$

As a consequence we can show now that

$$\lim_n \lambda_n = +\infty , \qquad \lim_n a_n = 0 .$$

First of all the sequence $\{a_n\}_{n\in N}$ is certainly bounded: indeed, if $\{a_n\}_{n\in N}$ is not bounded it would contain a subsequence tending to $+\infty$ and we could extract an infinite set H of integers such that the sequence $\{\lambda_n a_n\}_{n\in H}$ has a (finite or infinite) limit. If then $\varepsilon > 0$, we would have $\lambda_n(a_n - \varepsilon) \sim \lambda_n a_n$ for $n \to \infty$ in H, so that it would follow from (2) that

$$\lim_{n\in H} P(\xi_n > \varepsilon) = 1$$

which contradicts the convergence in probability $\xi_n \xrightarrow{P} 0$. Now, if $\{\lambda_n\}_{n\in N}$ does not tend to $+\infty$ we could find an infinite set of integers J such that both the subsequences $\{a_n\}_{n\in J}$ and $\{\lambda_n\}_{n\in J}$ admit a finite limit, respectively a and λ. If then $\varepsilon > 0$, we would have from (2) that

$$\lim_{n \in J} P(\xi_n > \varepsilon) = \frac{\ln(1 + e^{\lambda(a-\varepsilon)})}{\ln(1 + e^{\lambda a})} > 0$$

which again contradicts the convergence $\xi_n \xrightarrow{P} 0$. Finally, if the (bounded) sequence $\{a_n\}_{n \in N}$ is not infinitesimal we could always find an infinite set of integers K such that the subsequence $(a_n)_{n \in K}$ converges toward a finite number $a > 0$. Hence, if $0 < \varepsilon < a$, from (2) and from the fact that $\lambda_n \xrightarrow{n} +\infty$ we could get

$$\lim_{n \in K} P(\xi_n > \varepsilon) = \lim_{n \in K} \frac{\lambda_n(a_n - \varepsilon)}{\lambda_n a_n} = \frac{a - \varepsilon}{a} > 0\,,$$

still in contradiction with $\xi_n \xrightarrow{P} 0$.

Let us take now a real number $q > 0$ such that

$$\text{(3)} \qquad \ln(1 + e^x) < 2x\,, \qquad \forall x > q$$

and define

$$L = \{n \in N:\ \lambda_n a_n \leq q\}\,, \qquad M = \{n \in N:\ \lambda_n a_n > q\}\,, \qquad (L \cup M = N)\,.$$

Since

$$\sum_{n \in N} (a_n + \lambda_n^{-1}) = \sum_{n \in L} (a_n + \lambda_n^{-1}) + \sum_{n \in M} (a_n + \lambda_n^{-1}) \leq$$

$$\leq (1 + q^{-1}) \sum_{n \in M} a_n + (1 + q) \sum_{n \in L} \lambda_n^{-1}$$

to prove (*b*) we must only show that

$$\sum_{n \in M} a_n < +\infty\,, \qquad \sum_{n \in L} \lambda_n^{-1} < +\infty\,.$$

The first statement follows from the remark that, since $a_n \xrightarrow{n} 0$, we can always suppose, without loss of generality, that $a_n < 1$ for $n \in N$ so that from (3) and (II) of Lemma 2 we have

$$E(\xi_n \wedge 1) \geq \frac{1}{\lambda_n}\,\frac{1}{2\lambda_n a_n}\,\frac{\lambda_n^2 a_n^2}{4} = \frac{a_n}{8}\,.$$

To prove the second limitation, since $\lambda_n(1 - a_n) \xrightarrow{n} +\infty$, we should just remark that for $n \to \infty$ in L (with L supposed an infinite set) it follows from (II) of Lemma 2 that

$$E(\xi_n \wedge 1) \geq \frac{1}{\lambda_n \ln(1 + e^q)} \int_0^{\lambda_n(1-a_n)} \frac{t}{1 + e^t}\,dt \sim \frac{1}{\lambda_n \ln(1 + e^q)} \int_0^{+\infty} \frac{t}{1 + e^t}\,dt\,.$$

This completes the proof of the Theorem. ∎

This Theorem shows that it will always be possible to have (P-*a.s.*) convergent trigonometric series when the sequence of the amplitudes $\{\xi_n\}_{n\in N}$ is constituted of independent Fermi random variables with a suitable choice of their parameters.

3. - Estimates for expectation and variance

We will prove now a few results about the expectation and the variance of a Fermi random variable.

Proposition 3: *If ξ is a Fermi random variable, we have*

$$E\xi = a + \frac{1}{\lambda \ln(1+e^{\lambda a})}\left(\frac{\pi^2}{12} - I_1(\lambda a)\right),$$

$$V\xi = \frac{1}{\lambda^2 \ln(1+e^{\lambda a})}\left[\frac{3\zeta(3)}{2} + I_2(\lambda a) - \frac{1}{\ln(1+e^{\lambda a})}\left(\frac{\pi^2}{12} - I_1(\lambda a)\right)^2\right],$$

where

$$I_1(x) = \int_0^x \frac{t}{1+e^{-t}}\,dt, \quad I_2(x) = \int_0^x \frac{t^2}{1+e^{-t}}\,dt, \quad \zeta(3) = \sum_{k=1}^{\infty}\frac{1}{k^3},$$

and $\zeta(\cdot)$ is the Riemann ζ-function.

Proof: Since from the normalization integral and the change of variable $t = \lambda(x-a)$ we have that

$$1 = \frac{\lambda}{\ln(1+e^{\lambda a})}\int_0^{\infty}\frac{dx}{1+e^{\lambda(x-a)}} = \frac{1}{\ln(1+e^{\lambda a})}\int_{-\lambda a}^{\infty}\frac{dt}{1+e^t}$$

we have also that

$$E\xi = \frac{\lambda}{\ln(1+e^{\lambda a})}\int_0^{\infty}\frac{x}{1+e^{\lambda(x-a)}}\,dx = \frac{\lambda}{\ln(1+e^{\lambda a})}\left(\frac{a}{\lambda}\int_{-\lambda a}^{\infty}\frac{dt}{1+e^t} + \frac{1}{\lambda^2}\int_{-\lambda a}^{\infty}\frac{t}{1+e^t}\,dt\right) =$$

$$= a + \frac{1}{\lambda \ln(1+e^{\lambda a})}\left(\int_0^{\infty}\frac{t}{1+e^t}\,dt + \int_{-\lambda a}^{0}\frac{t}{1+e^t}\,dt\right).$$

Moreover, since (see for example [5], formula 3.411.3)

$$\int_0^{\infty}\frac{x^{\nu-1}}{1+e^x}\,dx = (1-2^{1-\nu})\,\Gamma(\nu)\,\zeta(\nu); \quad \operatorname{Re}\nu > 0 \tag{4}$$

and since (see [5], formula 9.542.1 and 9.71) $\Gamma(2) = 1$ and $\zeta(2) = \pi^2|B_2| = \pi^2/6$, we get the first relation

$$E\xi = a + \frac{1}{\lambda \ln(1+e^{\lambda a})}\left(\frac{\pi^2}{12} - I_1(\lambda a)\right).$$

With the same change of variables, and taking (4) into account we have also that

$$E\xi^2 = \frac{\lambda}{\ln(1+e^{\lambda a})}\int_0^\infty \frac{x^2}{1+e^{\lambda(x-a)}}\,dx =$$

$$= \frac{\lambda}{\ln(1+e^{\lambda a})}\left(\frac{a^2}{\lambda}\int_{-\lambda a}^\infty \frac{dt}{1+e^t} + \frac{2a}{\lambda^2}\int_{-\lambda a}^\infty \frac{t}{1+e^t}\,dt + \frac{1}{\lambda^3}\int_{-\lambda a}^\infty \frac{t^2}{1+e^t}\,dt\right) =$$

$$= a^2 + \frac{1}{\lambda \ln(1+e^{\lambda a})}\left(\frac{a\pi^2}{6} - 2aI_1(\lambda a) + \frac{3\zeta(3)}{2\lambda} + \frac{1}{\lambda}I_2(\lambda a)\right).$$

Hence the variance can also be calculated as

$$V\xi = E\xi^2 - (E\xi)^2 = \frac{1}{\lambda^2 \ln(1+e^{\lambda a})}\left[\frac{3\zeta(3)}{2} + I_2(\lambda a) - \frac{1}{\ln(1+e^{\lambda a})}\left(\frac{\pi^2}{12} - I_1(\lambda a)\right)^2\right].$$

This completes the proof of the proposition. ■

Since it is not possible to express the functions $I_1(x)$ and $I_2(x)$ in terms of elementary functions, it will be useful to obtain some estimates:

PROPOSITION 4: *For the functions $I_1(x)$ and $I_2(x)$ defined in the Proposition 1, the following inequalities hold $\forall x \geq 0$:*

$$-(1 - e^{-x} - xe^{-x}) < I_1(x) - \frac{x^2}{2} < 0,$$

$$-(2 - 2e^{-x} - 2xe^{-x} - x^2e^{-x}) < I_2(x) - \frac{x^3}{3} < 0;$$

moreover $\forall n \geq 1$ and $\forall x \geq 0$, even the following inequalities hold

$$\sum_{k=1}^{2n+1}(-1)^k \frac{1 - e^{-kx} - kxe^{-kx}}{k^2} < I_1(x) - \frac{x^2}{2} < \sum_{k=1}^{2n}(-1)^k \frac{1 - e^{-kx} - kxe^{-kx}}{k^2},$$

$$\sum_{k=1}^{2n+1}(-1)^k \frac{2 - 2e^{-kx} - 2kxe^{-kx} - k^2x^2e^{-kx}}{k^3} < I_2(x) - \frac{x^3}{3}$$

$$< \sum_{k=1}^{2n}(-1)^k \frac{2 - 2e^{-kx} - 2kxe^{-kx} - k^2x^2e^{-kx}}{k^3}.$$

PROOF: Since it is easy to see that $\forall t \geqslant 0$ and for $n = 0, 1, 2 \ldots$ we have

$$(1+e^{-t}) \sum_{k=0}^{2n} (-1)^k e^{-kt} = 1 + e^{-(2n+1)t} > 1,$$

$$(1+e^{-t}) \sum_{k=0}^{2n+1} (-1)^k e^{-kt} = 1 - e^{-2(n+1)t} < 1,$$

the following inequalities hold $\forall x \geqslant 0$ and for $n = 0, 1, 2, \ldots$:

$$\sum_{k=0}^{2n+1} (-1)^k \int_0^x t e^{-kt}\, dt < I_1(x) < \sum_{k=0}^{2n} (-1)^k \int_0^x t e^{-kt}\, dt,$$

$$\sum_{k=0}^{2n+1} (-1)^k \int_0^x t^2 e^{-kt}\, dt < I_2(x) < \sum_{k=0}^{2n} (-1)^k \int_0^x t^2 e^{-kt}\, dt.$$

Then the inequalities of the Proposition follow by elementary integration. ∎

A consequence of Proposition 4 is the following Corollary which gives the expectation of a Fermi random variable in terms of a series expansion (a similar result can be deduced for the variance):

COROLLARY 5: *If ξ is a Fermi random variable then its expectation has the form* $\mathbf{E}\,\xi = \lambda^{-1} g(a\lambda)$ *where*

$$g(x) = x + \frac{1}{\ln(1+e^x)} \left(\frac{\pi^2}{6} - \frac{x^2}{2} + \sum_{k=1}^{\infty} (-1)^k \frac{1+kx}{k^2} e^{-kx} \right).$$

PROOF: Since from Proposition 4 the quantity $I_1(x) - x^2/2$ always falls in between the even and odd terms of the sequence of the partial sums of a convergent series, it is immediate to recognize that it coincides with the sum of that series:

$$I_1(x) - \frac{x^2}{2} = \sum_{k=1}^{\infty} (-1)^k \frac{1 - e^{-kx} - kxe^{-kx}}{k^2}.$$

The result then immediately follows from Proposition 3 and the fact that (see for example [5], formulae 9.522.2, 9.542.1 and 9.71)

$$\sum_{k=1}^{\infty} (-1)^k \frac{1}{k^2} = -\frac{1}{2}\zeta(2) = -\frac{\pi^2}{12}.$$

This completes the proof of the Corollary. ∎

We remark that as a consequence of the Corollary 5 we could immediately deduce the relation $g(x) \sim x/2$ for $x \to +\infty$ that was used in the proof of the statement (I) of the Lemma 2. Moreover it is also possible to use the results of Proposition 3 and the inequalities of Proposition 4 in order to get upper bounds for $E\,\xi$ and $V\,\xi$. As an example we will only make use of the simplest among the inequalities of Proposition 4, even if in this way we will get only some very rough estimates. More precise results can be obtained by means of the other inequalities.

Corollary 6: *If ξ is a Fermi random variable the following inequalities are always verified*

$$E\,\xi < a\left(\frac{1}{2} + \frac{2}{\lambda^2 a^2}\right), \quad V\,\xi < a^2\left(\frac{1}{12} + \frac{2}{\lambda^2 a^2} + \frac{2}{\lambda^3 a^3}\right).$$

Proof: Since $\ln(1+e^s) > \ln e^s = s$, $\forall s > 0$, and (see for example [1], pag. 811) $\zeta(3) < 4/3$, taking into account the previous propositions we have:

$$E\,\xi = a + \frac{1}{\lambda \ln(1+e^{\lambda a})}\left(\frac{\pi^2}{12} - I_1(\lambda a)\right) <$$

$$< a\left[1 + \frac{1}{\lambda a \ln(1+e^{\lambda a})}\left(\frac{\pi^2}{12} - \frac{\lambda^2 a^2}{2} + 1 - e^{-\lambda a} - \lambda a e^{-\lambda a}\right)\right] <$$

$$< a\left[1 + \frac{1}{\lambda a \ln(1+e^{\lambda a})}\left(2 - \frac{\lambda^2 a^2}{2}\right)\right] < a\left[1 + \frac{1}{\lambda^2 a^2}\left(2 - \frac{\lambda^2 a^2}{2}\right)\right] = a\left(\frac{1}{2} + \frac{2}{\lambda^2 a^2}\right);$$

$$V\,\xi = \frac{1}{\lambda^2 \ln(1+e^{\lambda a})}\left[\frac{3\,\zeta(3)}{2} + I_2(\lambda a) - \frac{1}{\ln(1+e^{\lambda a})}\left(\frac{\pi^2}{12} - I_1(\lambda a)\right)^2\right] <$$

$$< \frac{1}{\lambda^2 \ln(1+e^{\lambda a})}\left[\frac{3\,\zeta(3)}{2} + \frac{\lambda^3 a^3}{3} - \frac{1}{\ln(1+e^{\lambda a})}\left(\frac{\lambda^4 a^4}{4} - \left(1 + \frac{\pi^2}{12}\right)\lambda^2 a^2\right)\right] <$$

$$< \frac{1}{\lambda^3 a}\left[2 + \frac{\lambda^3 a^3}{3} - \frac{1}{\lambda a}\left(\frac{\lambda^4 a^4}{4} - 2\lambda^2 a^2\right)\right] = a^2\left(\frac{1}{12} + \frac{2}{\lambda^2 a^2} + \frac{2}{\lambda^3 a^3}\right).$$

This completes the proof of the Corollary. ■

Acknowledgements. The author wants to thank Prof. Luca Pratelli for invaluable discussions and suggestions.

REFERENCES

[1] M. Abramowitz - I. Stegun, *Handbook of Mathematical Functions*, Dover, New York (1972).

[2] J. Backus, *The Acoustical Foundations of Music*, Norton, New York (1977).

[3] N. Cufaro Petroni - F. Degrassi - G. Pasquariello, *Detection of pitch in random acoustic signals by neural networks*, J. New Music Research, 23 (1994), 369-399.
[4] N. Cufaro petroni - M. Guarino - G. Pasquariello, *Neural networks and pitch detection in random acoustic signals*, Proceedings of the International Conference on Engineering Applications of Neural Networks, Helsinki (August 1995), p. 389.
[5] I. S. Gradshteyn - I. M. Ryzhik, *Table of Integrals, Series and Products*, Academic Press, San Diego (1980).
[6] J.P. Kahane, *Some Random Series of Functions*, D.C. Heath, Lexington (1968).
[7] L. Landau - E. Lifchitz, *Physique statistique*, MIR, Moscou (1967).