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Quasilinear Parabolic Variational Inequalities with Discontinuous Coefficients (**)

SUMMARY. — We obtain an existence theorem for the problem (*) assuming that the coefficients $a_{i,j}(x,t,s)$ satisfy hypotheses weaker than the continuity with respect to the variable s.

Disequazioni variazionali quasilineari di tipo parabolico con coefficienti discontinui

SUNTO. — Si ottiene un teorema di esistenza per il problema (*) supponendo che i coefficienti $a_{i,j}(x,t,s)$ verifichino ipotesi più deboli della continuità rispetto alla variabile s.

1. - INTRODUCTION

Certain free boundary problems related to diffusion processes lead to the evolution variational inequality

$$(*)\qquad \int_Q \sum_{i,j}^{m} a_{i,j}(x,t,u)\frac{\partial u}{\partial x_j}\frac{\partial(v-u)}{\partial x_i}\,dx\,dt + \left(\frac{\partial v}{\partial t}, v-u\right) \geq (f, v-u) - \frac{1}{2}\int_Q [v(x,0)-u_0(x)]^2\,dx$$

for all v belonging to a closed convex non empty subset of $L^2(0,T;H^1(\Omega)) \cap L^{2,\infty}(Q)$, in an open cylinder $Q = \Omega \times]0,T[$, when the coefficients $a_{i,j}(x,t,s)$ are not functions of Carathéodory type but may depend discontinuously on s. The purpose

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of this report is to give an existence result for a weak solution of the above variational inequality assuming that the coefficients satisfy some usual conditions with respect to the variables (x, t) and an hypothesis weaker than the continuity with respect to the variable s. Precisely, we make the following assumptions:

Let Ω be an open bounded subset of $\mathbb{R}^m$ and T a positive number:

(a_1) *the functions $a_{i,j}(x, t, s)$ are Borel measurable in $Q \times \mathbb{R}$, bounded and such that*

$$(1.0) \qquad \sum_{i,j=1}^{m} a_{i,j}(x, t, s)\,\xi_i \xi_j \geq \lambda \sum_{i=1}^{m} \xi_i^2 \quad \textit{for every } (x, t, s, \xi) \in Q \times \mathbb{R} \times \mathbb{R}^m$$

with $\lambda > 0$;

(a_2) *for almost every t in $]0, T[$, for every $\varepsilon > 0$ there exists a compact subset $K_\varepsilon(t) \subset \subset \Omega$ with $\operatorname{meas}(\Omega \setminus K_\varepsilon(t)) < \varepsilon$ such that, for every $r > 0$, the functions of the family $\{a_{i,j}(\cdot, t, s)\}_{|s| \leq r,\, i,j = 1, 2, \ldots, m}$ are equicontinuous on $K_\varepsilon(t)$.*

Clearly, the main difficult in applying classical results is the presence of integral functionals of the type

$$(**) \qquad F(u) = \int_\Omega f(x, u, Du)\,dx$$

where $f(x, s, z)$ is possibly discontinuous on s. When $h(s, z)$: $\mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ is a discontinuous function on s, some properties concerning integral functionals of the form

$$G(u) = \int_\Omega h(u, Du)\,dx$$

have been studied in [DBD] and next generalized in [A] for the case (**), by introducing a condition similar to assumption (a_2).

Let us consider the following closed convex set

$$k = \{u(x) \in H^1(\Omega)\colon u(x) \geq 0 \text{ on } \partial\Omega\}$$

and let us set

$$K = \{u(x, t) \in L^2(0, T; H^1(\Omega))\colon u(x, t) \in k \text{ a.e. on }]0, T[\}.$$

In the section 2 we will show the following:

THEOREM 1.1: *If f and u_0 are functions belonging to $L^2(0, T; (H^1(\Omega))^*)$ and to k respectively and the assumptions (a_1), (a_2), hold, then there exists a function $u(x, t) \in$*

$\in K \cap C^0([0,T]; L^2(\Omega))$ *such that*

$$\text{(1.1)} \qquad \int_Q \sum_1^m{}_{i,j} a_{i,j}(x,t,u)\,\frac{\partial u}{\partial x_j}\,\frac{\partial(v-u)}{\partial x_i}\,dx\,dt\, +$$

$$+ \int_0^T \left\langle \frac{\partial v}{\partial t}, v-u \right\rangle dt \geq \int_0^T \langle f, v-u\rangle - \frac{1}{2}\int_\Omega [v(x,0)-u_0]^2\,dx$$

for all $v \in K$ *such that* $\partial v/\partial t \in L^2(0,T; (H^1(\Omega))^*)$. *Moreover* $u(x,0) = u_0$ *on* Ω.

A similar question has been studied in [BB] for quasilinear elliptic equations of the form:

$$\begin{cases} -\operatorname{div}(A(x,u)Du) = f, \\ u \in H_0^1(\Omega); \end{cases}$$

here the matrix $A(x,s)$ of the coefficients can be discontinuous. Next, in the more general case of quasilinear elliptic variational inequalities, this author obtained in [B] some results assuming a degenerate ellipticity condition, that is the ellipticity constant λ is a positive function of x.

Finally we observe that, for the solutions of (1.1), the maximum principle shown in [M$_1$] holds.

2. - Proof of Theorem 1.1

By hypothesis (a_1), for all t in $]0,T[$, the operator $A(t)$ set by putting for every $u, v \in H^1(\Omega)$

$$\langle A(t)u, v\rangle = \int_\Omega \sum_1^m{}_{ij} a_{i,j}(x,t,u(x))\,\frac{\partial u(x)}{\partial x_j}\,\frac{\partial v(x)}{\partial x_i}\,dx\,,$$

is well defined between $H^1(\Omega)$ and $(H^1(\Omega))^*$; moreover it is easy to check that

$$\langle A(t)u, u\rangle \geq \lambda\|u\|_1^2 - \lambda|u|_2^2,$$

$$\|A(t)u\|_* \leq \Lambda(1+|u|_2)\|u\|_1 \text{ (}^1\text{)}$$

for every $u \in H^1(\Omega)$ and $t \in]0,T[$; here $\Lambda = \sup_{Q\times\mathbb{R}} \sum_1^m{}_{i,j} |a_{i,j}(x,t,s)|$.

(1) We write $\|\cdot\|_1$, $|\cdot|_2$, $\|\cdot\|_*$, to denote the norm in $H^1(\Omega)$, $L^2(\Omega)$, $(H^1(\Omega))^*$, respectively.

In the sequel, to shorten notation, we will say that a sequence $\{u_h\} \subset L^2(0, T; H^1(\Omega))$ ($h \in \mathbb{N}$) satisfies the condition (Π) if:

$$u_h \rightharpoonup u \quad \text{in } L^2(0, T; H^1(\Omega)).$$

$$u_h \to u \quad \text{in } L^1(\Omega).$$

We have divided the proof into a sequence of lemmas.

LEMMA 2.1: *If the assumptions* (a_1), (a_2), *hold, and if the sequence* $\{u_h\} \subset L^2(0, T; H^1(\Omega))$ ($h \in \mathbb{N}$) *satisfies the condition* (Π), *then*

$$\liminf_{h\to\infty} \int_0^T \langle A(t)u_h, u_h\rangle\, dt \geq \int_0^T \langle A(t)u, u\rangle\, dt\,. \tag{2.1}$$

PROOF: In Th. 4.6 of [A] it has been shown that in the assumptions (a_1), (a_2) the functional

$$\omega \to F_t(\omega) = \int_\Omega \sum_{i,j}^m a_{i,j}(x, t, \omega)\,\frac{\partial\omega}{\partial x_i}\,\frac{\partial\omega}{\partial x_j}\,dx$$

is sequentially weakly lower semicontinuous on $W^{1,1}(\Omega)$, a.e. $t \in]0, T[$.

Also, we claim that if ω_h, $\omega \in W^{1,1}(\Omega)$ ($h \in \mathbb{N}$) are such that $\omega_h \to \omega$ in $L^1(\Omega)$, then

$$\liminf_{h\to\infty} F_t(\omega_h) \geq F_t(\omega), \qquad \text{a.e. } t \in]0, T[\,. \tag{2.2}$$

Indeed, if $\liminf_{h\to\infty} F_t(\omega_h) < +\infty$ (the other case is obvious) then there exists a subsequence $\{v_k\}$ such that

$$\lim_{k\to\infty} F_t(v_k) = \liminf_{h\to\infty} F_t(\omega_h).$$

Hence, according to (1.0) and Dunford-Pettis Theorem, we have that

$$\nabla_x v_k \rightharpoonup \bar{\omega} \quad \text{in } L^1(\Omega; \mathbb{R}^m)\,(^2).$$

We proceed to show that

$$\bar{\omega} \equiv \nabla_x \Leftrightarrow \bar{\omega}_i = \frac{\partial\omega}{\partial x_i} \quad \text{for every } i = 1, 2, \ldots, m\,. \tag{2.3}$$

(2) The symbol $\nabla_x v$ stands for $(\partial v/x_1, \ldots, \partial v/\partial x_m)$.

We see at once that

$$(2.4)\qquad \lim_{k\to\infty}\int_\Omega v_k \frac{\partial\varphi}{\partial x_i}\,dx = \int_\Omega \omega \frac{\partial\varphi}{\partial x_i}\,dx \quad \text{for every } \varphi \in C_0^\infty(\Omega),$$

which is clear from convergence of v_k to ω in $L^1(\Omega)$.

In addition, we have:

$$(2.5)\qquad \lim_{k\to\infty}\int_\Omega \frac{\partial v_k}{\partial x_i}\varphi\,dx = \int_\Omega \bar{\omega}_i \varphi\,dx \quad \text{for every } \varphi \in C_0^\infty(\Omega).$$

By the definition of derivative in distributional sense, on account of (2.4) and (2.5), we obtain (2.3).

Consequently $v_k \rightharpoonup \omega$ in $W^{1,1}(\Omega)$ and finally, using the above mentioned property of functional F_t, we deduce that

$$\liminf_{k\to\infty} F_t(v_k) \geq F_t(\omega).$$

To achieve (2.2), let us suppose that

$$(2.6)\qquad \liminf_{h\to\infty} F_t(\omega_h) < F_t(\omega);$$

then we can find a subsequence $\{\omega_{h_k}\}$ such that

$$\lim_{k\to\infty} F_t(\omega_{h_k}) = \liminf_{h\to\infty} F_t(\omega_h) < F_t(\omega).$$

Since $\omega_{h_k} \to \omega$ in $L^1(\Omega)$, using the above argument, we can get a subsequence $\{\omega_{h_{k_r}}\}$ such that

$$\liminf_{r\to\infty} F_t(\omega_{h_{k_r}}) \geq F_t(\omega)$$

and hence

$$F_t(\omega) \leq \liminf_{r\to\infty} F_t(\omega_{h_{k_r}}) = \lim_{k\to\infty} F_t(\omega_{h_k}) < F_t(\omega)$$

from which it follows that (2.6) can not occur.

Let be now $\{u_h\}$ a sequence of functions belonging to $L^2(0,T;H^1(\Omega))$ and satisfying the condition (Π).

We observe that $u_h, u \in W^{1,1}(\Omega)$ and $u_h \to u$ in $L^1(\Omega)$ a.e. $t\in]0,T[$; therefore setting in (2.2) $\omega_h = u_h$, $\omega = u$ we deduce that:

$$\liminf_{h\to\infty} \langle A(t)u_h, u_h\rangle \geq \langle A(t)u, u\rangle \quad \text{a.e. } t\in]0,T[.$$

By means of the Fatou's Lemma we get (2.1). ∎

Lemma 2.2: *If the assumptions* (a_1), (a_2), *hold, and if the sequence* $\{u_h\} \subset$ $\subset L^2(0,T;H^1(\Omega))$ $(h\in \mathbb{N})$ *satisfies the condition* (Π), *then, for every* $i,j = 1, 2, \ldots, m$

and for all $\varphi \in C_0^\infty(\Omega)$,

$$\liminf_{h \to \infty} \int_Q a_{i,j}(x, t, u_h) \frac{\partial u_h}{\partial x_j} \varphi \, dx \, dt \geq \int_Q a_{i,j}(x, t, u) \frac{\partial u}{\partial x_j} \varphi \, dx \, dt . \tag{2.7}$$

PROOF: Define for every $m \in \mathrm{N}$ and every $(x, t, s, z) \in Q \times \mathrm{R} \times \mathrm{R}^m$

$$f_n(x, t, s, z) = \max[a_{i,j}(x, t, s) z_j \varphi(x, t), -n] + n ,$$

$$f_{n,t}(x, s, z) = f_n(x, t, s, z), \qquad g_{n,t}(x, s, z) = f_{n,t}^{*}(x, s, z) + \gamma \sum_i^m z_i^2 \quad (\gamma > 0).$$

It is easy to check that $g_{n,t}(x, s, z)$ satisfies the hypotheses of Th. 4.15 of [A] for every $n \in \mathrm{N}$ and a.e. $t \in]0, T[$, consequently the functional

$$\omega \to H_{n,t}(\omega) = \int_\Omega g_{n,t}(x, \omega, \nabla_x \omega) \, dx$$

is sequentially weakly lower semicontinuous on $W^{1,1}(\Omega)$, for every $n \in \mathrm{N}$ and a.e. $t \in]0, T[$.

The same arguments used in the proof of Lemma 2.1 can be applied to conclude that if ω_h, $\omega \in W^{1,1}(\Omega)$ are such that $\omega_h \to \omega$ in $L^1(\Omega)$, then

$$\liminf_{h \to \infty} \int_\Omega g_{n,t}(x, \omega_h, \nabla_x \omega_h) \, dx \geq \int_\Omega g_{n,t}(x, \omega, \nabla_x \omega) \, dx .$$

Therefore, if $\{u_h\} \subset L^2(0, T; H^1(\Omega))$ $(h \in \mathrm{N})$ satisfies the condition (Π) we obtain

$$\liminf_{h \to \infty} \int_\Omega \{ f_{n,t}(x, u_h, \nabla_x u_h) + \gamma |\nabla_x u_h|^2 \} \, dx \geq$$

$$\geq \int_\Omega \{ f_{n,t}(x, u, \nabla_x u) + \gamma |\nabla_x u|^2 \} \, dx \qquad \text{a.e. } t \in]0, T[.$$

By integration on $]0, T[$, taking into account of Fatou's Lemma, we can assert that

$$\liminf_{h \to \infty} \int_Q \{ f_n(x, t, u_h, \nabla_x u_h) + \gamma |\nabla_x u_h|^2 \} \, dx \, dt \geq \int_Q \{ f_n(x, t, u, \nabla_x u) + \gamma |\nabla_x u|^2 \} \, dx \, dt .$$

Since

$$\int_Q |\nabla_x u_h|^2 \, dx \, dt \leq C \quad \text{for every } h \in \mathrm{N} ,$$

by weak convergence of u_h to u in $L^2(0,T;H^1(\Omega))$, for all $\gamma>0$ we have

$$\liminf_{h\to\infty}\left\{\int_Q f_n(x,t,u_h,\nabla_x u_h)\,dx\,dt\right\}+C\gamma\geq$$

$$\geq \liminf_{h\to\infty}\left\{\int_Q f_n(x,t,u_h,\nabla_x u_h)\,dx\,dt+\gamma\int_Q|\nabla_x u_h|^2\,dx\,dt\right\}\geq$$

$$\geq\int_Q\{f_n(x,t,u,\nabla_x u)+\gamma|\nabla_x u|^2\}\,dx\,dt\geq\int_Q\{f_n(x,t,u,\nabla_x u)\,dx\,dt.$$

Letting $\gamma\to 0$, we get:

$$\liminf_{h\to\infty}\int_Q f_n(x,t,u_h,\nabla_x u_h)\,dx\,dt\geq\int_Q\{f_n(x,t,u,\nabla_x u)\,dx\,dt$$

and immediatly

$$\text{(2.8)}\quad \liminf_{h\to\infty}\int_Q\max\left[a_{i,j}(x,t,u_h)\frac{\partial u_h}{\partial x_j}\varphi,-n\right]dx\,dt\geq$$

$$\geq\int_Q\max\left[a_{i,j}(x,t,u)\frac{\partial u}{\partial x_j}\varphi,-n\right]dx\,dt\,.$$

Let us now introduce the sets

$$Q_{1,n}=\left\{(x,t)\in Q\colon \varphi(x,t)a_{i,j}(x,t,u_h(x,t))\frac{\partial u_h(x,t)}{\partial x_j}<-n\right\},$$

$$Q_{2n}=Q\setminus Q_{1,n}\,.$$

It easily seen that:

$$\text{(2.9)}\quad \int_Q\max\left[a_{i,j}(x,t,u_h)\frac{\partial u_h}{\partial x_j}\varphi,-n\right]dx\,dt\leq$$

$$\leq\int_Q a_{i,j}(x,t,u_h)\frac{\partial u_h}{\partial x_j}\varphi\,dx\,dt+A\int_{Q\setminus Q_{1,n}}|\varphi|\left|\frac{\partial u_h}{\partial x_j}\right|dx\,dt\leq$$

$$\leq\int_Q a_{i,j}(x,t,u_h)\frac{\partial u_h}{\partial x_j}\varphi\,dx\,dt+\bar c[\operatorname{meas}(Q\setminus Q_{2,n})]^{1/2}\,.$$

It results

$$\text{meas}(Q \setminus Q_{2,n}) \leq \text{meas}\left\{(x,t) \in Q\colon \Lambda|\varphi| \left|\frac{\partial u_h}{\partial x_j}\right| > n\right\} \leq \frac{L}{n} \text{ (}^3\text{)} \tag{2.10}$$

consequently, by (2.8), (2.9) and (2.10), we get

$$\int_Q a_{i,j}(x,t,u)\frac{\partial u}{\partial x_j}\varphi\,dx\,dt \leq \liminf_{h\to\infty}\int_Q \max\left[a_{i,j}(x,t,u_h)\frac{\partial u_h}{\partial x_j}\varphi, -n\right]dx\,dt \leq$$

$$\leq \liminf_{h\to\infty}\int_Q a_{i,j}(x,t,u_h)\frac{\partial u_h}{\partial x_j}\varphi\,dx\,dt + \frac{c}{\sqrt{n}}, \qquad \text{for every } n \in \mathbb{N}.$$

From this, letting $n \to \infty$, we have (2.7). ■

Corollary 2.3: *Under the same hypothesis of previous Lemmas, we can assert that for every* $i, j = 1, 2, \ldots, m$ *and for all* $\varphi \in L^2(0,T; H^1(\Omega))$:

$$\lim_{h\to\infty}\int_Q a_{i,j}(x,t,u_h)\frac{\partial u_h}{\partial x_j}\frac{\partial \varphi}{\partial x_i}dx\,dt = \int_Q a_{i,j}(x,t,u)\frac{\partial u}{\partial x_j}\frac{\partial \varphi}{\partial x_i}dx\,dt.$$

The assertion it is evident from (2.7) interchanging $a_{i,j}(x,t,s)$ and $-a_{ij}(x,t,s)$ and using the density of $C_0^\infty(Q)$ into $L^2(Q)$. ■

Finally, observing that if $\{u_h\} \subset L^2(0,T; H^1(\Omega))$ $(h \in \mathbb{N})$ satisfies the condition (Π) it results:

$$\liminf_{h\to\infty}\int_0^T \langle A(t)u_h, u_h - w\rangle dt =$$

$$= \liminf_{h\to\infty}\int_0^T \langle A(t)u_h, u_h\rangle dt - \lim_{h\to\infty}\int_Q \sum_{i,j}^m a_{i,j}(x,t,u_h)\frac{\partial u_h}{\partial x_j}\frac{\partial w}{\partial x_i}dx\,dt \geq$$

$$\geq \int_0^T \langle A(t)u, u\rangle dt - \int_Q \sum_{i,j}^m a_{i,j}(x,t,u_h)\frac{\partial u}{\partial x_j}\frac{\partial w}{\partial x_i}dx\,dt =$$

$$= \int_0^T \langle A(t)u, u - w\rangle dt \qquad \text{for all } w \in L^2(0,T; H^1(\Omega))$$

the existence of a solution is a consequence of the following

(3) See, for more details, [B], p. 60.

THEOREM: *Let* $\{\chi(t)\}$ $(t \in]0, T[)$ *be a family of operators from* $H^1(\Omega)$ *into* $(H^1(\Omega))^*$ *such that:*

a) $\|\chi(t)u\|_* \leq c_1(1 + \|u\|_1)$ *for every* $t \in]0, T[$ *and all* $u \in H^1(\Omega)$ $(c_1 = \text{const.} \geq 0)$;

b) the function $t \to \chi(t)u(t)$ *is strongly measurable on* $]0, T[$ *for all* $u \in L^2(0, T; H^1(\Omega))$; *moreover*

c) $\|\chi(t)u\|_* \leq c_2(1 + |u|_2)\|u\|_1$ *for every* $t \in]0, T[$ *and all* $u \in H^1(\Omega)$ $(c_2 = \text{const.})$;

d) $\langle \chi(t)u, u\rangle \geq c_3\|u\|_1^2 - c_4|u|_2^2 - c_5(t)$ *for every* $t \in]0, T[$ *and all* $u \in H^1(\Omega)$ $(c_3 = \text{const.} > 0,\ c_4 = \text{const.} \geq 0,\ c_5(t) \in L^1(0, T))$;

e) for every $\{v_n\} \subset L^2(0, T; H^1(\Omega)) \cap L^{2,\infty}(Q)$ $(n \in \mathbb{N})$ *with* $v_n \rightharpoonup v$ *in* $L^2(0, T; H^1(\Omega))$, $v_n \to v$ *in* $L^2(Q)$, $\operatorname{ess\,sup}_{]0,T[} |v_n(t)|_2 \leq \text{const.}$ *and*

$$\limsup_{n\to\infty} \int_0^T \langle \chi(t)v_n, v_n - v\rangle dt \leq 0$$

one has

$$\int_0^T \langle \chi(t)v, v - w\rangle dt \leq \liminf_{n\to\infty} \int_0^T \langle \chi(t)v_n, v_n - w\rangle dt$$

for every $w \in L^2(0, T; H^1(\Omega)) \cap L^{2,\infty}(Q)$.

Hence, if f and u_0 *are functions belonging to* $L^2(0, T; (H^1(\Omega))^*)$ *and to k respectively, then there exists a function* $u \in L^2(0, T; H^1(\Omega))$:

$$\frac{\partial u}{\partial t} \in L^2(0, T; H^{-1}(\Omega)),$$

$$\int_0^T \left\langle \frac{\partial v}{\partial t} + \chi(t)u - f, v - u\right\rangle dt \geq -\frac{1}{2}\int_\Omega [v(0) - u_0]^2 dx$$

for every $v \in L^2(0, T; H^1(\Omega))$ *with* $\dfrac{\partial v}{\partial t} \in L^2(0, T; (H^1(\Omega))^*)$,

$$u(0) = u_0 .$$

(See [N, Th. 3.3 chap.2, p. 61], [M], for a more general proof of this theorem.) ■

REMARK 2.4: We observe that the operator $\mathcal{B}u$, defined by putting for every $u \in L^2(0, T; H^1(\Omega)) \cap L^{2,\infty}(Q)$

$$(\mathcal{B}u, v) = \int_Q \sum_{i,j}^{m} a_{i,j}(x, t, u)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}\,dx\,dt, \qquad v \in L^2(0, T; H^1(\Omega))$$

is pseudomonotone on

$$\mathcal{W} = \left\{ w \in L^2(0, T; H^1(\Omega)) \colon \frac{\partial w}{\partial t} \in L^2(0, T; (H^1(\Omega))^*) \right\},$$

i.e. if u_n, $u \in \mathcal{W}$, $u_n \rightharpoonup u$ in $\mathcal{W}$ and

$$\limsup_{n \to \infty} \langle \mathcal{B}u_n, u_n - u \rangle \leq 0$$

then

$$\liminf_{n \to \infty} \langle \mathcal{B}u_n, u_n - v \rangle \geq \langle \mathcal{B}u, u - v \rangle \quad \text{for every } v \in \mathcal{W}\text{(4)}.$$

REMARK 2.5: It is an open question if Theorem 1.1 can be shown, as in the elliptic case, when in (1.0) the constant λ depends on x and t, more precisely $\lambda = \nu(x)\psi(t)$ with $\nu(x)$, $\psi(t)$ satisfying hypotheses like ones of [Ni].

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(4) We note that the imbedding of $\mathcal{W}$ into $L^2(Q)$ is compact.

REFERENCES

[A] L. AMBROSIO, *New lower semicontinuity results for integral functional*, Rend. Accad. Naz. Sci. XL, Mem. Mat., 11 (1987), 1-42.

[B] S. BONAFEDE, *Quasilinear degenerate elliptic variational inequalities with discontinuous coefficients*, Comment. Math. Univ. Carolinae 34, 1 (1993), 55-61.

[BB] L. BOCCARDO - G. BUTTAZZO, *Quasilinear elliptic equations with discontinuous coefficients*, Atti Acc. Lincei Rend. Fis. (8), LXXXII (1988), 21-28.

[DBD] E. DE GIORGI - G. BUTTAZZO G. - G. DAL MASO, *On the lower semicontinuity of certain integral functionals*, Atti Acc. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., 74 (1983), 274-282.

[M] A. MAUGERI, *Sulle disequazioni variazionali di tipo parabolico*, Boll. Un. Mat. Ital. (5), 17-B (1980), 492-513.

[M_1] A. MAUGERI, *Un principio di massimo per le soluzioni di alcune disequazioni variazionali di tipo parabolico*, Le Matematiche, 33 (1978), 26-31.

[N] J. NAUMANN, *Einführung in die Theorie parabolischer Variationsungleichungen*, Teubner-Texte zur Mathematik, Band 64, Leipzing (1984).

[Ni] F. NICOLOSI, *Soluzioni deboli dei problemi al contorno per operatori parabolici che possono degenerare*, Ann. di Matem. (4), 125 (1980), 135-155.

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