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Some Results on Minimal Barriers in the Sense of De Giorgi Applied to Driven Motion by Mean Curvature (**)(***)

ABSTRACT. — We prove some properties of the minimal barriers in the sense of De Giorgi for the driven mean curvature flow in codimension one. We compare the resulting evolution with an abstract evolution, and in particular with the evolution defined with the methods of Evans-Spruck, Chen-Giga-Goto, and Giga-Goto-Ishii-Sato.

Alcuni risultati sulle minime barriere secondo De Giorgi per il movimento per curvatura media con termine forzante

RIASSUNTO. — Vengono dimostrate alcune proprietà relative alle minime barriere secondo De Giorgi per il movimento secondo la curvatura media con termine forzante. Tale evoluzione viene confrontata con una evoluzione astratta e in particolare con il movimento definito con i metodi di Evans-Spruck, Chen-Giga-Goto, Giga-Goto-Ishii-Sato.

1. - INTRODUCTION

In the last few years several definitions of generalized evolution by mean curvature have been proposed in different contexts, such as geometric measure theory and the theory of viscosity solutions for parabolic equations. Such generalized approaches arise since smooth hypersurfaces evolving by mean curvature can develop singularities after a

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finite time. Among the generalized theories considered to treat the mean curvature evolution even past singularities we recall: the approach of Brakke [6], which studies the mean curvature evolution in the context of varifolds theory; the approach of Osher-Sethian [27], Evans-Spruck [17,18,19], Chen-Giga-Goto [8], Giga-Goto-Ishii-Sato [20], which consider the level sets of the solution, in the viscosity sense, of a suitable non linear parabolic partial differential equation; the solutions that can be obtained as asymptotic limits of the scaled Allen-Cahn equation [3,7,11,12,15,16]; the variational approach of Almgren-Taylor-Wang [1] and its possible generalizations by means of minimizing movements in the sense of De Giorgi [13]; the elliptic regularization method of Ilmanen [23]; the method of the minimal barriers of De Giorgi [14,2].

The aim of this paper is to show some general properties of minimal barriers applied to the driven motion by mean curvature of oriented boundaries, and to compare the resulting evolution with an abstract evolution satisfying suitable properties. In particular, the comparison between the barriers and the viscosity evolution is in the spirit of the theory of set-theoretic subsolutions of Ilmanen [23], who considered closed evolutions without driving force.

Let us briefly describe the content of the paper. In Section 2 we recall the abstract definitions of barrier and minimal barrier. In Section 3 such definitions are particularized for the driven mean curvature evolution. The minimal barrier $\mathrm{mibar}(E, \mathcal{F}_g)$ is introduced in Section 3.2, and the two lower and upper regularized minimal barriers $\mathrm{mibar}_\star(E, \mathcal{F}_g)$, $\mathrm{mibar}^\star(E, \mathcal{F}_g)$ are defined in Section 3.3. Due to the presence of the forcing term g, the equation describing the evolution is no more translation invariant; to overcome this difficulty in Section 4 we prove some results concerning the distance between barriers (see Theorem 4.1). In Section 5 we show some general properties of the minimal barriers. In particular in (5.15) we prove the equality

$$\mathrm{mibar}_\star(\mathbf{R}^n \setminus E, \mathcal{F}_{-g}) = \mathbf{R}^n \setminus \mathrm{mibar}^\star(E, \mathcal{F}_g),$$

which shows the connection between the barriers starting from the set E with forcing term g and the barriers starting from the set $\mathbf{R}^n \setminus E$ with forcing term $-g$.

In Theorem 6.1 we compare the resulting evolution with an abstract evolution of sets extending the smooth evolution, satisfying a semigroup property and a comparison principle. In particular we prove that $\mathrm{mibar}_\star(E, \mathcal{F}_g)$ and $\mathrm{mibar}^\star(E, \mathcal{F}_g)$ yield respectively a lower and an upper bound for any abstract evolution satisfying the previous properties. As a particular case, the comparison between the barriers and the viscosity evolution is given in Section 7.

Most of the results proved in this note have been announced in [5].

2. - Notations and general definition of barrier and minimal barrier

We choose the following conventions: if $E \subseteq \mathbf{R}^n$ is a set with compact smooth boundary, then ∂E is oriented by the outer unit normal vector ν_E; hence if d is the signed distance function to ∂E negative inside E, i.e., $d(x) = \mathrm{dist}(x, E) - \mathrm{dist}(x, R^n \setminus E)$, we have $\nu_E = \nabla d$ on ∂E. The mean curvature $\kappa(x)$ of ∂E at $x \in \partial E$ is $(n-1)^{-1} \Delta d(x)$, and is non-negative for convex sets. Concerning the evolution, we choose the normal velocity V to be positive for expanding sets E; a positive driving force g will correspond to an expansion of E. The evolution law we are interested in reads as $V = -(n-1)\kappa + g$.

In the following we fix the interval $I = [t_0, +\infty[$, for $t_0 \in R$.

We denote by $\mathcal{P}(R^n)$ the family of all subsets of R^n, $n \geq 1$. Given a set $C \subseteq R^n$, we denote by $\mathrm{int}(C)$, $\overline{C}$, and ∂C the interior part, the closure, and the boundary of C, respectively. In the sequel we denote by $g \in \mathcal{C}^\infty(R^n \times I) \cap L^\infty(R^n \times I)$ a function satisfying the following property: there exists a constant $G > 0$ such that

$$|g(x,t) - g(y,t)| \leq G|x-y|, \qquad \forall x, y \in R^n, \quad \forall t \in I. \tag{2.1}$$

The function g will stand for a driving force.

We recall the following result, which can be proved reasoning as in [21].

Lemma 2.1: *Let $\Sigma \subseteq R^n$ be a compact hypersurface of class $\mathcal{C}^\infty$. Then there exists $\tau > 0$ depending on the L^∞ norm of the second fundamental form of Σ and on the $W^{1,\infty}$ norm of g, such that the evolution of Σ by mean curvature with forcing term g is of class $\mathcal{C}^\infty$ for any $t \in [t_0, t_0 + \tau]$.*

Let us recall the general definition of barrier and minimal barrier in the sense of De Giorgi [14].

Definition 2.1: *Let S be a set and let $r \subseteq S^2$. Assume that $S = \bigcap\{E : r \subseteq E^2\}$ (that is, S is the ambient of the binary relation r). Let $\mathcal{F}$ be a family of functions of a real variable which satisfy the following property: for any $f \in \mathcal{F}$ there exist two real numbers a, b such that $a < b$ and $f : [a, b] \to S$. We say that a function ϕ is a barrier associated to the pair $(r, \mathcal{F})$, and we shall write $\phi \in \mathrm{Barr}(r, \mathcal{F})$, if there exists a convex set $J \subseteq I$ such that $\phi : J \to S$ and, whenever a, b, f satisfy the conditions*

$$[a, b] \subseteq J, \qquad f : [a, b] \to S, \qquad f \in \mathcal{F}, \qquad (f(a), \phi(a)) \in r,$$

then

$$(f(b), \phi(b)) \in r.$$

For any $E \subseteq S$ set

$$\mathrm{Minor}(r, E) = \{x \in S : (x, y) \in r \ \ \forall y \in E\},$$

$$\mathrm{Maior}(r, E) = \{x \in S : (y, x) \in r \ \ \forall y \in E\},$$

$$\mathrm{Mini}(r, E) = E \cap \mathrm{Minor}(r, E), \qquad \mathrm{Maxi}(r, E) = E \cap \mathrm{Maior}(r, E).$$

If the set $\operatorname{Mini}(r, E)$ (resp. $\operatorname{Maxi}(r, E)$) contains a unique element, this element will be denoted by $\min(r, E)$ (resp. $\max(r, E)$).

Let us define now the minimal barrier. The following definition is a slight generalization of the definition given in [14], where $I = [0, +\infty[$.

Definition 2.2: *Let $x \in S$; if there exists a function $\sigma: I \to S$ defined, for any $t \in I$, by the formula*

$$(2.2) \qquad \sigma(t) = \min(r, \{\phi(t): \quad \phi: I \to S, \quad \phi \in \operatorname{Barr}(r, \mathcal{F}), \quad (x, \phi(t_0)) \in r\}),$$

we shall say that σ is the minimal barrier associated to x, $\mathcal{F}$, r, I, and we shall write $\sigma = \operatorname{mibar}(x, \mathcal{F}, r, I)$.

We stress that the definition of barrier is very general and, concerning the mean curvature evolution, it can be applied for the motion of manifolds of arbitrary codimension (see [14, 2]).

3. - Barriers for the driven mean curvature evolution

In order to obtain the definition of barrier and minimal barrier for the evolution of a hypersurface by its mean curvature with forcing term g, we choose in Definition 2.1

$$(3.1) \qquad r = \{(E, L): E \subseteq L \subseteq \mathbf{R}^n\}, \qquad S = \mathcal{P}(\mathbf{R}^n)$$

and we choose the family $\mathcal{F}$, which we shall denote by $\mathcal{F}_g$, as follows.

Definition 3.1: *Let $a, b \in \mathbf{R}$, $a < b$, $[a, b] \subseteq I$; a function $f: [a, b] \to \mathcal{P}(\mathbf{R}^n)$ belongs to $\mathcal{F}_g$ if and only if the following three conditions hold:*

(1) *the set $\{(x, t): a \leq t \leq b, x \in f(t)\} \subseteq \mathbf{R}^{n+1}$ is closed with compact boundary;*

(2) *if $d(\cdot, t)$ denotes the signed-distance function to the set $f(t)$ negative inside $f(t)$ for $t \in [a, b]$, i.e.,*

$$d(x, t) = \operatorname{dist}(x, f(t)) - \operatorname{dist}(x, \mathbf{R}^n \setminus f(t)) \qquad \forall x \in \mathbf{R}^n, \ \forall t \in [a, b],$$

then there exists an open set $A \subseteq \mathbf{R}^n$ such that $d \in \mathcal{C}^\infty(A \times [a, b])$ and $\partial f(t) \subseteq A$ for any $t \in [a, b]$;

(3) *the following equation in d is verified on $\partial f(t)$:*

$$(3.2) \qquad \frac{\partial d}{\partial t} - \Delta d + g = 0 \qquad \forall t \in [a, b], \ \forall x \in \partial f(t).$$

Observe that condition (2) requires that the set $f(t)$ is of class $\mathcal{C}^\infty$ for any $t \in [a, b]$; condition (3) implies that $\partial f(t)$ smoothly evolves in time $t \in [a, b]$ by mean curvature (multiplied by $-(n-1)$) with forcing term g, since the expanding velocity is actually $-\partial d/\partial t$.

The class $\mathcal{F}_g$ is therefore the family of all (local in time) such smooth evolutions. Note that if $f \in \mathcal{F}_g$ then $\partial f(t)$ is a compact subset of $\mathbf{R}^n$ for any $t \in [a, b]$; note also that the smooth evolution $f^c = \mathbf{R}^n \setminus f$ satisfies conditions (1)-(3) of Definition 3.1 with g replaced by $-g$, so that $f^c \in \mathcal{F}_{-g}$.

3.1. *The classes* Barr$(\mathcal{F}_g)$ *and* Barr$^-(\mathcal{F}_g)$.

In the sequel the relation r will be the inclusion of sets as in (3.1); hence in the notation we shall drop the dependence on r, thus denoting the class Barr$(\subseteq, \mathcal{F}_g)$ by Barr$(\mathcal{F}_g)$. From the previous definitions the notions of barrier and minimal barrier with respect to the inclusion of sets and to the family $\mathcal{F}_g$ read as follows.

Definition 3.2: *A function ϕ is a barrier for the mean curvature evolution with forcing term g, and we shall write $\phi \in$ Barr$(\mathcal{F}_g)$, if and only if there exists a convex set $J \subseteq I$ such that $\phi: J \to \mathcal{P}(\mathbf{R}^n)$ and the following condition holds: if $f: [a, b] \subseteq J \to \mathcal{P}(\mathbf{R}^n)$ belongs to $\mathcal{F}_g$ and $f(a) \subseteq \phi(a)$ then $f(b) \subseteq \phi(b)$.*

The following observation is a direct consequence of the comparison principle between smooth evolutions, and will be useful in the sequel.

Remark 3.1: *Let $f_g: [a, b] \subseteq I \to \mathcal{P}(\mathbf{R}^n)$, $f_g \in \mathcal{F}_g$ and let $h \in \mathcal{C}^\infty(\mathbf{R}^n \times I) \cap$ $\cap L^\infty(\mathbf{R}^n \times I)$, $f_h: [a, c] \subseteq I \to \mathcal{P}(\mathbf{R}^n)$, $f_h \in \mathcal{F}_h$. Then*

$$h \leq g, \; f_h(a) \subseteq f_g(a) \Rightarrow f_h(t) \subseteq f_g(t) \qquad \forall t \in [a, \min(b, c)]. \tag{3.3}$$

Let us now introduce the class Barr$^-(\mathcal{F}_g)$, which will be useful in the sequel.

Definition 3.3: *We write $\phi \in$ Barr$^-(\mathcal{F}_g)$ if and only if there exists a convex set $J \subseteq I$ such that $\phi: J \to \mathcal{P}(\mathbf{R}^n)$ and the following condition holds: if $f: [a, b] \subseteq J \to \mathcal{P}(\mathbf{R}^n)$ belongs to $\mathcal{F}_h$ for any $h \in \mathcal{C}^\infty(\mathbf{R}^n \times I) \cap L^\infty(\mathbf{R}^n \times I)$ with $h \leq g$ and $f(a) \subseteq \phi(a)$ then $f(b) \subseteq$ $\subseteq \phi(b)$.*

Note that Barr$^-(\mathcal{F}_g)$ can be defined under less regularity assumptions on g, for instance by requiring only that g is continuous and verifies (2.1).

It is clear that the definition of Barr$^-(\mathcal{F}_g)$ is more restrictive then Barr$(\mathcal{F}_g)$, so that Barr$^-(\mathcal{F}_g) \subseteq$ Barr$(\mathcal{F}_g)$. The following lemma shows that these two classes actually coincide. In the sequel we shall then use Barr$^-(\mathcal{F}_g)$ in place of Barr$(\mathcal{F}_g)$ when necessary.

Lemma 3.1: *We have*

$$\mathrm{Barr}(\mathcal{F}_g) = \mathrm{Barr}^-(\mathcal{F}_g).$$

Proof: We have to show that $\mathrm{Barr}(\mathcal{F}_g) \subseteq \mathrm{Barr}^-(\mathcal{F}_g)$. Let $\phi: I \to \mathcal{P}(\mathbf{R}^n)$, $\phi \in \mathrm{Barr}(\mathcal{F}_g)$; let $b \in \mathcal{C}^\infty(\mathbf{R}^n \times I) \cap L^\infty(\mathbf{R}^n \times I)$, $b \leq g$. Let $f_b : [a, b] \subseteq I \to \mathcal{P}(\mathbf{R}^n)$, $f_b \in \mathcal{F}_b$, with $f_b(a) \subseteq \phi(a)$. We have to prove that $f_b(b) \subseteq \phi(b)$. Let $s \in [a, b]$; we apply Lemma 2.1 with $\Sigma = \partial f_b(s)$. By definition of $\mathcal{F}_b$ the set $\partial f_b(s)$ is a smooth hypersurface for any $s \in [a, b]$, hence there is a bound on the L^∞ norm of the second fundamental form of $\partial f_b(s)$ which is uniform with respect to $s \in [a, b]$. Then the number τ given by Lemma 2.1 (depending also on $\|g\|_{W^{1,\infty}}$) does not depend on $s \in [a, b]$. Write $[a, b] = \bigcup_{i=1}^{m} [t_i, t_{i+1}]$ where $a = t_1 < \ldots < t_{m+1} = b$ and $t_{i+1} - t_i \leq \tau$. Let us denote by $f_g^i(t)$ the mean curvature evolution of $f_b(t_i)$ with forcing term g when t belongs to the interval $[t_i, t_{i+1}]$. By (3.3) we have $f_b(t_{i+1}) \subseteq f_g^i(t_{i+1})$. Then, reasoning by induction on i, each f_g^i satisfies $f_g^i(t_i) \subseteq \phi(t_i)$, hence $f_g^i(t_{i+1}) \subseteq \phi(t_{i+1})$, and therefore $f_b(t_{i+1}) \subseteq \phi(t_{i+1})$. For $i = m$ the assertion follows. ∎

3.2. *The minimal barrier* $\mathrm{mibar}(E, \mathcal{F}_g)$.

Given an arbitrary set $E \subseteq \mathbf{R}^n$ it is immediate to verify that the set

$$\bigcap \{\phi(t):\ \phi: I \to \mathcal{P}(\mathbf{R}^n),\ \phi \in \mathrm{Barr}(\mathcal{F}_g),\ \phi(t_0) \supseteq E\}$$

is a barrier, which implies the existence of the minimum defined in (2.2). Hence the minimal barrier $\mathrm{mibar}(E, \mathcal{F}_g, I)$ starting from an arbitrary set $E \subseteq \mathbf{R}^n$ with forcing term g on the time interval I, i.e., the generalized motion of E by mean curvature with forcing term g defined in I, reads as follows.

Definition 3.4: *Let $E \subseteq \mathbf{R}^n$. The minimal barrier* $\mathrm{mibar}(E, \mathcal{F}_g, I): I \to \mathcal{P}(\mathbf{R}^n)$ *is defined as*

$$\mathrm{mibar}(E, \mathcal{F}_g, I)(t) = \bigcap \{\phi(t):\ \phi: I \to \mathcal{P}(\mathbf{R}^n),\ \phi \in \mathrm{Barr}(\mathcal{F}_g),\ \phi(t_0) \supseteq E\} \tag{3.4}$$

for any $t \in I$.

Since in what follows the interval I is fixed, for simplicity we drop the dependence on I in the notation of the minimal barrier, thus denoting the evolution $\mathrm{mibar}(E, \mathcal{F}_g, I)$ by $\mathrm{mibar}(E, \mathcal{F}_g)$.

Observe that the set defined by E if $t = t_0$ and by $\mathrm{mibar}(E, \mathcal{F}_g)(t)$ if $t > t_0$, $t \in I$, is still a barrier, so that $\mathrm{mibar}(E, \mathcal{F}_g)(t_0) = E$.

Obviously if $E, F \in \mathcal{P}(\mathbf{R}^n)$ then

$$E \subseteq F \Rightarrow \mathrm{mibar}(E, \mathcal{F}_g)(t) \subseteq \mathrm{mibar}(F, \mathcal{F}_g)(t) \quad \forall t \in I. \tag{3.5}$$

The following remark shows that if E is smooth and admits a smooth evolution then this evolution coincides with the minimal barrier.

Remark 3.2: *Assume that E is smooth and that it admits a classical evolution $C_E(t)$ by* *mean curvature with forcing term g for any t in some time interval* $[t_0, t_1]$. *Then* $\mathrm{mibar}(E, \mathcal{F}_g)(t) = C_E(t)$ *for all* $t \in [t_0, t_1]$.

PROOF: By the comparison principle between smooth evolutions, one has that $C_E: [t_0, t_1] \to \mathcal{P}(\mathbf{R}^n)$ is a barrier, so that $\mathrm{mibar}(E, \mathcal{F}_g)(t) \subseteq C_E(t)$ for all $t \in [t_0, t_1]$. Conversely, since $C_E \in \mathcal{F}_g$, $C_E(t_0) = E = \mathrm{mibar}(E, \mathcal{F}_g)(t_0)$, and $\mathrm{mibar}(E, \mathcal{F}_g) \in \mathrm{Barr}(\mathcal{F}_g)$, by definition of barrier we must have $C_E(t) \subseteq \mathrm{mibar}(E, \mathcal{F}_g)(t)$ for all $t \in [t_0, t_1]$. ■

To clarify the previous definitions, let us consider some one-dimensional examples.

EXAMPLE 3.1: Let $n = 1$, $f: [a, b] \subseteq I \to \mathcal{P}(\mathbf{R})$, $f \in \mathcal{F}_g$, and let d, A be as in (2) of Definition 3.1. As $\partial f(t)$ is a compact set contained in A and the signed-distance function d is of class $\mathcal{C}^\infty(A \times [a, b])$, it follows that $\partial f(t)$ is a finite union of points, so that $f(t)$ is a finite union of intervals, evolving in a smooth way. Moreover d is linear, and hence $\Delta d = 0$. Assume that $[x^-(t), x^+(t)]$ is one of the intervals composing $f(t)$ for any $t \in [a, b]$. Note that

$$\frac{\partial d}{\partial t}(x^-(t), t) = \frac{dx^-}{dt}(t), \qquad \frac{\partial d}{\partial t}(x^+(t), t) = -\frac{dx^+}{dt}(t) \qquad \forall t \in [a, b].$$

Hence by (3) of Definition 3.1 we get

$$\frac{dx^-}{dt}(t) = -g(x^-(t), t), \qquad \frac{dx^+}{dt}(t) = g(x^+(t), t) \qquad \forall t \in [a, b].$$

EXAMPLE 3.2: Let $n = 1$ and let E be the union of two disjoint intervals, $E = E_1(t_0) \cup E_2(t_0)$, where $E_i(t_0) = [x_i^-(t_0), x_i^+(t_0)]$, $i = 1, 2$, and $x_1^+(t_0) < x_2^-(t_0)$. Let us assume that the sets $E_i(t_0)$ smoothly evolve in time by mean curvature with forcing term g for any $t \in [t_0, t^\star]$ (see Example 3.1), and denote by $E_i(t) = [x_i^-(t), x_i^+(t)]$ such evolutions, $i = 1, 2$ (see Remark 3.2). Obviously we assume that $E_i(t^\star) \neq \emptyset$. We also assume $E_1(t) \cap E_2(t) = \emptyset$ for any $t \in [t_0, t^\star[$, and $x_1^+(t^\star) = x_2^-(t^\star)$. Then

$$\mathrm{mibar}(E, \mathcal{F}_g)(t^\star + \tau) = [x_1^-(t^\star + \tau), x_2^+(t^\star + \tau)] \tag{3.6}$$

for any $\tau > 0$ small enough. Indeed, as $E_i(t_0) \subseteq E$, $E_i: [t_0, t^\star] \to \mathcal{P}(\mathbf{R})$, $E_i \in \mathcal{F}_g$, and $\mathrm{mibar}(E, \mathcal{F}_g) \in \mathrm{Barr}(\mathcal{F}_g)$, we have $E_i(t) \subseteq \mathrm{mibar}(E, \mathcal{F}_g)(t)$ for any $t \in [t_0, t^\star]$, so that

$$E_1(t^\star) \cup E_2(t^\star) = [x_1^-(t^\star), x_2^+(t^\star)] \subseteq \mathrm{mibar}(E, \mathcal{F}_g)(t^\star). \tag{3.7}$$

Then (3.6) follows from (3.7) choosing $E_1(t^\star) \cup E_2(t^\star)$ as starting smooth set for the evolution.

Different definitions of barriers and minimal barriers (and hence of generalized evolutions of the set E) could be obtained by replacing the family $\mathcal{P}(R^n)$ with other families of sets. For instance, one could introduce barriers using the family of all compact subsets of R^n; with this choice the corresponding generalized evolution is a compact set. Note that if X, Y are two such families and if $X \subseteq Y$, then the generalized evolution of the set E corresponding to X contains the generalized evolution of E corresponding to Y.

Finally, observe that the set $\mathrm{mibar}(E, \mathcal{F}_g)$ is very sensible to modifications of the original set E on sets of zero Lebesgue measure, as showed in Examples 5.1, 5.2.

3.3. *The evolutions* $\mathrm{mibar}_\star(E, \mathcal{F}_g)$, $\mathrm{mibar}^\star(E, \mathcal{F}_g)$.

It is useful to give the following definitions. Let $E \subseteq R^n$; for any $\varrho > 0$ set

$$E_\varrho^- = R^n \setminus \{x \in R^n : \mathrm{dist}(x, R^n \setminus E) < \varrho\}, \tag{3.8}$$

$$E_\varrho^+ = \{x \in R^n : \mathrm{dist}(x, E) < \varrho\}, \tag{3.9}$$

and let us define the functions $\mathrm{mibar}_\star(E, \mathcal{F}_g)$, $\mathrm{mibar}^\star(E, \mathcal{F}_g)$ as follows: if $t \in I$ we set

$$\begin{aligned} \mathrm{mibar}_\star(E, \mathcal{F}_g)(t) &= \bigcup_{\varrho > 0} \mathrm{mibar}(E_\varrho^-, \mathcal{F}_g)(t), \\ \mathrm{mibar}^\star(E, \mathcal{F}_g)(t) &= \bigcap_{\varrho > 0} \mathrm{mibar}(E_\varrho^+, \mathcal{F}_g)(t). \end{aligned} \tag{3.10}$$

REMARK 3.3: *For any* $t \in I$ *we have*

$$\mathrm{mibar}_\star(E, \mathcal{F}_g)(t) \subseteq \mathrm{mibar}(E, \mathcal{F}_g)(t) \subseteq \mathrm{mibar}^\star(E, \mathcal{F}_g)(t), \tag{3.11}$$

and such inclusions can be strict.

PROOF: (3.11) follows immediately from (3.5). To show that inclusions (3.11) can be strict, let $n = 2$, $g = 0$, and simply let evolve respectively the two sets $E = \{x \in R^2 : |x| \leq 1\}$, and $\mathrm{int}(E)$. Then, if $t > t_0$ is less than the extinction time, the set $\mathrm{mibar}(E, \mathcal{F}_0)(t)$ is closed (see Remark 3.2), while one can directly check that $\mathrm{mibar}_\star(E, \mathcal{F}_0)(t)$ is open, so that $\mathrm{mibar}(E, \mathcal{F}_0)(t) \supsetneq \mathrm{mibar}_\star(E, \mathcal{F}_0)(t)$. On the other hand $\mathrm{mibar}(\mathrm{int}(E), \mathcal{F}_0)(t)$ is open and $\mathrm{mibar}^\star(\mathrm{int}(E), \mathcal{F}_0)(t)$ is closed, hence $\mathrm{mibar}(\mathrm{int}(E), \mathcal{F}_0)(t) \subsetneq \mathrm{mibar}^\star(\mathrm{int}(E), \mathcal{F}_0)(t)$. ∎

As we shall see in Sections 5, 6, 7, the set $\mathrm{mibar}(E, \mathcal{F}_g)$ does not easily compare with other notions of generalized mean curvature evolution; this is not the case for the evolutions defined in (3.10), where the comparison is more natural.

4. - The exponential estimate

The main result of this section is Theorem 4.1. We need some preliminaries.

Definition 4.1: *We define the function* $\varrho : [0, +\infty[\to [0, +\infty[$ *as follows: if* $t \in [0, +\infty[$ *then* $\varrho(t)$ *is the radius at time* $-t$ *of a ball shrinking at* $t = 0$ *under the law «normal velocity* $= -(n-1)/\varrho(t) - \|g\|_{L^\infty}$*».*

The function ϱ is then an increasing continuous function which is $(1/2)$-Hölder continuous at $t = 0$, and with $\varrho(0) = 0$.

The following lemma shows that a barrier cannot shrink too fast.

Lemma 4.1: *Let* $\phi : I \to \mathscr{P}(R^n)$, $\phi \in \mathrm{Barr}(\mathscr{F}_g)$, $s, t \in I$, $s > t$. *Then*

$$\{x \in R^n : \mathrm{dist}(x, R^n \setminus \phi(t)) > \varrho(s-t)\} \subseteq \mathrm{int}(\phi(s)).$$

Proof: Let $x \in \phi(t)$ be such that $\mathrm{dist}(x, R^n \setminus \phi(t)) > \overline{\varrho} > \varrho(s-t)$. Denote by $B(x, \overline{\varrho})$ the ball centered at x with radius $\overline{\varrho}$, and denote by $B(s)$ the evolution of $B(x, \overline{\varrho})$ by mean curvature with forcing term $h = -\|g\|_{L^\infty}$. Thanks to Lemma 3.1 we have $\phi \in \mathrm{Barr}^-(\mathscr{F}_h)$ so that $x \in B(s) \subseteq \phi(s)$. Since $\overline{\varrho} > \varrho(s-t)$, we have $x \in \mathrm{int}(\phi(s))$. ■

Corollary 4.1: *Let* $f : [a, b] \subseteq I \to \mathscr{P}(R^n)$, $f \in \mathscr{F}_g$, $s, t \in [a, b]$, $s > t$. *Then*

$$f(s) \subseteq \{x \in R^n : \mathrm{dist}((x, f(t)) \leq \varrho(s-t)\}.$$

Proof: Let $f^c(t) = \overline{R^n \setminus f(t)}$, $t \in [a, b]$. Then $f^c \in \mathrm{Barr}(\mathscr{F}_{-g})$. Applying Lemma 4.1 with ϕ and g replaced by f^c and $-g$ respectively, we have the assertion. ■

We recall that if g does not depend on x, the translation invariance of equation (3.2) provides the following useful property (see [23,24]): let $\phi : I \to \mathscr{P}(R^n)$, $\phi \in \mathrm{Barr}(\mathscr{F}_g)$, and let $f : [a, b] \subseteq I \to \mathscr{P}(R^n)$, $f \in \mathscr{F}_g$ be such that $f(a) \subseteq \phi(a)$. Then

$$\mathrm{dist}(f(t), R^n \setminus \phi(t)) \geq \mathrm{dist}(f(a), R^n \setminus \phi(a)) \qquad \forall t \in [a, b].$$

In the general case in which we allow the forcing term to depend on x we can prove the following result, which states that $\mathrm{dist}(f(t), R^n \setminus \phi(t))$ could decrease, but in a controlled way.

Lemma 4.2: *Let* $\phi : I \to \mathscr{P}(R^n)$, $\phi \in \mathrm{Barr}(\mathscr{F}_g)$, *and let* $f : [a, b] \subseteq I \to \mathscr{P}(R^n)$, $f \in \mathscr{F}_g$, *be such that* $f(a) \subseteq \phi(a)$. *Set*

$$\delta(t) = \mathrm{dist}(f(t), R^n \setminus \phi(t)) \qquad \forall t \in [a, b].$$

Then, recalling the definition (2.1) *of* G, *we have*

$$\delta(t) \geq \delta(a) \exp(-G(t-a)) \qquad \forall t \in [a, b]. \tag{4.1}$$

PROOF: *Step* 1. We shall prove that

$$\delta(s) \geq \delta(t) - 2\varrho(s-t) \qquad \forall t, s \in [a,b],\ s > t, \tag{4.2}$$

where ϱ is given by Definition 4.1. Let $t, s \in [a,b]$, $s > t$; choose $x \in f(s)$ and $y \in R^n \setminus \phi(s)$ in such a way that $|x - y| = \delta(s)$. By Lemma 4.1 and Corollary 4.1 we have

$$\operatorname{dist}(y, R^n \setminus \phi(t)) \leq \varrho(s-t), \qquad \operatorname{dist}(x, f(t)) \leq \varrho(s-t).$$

Using the triangular property of δ we then have

$$\delta(t) \leq \delta(s) + \operatorname{dist}(x, f(t)) + \operatorname{dist}(y, R^n \setminus \phi(t)) \leq \delta(s) + 2\varrho(s-t),$$

and Step 1 is proved.

Step 2. We shall prove that

$$\limsup_{\tau \to 0^+} \frac{\delta(t) - \delta(t+\tau)}{\tau} \leq G\delta(t) \qquad \forall t \in [a,b].$$

Given $a \leq t < b$, for any $\tau > 0$ such that $t + \tau \leq b$ let $p_\tau \in f(t+\tau)$, $q_\tau \in \overline{R^n \setminus \phi(t+\tau)}$ be such that $|p_\tau - q_\tau| = \delta(t+\tau)$. By compactness we can extract a sequence $\{\tau_n\}$ converging to zero as $n \to +\infty$ such that $\lim_{n\to+\infty} p_{\tau_n} = p \in f(t)$ and $\lim_{n\to+\infty} q_{\tau_n} = q \in \overline{R^n \setminus \phi(t)}$ with $|p - q| = \delta(t)$. Given $\gamma < 1$ let $\hat{f}_\gamma(t)$ be the translation of $f(t)$ by the vector $\gamma(q-p)$, and let $\hat{f}_\gamma(t+\tau)$ be the mean curvature evolution of $\hat{f}_\gamma(t)$ with forcing term g for a small time $\tau > 0$. Since $\hat{f}_\gamma \in \mathcal{F}_g$ and $\hat{f}_\gamma(t) \subseteq \phi(t)$, we get $\hat{f}_\gamma(t+\tau) \subseteq \phi(t+\tau)$, and by taking union with respect to γ we have $\operatorname{int}(\hat{f}_1(t+\tau)) \subseteq \phi(t+\tau)$ for any small $\tau > 0$, so that, passing to the interiors, $\operatorname{int}(\hat{f}_1(t+\tau)) \subseteq \operatorname{int}(\phi(t+\tau))$.

Denoting by $\nu = (q-p)/\delta(t)$ the outer unit normal to $f(t)$ at p, we have that ν is also normal to $\hat{f}_1(t)$ at q. Since $p_\tau \in \partial f(t+\tau)$ and $\partial f(t+\tau)$ is a regularly evolving smooth surface we have that

$$\lim_{\tau\to 0^+} \frac{p_\tau - p}{\tau} \cdot \nu = V, \tag{4.3}$$

where V is the outer normal velocity of $f(t)$ at p. Also, since $q_\tau \notin \operatorname{int}(\hat{f}_1(t+\tau))$ and $q \in \in \hat{f}_1(t)$ we get the inequality

$$\liminf_{\tau\to 0^+} \frac{q_\tau - q}{\tau} \cdot \nu \geq \hat{V}, \tag{4.4}$$

where $\hat{V}$ denotes the outer normal velocity of $\hat{f}_1(t)$ at q. Now

$$\delta(t+\tau) - \delta(t) = |q_\tau - p_\tau| - |q - p| \geq (q_\tau - p_\tau)\cdot\nu - (q-p)\cdot\nu =$$
$$= (q_\tau - q)\cdot\nu - (p_\tau - p)\cdot\nu.$$

By dividing by τ and taking the lim inf, using (4.3) and (4.4) we get

$$\liminf_{\tau\to 0^+} \frac{\delta(t+\tau)-\delta(t)}{\tau} \geq \tilde{V} - V = g(q) - g(p),$$

and Step 2 follows.

Suppose by contradiction that we can find a time $t_1 \in]a, b]$ such that $\delta(t_1) < \delta(a)\exp(-G(t_1 - a))$. Let $\mu(s) = P(s)\exp(-G(s-a))$, where P is a linear decreasing polynomial such that $\mu(a) = \delta(a)$ and $\mu(t_1) > \delta(t_1)$. Define

$$s^* = \inf\{s \in [a, b]: \delta(s) \leq \mu(s)\}.$$

By Step 1 we have $\mu(s^*) = \delta(s^*)$, hence $s^* < b$, and by definition of s^*

$$\liminf_{\tau\to 0^+} \frac{\delta(s^*+\tau)-\delta(s^*)}{\tau} \leq \mu'(s^*) < -G\delta(s^*),$$

which contradicts Step 2. ∎

Lemma 4.2 in general does not hold if f is replaced by a barrier, since in this case the barrier could grow instantly.

The following result generalizes the avoidance principle of Ilmanen [23, 4E].

Theorem 4.1: *Let $\phi \in \mathrm{Barr}(\mathcal{F}_\xi)$, $\psi \in \mathrm{Barr}(\mathcal{F}_{-\xi})$ be two barriers defined on $I = [t_0, +\infty[$. Assume that either $R^n \setminus \mathrm{int}(\psi(t_0))$ or $R^n \setminus \mathrm{int}(\phi(t_0))$ is compact. Set*

$$\eta(t) = \mathrm{dist}(R^n \setminus \phi(t), R^n \setminus \psi(t)) \qquad \forall t \in I.$$

Then

$$\eta(t) \geq \eta(t_0)\exp(-G(t-t_0)) \qquad \forall t \in I. \tag{4.5}$$

Proof: We assume that $\eta(t_0) > 0$, otherwise the result is trivial. Also we assume that $R^n \setminus \mathrm{int}(\psi(t_0))$ is compact. Let $t \in I$; by Lemma 4.1, for any $s \in I$, $s > t$, we have

$$R^n \setminus \mathrm{int}(\phi(s)) \subseteq \{x \in R^n: \mathrm{dist}(x, R^n \setminus \phi(t)) \leq \varrho(s-t)\},$$

$$R^n \setminus \mathrm{int}(\psi(s)) \subseteq \{x \in R^n: \mathrm{dist}(x, R^n \setminus \psi(t)) \leq \varrho(s-t)\}$$

(in particular $R^n \setminus \mathrm{int}(\psi(s))$ is compact). Using the triangular property of η (see the proof of Step 1 in Lemma 4.2) we have

$$\eta(s) \geq \eta(t) - 2\varrho(s-t) \qquad \forall s > t. \tag{4.6}$$

Assume by contradiction that (4.5) is false, and let

$$t^* = \inf\{t \in I: \eta(t) < \eta(t_0)\exp(-G(t-t_0))\} < +\infty.$$

By (4.6) it follows that

$$\eta(t^\star) \geq \eta(t_0)\exp\left(-G(t^\star - t_0)\right). \tag{4.7}$$

Following [23, 4E] we can find a family $\{Q^\varepsilon_{t^\star}\}$ of smooth hypersurfaces whose L^∞ norm of the second fundamental form is uniformly bounded with respect to ε, and satisfying the following property:

$$\operatorname{dist}\left(R^n \setminus \operatorname{int}(\phi(t^\star)), Q^\varepsilon_{t^\star}\right) + \operatorname{dist}\left(Q^\varepsilon_{t^\star}, R^n \setminus \operatorname{int}(\psi(t^\star))\right) \geq \eta(t^\star) - \varepsilon. \tag{4.8}$$

Write R^n as union of three mutually disjoint sets, as $R^n = Q^\varepsilon_{t^\star} \cup I^\varepsilon_{t^\star} \cup O^\varepsilon_{t^\star}$, where $I^\varepsilon_{t^\star}$ is the connected component of $R^n \setminus Q^\varepsilon_{t^\star}$ which is contained in $\phi(t^\star)$. Then $O^\varepsilon_{t^\star} \cup Q^\varepsilon_{t^\star} \subseteq \psi(t^\star)$.

Let τ be given by Lemma 2.1 applied to $Q^\varepsilon_{t^\star}$; τ may depend on $t^\star$ but can be chosen independent of ε. Hence each mean curvature flow Q^ε_t (resp. I^ε_t, O^ε_t) with forcing term g starting from $Q^\varepsilon_{t^\star}$ (resp. from $I^\varepsilon_{t^\star}$, $O^\varepsilon_{t^\star}$) remains smooth for $t \in [t^\star, t^\star + \tau]$. Therefore $t \in [t^\star, t^\star + \tau] \to I^\varepsilon_t \cup Q^\varepsilon_t$ belongs to $\mathcal{F}_g$ and $t \in [t^\star, t^\star + \tau] \to O^\varepsilon_t \cup Q^\varepsilon_t$ belongs to $\mathcal{F}_{-g}$. Using the triangular property, Lemma 4.2, and (4.8) we have

$$\eta(t) \geq \operatorname{dist}\left(R^n \setminus \phi(t), I^\varepsilon_t \cup Q^\varepsilon_t\right) + \operatorname{dist}\left(O^\varepsilon_t \cup Q^\varepsilon_t, R^n \setminus \psi(t)\right) \geq$$

$$\geq \left[\operatorname{dist}\left(R^n \setminus \phi(t^\star), I^\varepsilon_{t^\star} \cup Q^\varepsilon_{t^\star}\right) + \operatorname{dist}\left(O^\varepsilon_{t^\star} \cup Q^\varepsilon_{t^\star}, R^n \setminus \psi(t^\star)\right)\right]\exp\left(-G(t - t^\star)\right) \geq$$

$$\geq \left(\eta(t^\star) - \varepsilon\right)\exp\left(-G(t - t^\star)\right),$$

for any $t \in [t^\star, t^\star + \tau]$. Letting $\varepsilon \to 0$ and using (4.7) we get

$$\eta(t_0)\exp\left(-G(t^\star - t_0)\right) \leq \eta(t^\star) \leq \eta(t)\exp\left(G(t - t^\star)\right).$$

This implies that

$$\eta(t) \geq \eta(t_0)\exp\left(-G(t - t_0)\right) \qquad \forall t \in [t^\star, t^\star + \tau],$$

which contradicts the definition of $t^\star$. ∎

Observe that if $t \in I$

$$\eta(t) > 0 \Leftrightarrow \phi(t) \supseteq R^n \setminus \operatorname{int}(\psi(t)). \tag{4.9}$$

5. - Some general properties of the minimal barriers

Observe that if $E, F \subseteq R^n$ and if any smooth subset of E with compact boundary is also a subset of F, then $\operatorname{mibar}(E, \mathcal{F}_g)(t) \subseteq \operatorname{mibar}(F, \mathcal{F}_g)(t)$ for any $t \in I$. In particular, if E, F contain the same smooth sets with compact boundary, then $\operatorname{mibar}(E, \mathcal{F}_g) = \operatorname{mibar}(F, \mathcal{F}_g)$.

Remark 5.1: *For any $E \subseteq \mathbf{R}^n$ we have*

$$\operatorname{mibar}(E, \mathcal{F}_\zeta)(t) \subseteq \operatorname{mibar}(\overline{\operatorname{int}(E)}, \mathcal{F}_\zeta)(t) \qquad \forall t \in I,\ t > t_0. \tag{5.1}$$

If in addition E is closed then

$$\operatorname{mibar}(E, \mathcal{F}_\zeta) = \operatorname{mibar}(\overline{\operatorname{int}(E)}, \mathcal{F}_\zeta)(t) \qquad \forall t \in I,\ t > t_0. \tag{5.2}$$

Proof: If $V \subseteq E$ is a smooth set with compact boundary, then $V = \overline{\operatorname{int}(V)}$, and therefore V is contained in $\overline{\operatorname{int}(E)}$, and this implies (5.1).

If E is closed, then $E \supseteq \overline{\operatorname{int}(E)}$, so that E and $\overline{\operatorname{int}(E)}$ contain the same smooth sets, and (5.2) follows. ■

Observe that if E is such that $\operatorname{int}(E) = \emptyset$, then (5.1) implies $\operatorname{mibar}(E, \mathcal{F}_\zeta)(t) = \emptyset$ for any $t > t_0$, $t \in I$.

Theorem 5.1: *The following properties hold:*

$$\text{if } A \subseteq \mathbf{R}^n \text{ is open then } \operatorname{mibar}(A, \mathcal{F}_\zeta)(t) \text{ is open for any } t \in I. \tag{5.3}$$

If $A \subseteq \mathbf{R}^n$ is open and if $\{A_\varepsilon\}$ is a family of open sets such that $A_\varepsilon \uparrow A$ as $\varepsilon \to 0$, then for any $t \in I$ we have

$$\operatorname{mibar}(A_\varepsilon, \mathcal{F}_\zeta)(t) \uparrow \operatorname{mibar}(A, \mathcal{F}_\zeta)(t) \qquad \text{as } \varepsilon \to 0. \tag{5.4}$$

In particular

$$\text{if } A \subseteq \mathbf{R}^n \text{ is open then } \operatorname{mibar}_*(A, \mathcal{F}_\zeta) = \operatorname{mibar}(A, \mathcal{F}_\zeta). \tag{5.5}$$

In addition, if $A \subseteq \mathbf{R}^n$ is an open set and $K \subseteq A$ is a closed set with compact boundary, then

$$\operatorname{mibar}^*(K, \mathcal{F}_\zeta) \subseteq \operatorname{mibar}(A, \mathcal{F}_\zeta), \tag{5.6}$$

while (5.6) does not hold in general if $K \subseteq A$ is a closed set.

Finally, if $C \in \mathbf{R}^n$ is a closed set, in general $\operatorname{mibar}(C, \mathcal{F}_\zeta)(t)$ is neither closed nor open.

Proof: Let $A \subseteq \mathbf{R}^n$ be an open set. Let $\phi: I \to \mathcal{P}(\mathbf{R}^n)$, $\phi \in \operatorname{Barr}(\mathcal{F}_\zeta)$, $\phi(t_0) = A$. We claim that $\operatorname{int}(\phi) \in \operatorname{Barr}(\mathcal{F}_\zeta)$. This is equivalent to say that if $f: [a, b] \subseteq I \to \mathcal{P}(\mathbf{R}^n)$, $f \in \mathcal{F}_\zeta$, $f(a) \subseteq \operatorname{int}(\phi(a))$, then $f(b) \subseteq \operatorname{int}(\phi(b))$. We have $\operatorname{dist}(f(a), \mathbf{R}^n \setminus \phi(a)) > 0$, so that by Lemma 4.2 we deduce $\operatorname{dist}(f(b), \mathbf{R}^n \setminus \phi(b)) > 0$. Hence $f(b) \subseteq \operatorname{int}(\phi(b))$, and the claim is proved. Hence, as $\operatorname{mibar}(A, \mathcal{F}_\zeta)$ is a barrier, also $\operatorname{int}(\operatorname{mibar}(A, \mathcal{F}_\zeta))$ is a barrier, and (5.3) follows.

Assertion (5.4) is a consequence of (3.5) and (5.3).

Let A be open and $K \subseteq A$ be compact; then (5.6) is a consequence of the following observation: there exists $\bar{\varrho} > 0$ such that $K_\varrho^+ \subseteq A$ for any $0 < \varrho < \bar{\varrho}$, where K_ϱ^+ is defined as in (3.9). To prove that inclusion (5.6) does not hold for just closed K, take $n = 2, g = 0, A = \{(x_1, x_2) \in \mathbf{R}^2 : |x_2| < e^{-|x_1|^2}\}$, and $K = \{(x_1, x_2) \in \mathbf{R}^2 : x_2 = 0\}$. Any smooth set contained in A is going to shrink to a point at finite time $t^\dagger$, and $t^\dagger$ is related to the Lebesgue measure of A, which is finite. As a consequence $\operatorname{mibar}(A, \mathcal{F}_0)(t) = \emptyset$ for any $t > t^\dagger$ (see [23, Section 6], [22, 7.3]). On the contrary, for any $\varrho > 0$ the set K_ϱ^+ is an infinite strip which stays constant, so that $\operatorname{mibar}(K_\varrho^+, \mathcal{F}_0)(t) = K_\varrho^+$ for any $t > t_0$. Hence $\operatorname{mibar}^\star(K, \mathcal{F}_0)(t) = K$ for any $t > t_0$.

It remains to show that there is a closed set C so that $\operatorname{mibar}(C, \mathcal{F}_g)(t)$ is not closed. Take $n = 2$, $g = 0$, $C = \{x = (x_1, x_2) \in \mathbf{R}^2 : |x_1| \leqslant 1, \ |x_2| \leqslant 1\}$. We shall prove that

$$(5.7) \qquad \operatorname{mibar}(C, \mathcal{F}_0)(t) = \operatorname{mibar}(\operatorname{int}(C), \mathcal{F}_0)(t) \qquad \forall t \in I, \ t > t_0,$$

so that $\operatorname{mibar}(C, \mathcal{F}_0)(t)$ is open for any $t > t_0$ (see (5.3)), and this will conclude the proof of the theorem.

To prove (5.7) it is enough to show that

$$(5.8) \qquad \begin{cases} \operatorname{mibar}(\operatorname{int}(C), \mathcal{F}_0)(t) & \text{if } t \in I, \ t > t_0 \\ C & \text{if } t = t_0 \end{cases} \quad \text{is a barrier}.$$

Denote by $\{C^\varepsilon\}_{\varepsilon \in]0,1[}$ an increasing family of convex subsets of C of class $\mathcal{C}^\infty$ symmetric with respect to the x_1 and x_2 axes, with the following properties:

$$\bigcup_{\varepsilon \in]0,1[} C^\varepsilon \supseteq \operatorname{int}(C),$$

$$\partial C^\varepsilon \cap \partial C = ([-1+\varepsilon, 1-\varepsilon] \times \{-1, 1\}) \cup (\{-1, 1\} \times [-1+\varepsilon, 1-\varepsilon]),$$

and

$$\partial C^\varepsilon \cap \partial C^{\varepsilon'} \cap \operatorname{int}(C) = \emptyset \qquad \forall 0 < \varepsilon' < \varepsilon.$$

Observe that if S is a smooth set contained in C, since S cannot contain any corner of C, it follows that $S \subseteq C^\varepsilon$ for some $\varepsilon \in]0, 1]$. Consequently, to prove (5.8) it is enough to show that, for any $\varepsilon \in]0, 1[$, denoting by $C^\varepsilon(t)$, $t \in [a, b] \subseteq I$, $a > t_0$, the smooth evolution of $C^\varepsilon = C^\varepsilon(a)$ by mean curvature at time t, then

$$(5.9) \qquad C^\varepsilon(b) \subseteq \operatorname{mibar}(\operatorname{int}(C), \mathcal{F}_0)(b).$$

Let us fix $\varepsilon \in]0, 1]$. We claim that

$$(5.10) \qquad \operatorname{dist}(\partial C^\varepsilon(t), \partial C^{\varepsilon/2}(t)) > 0 \qquad \forall t \in [a, b[.$$

Fix $0 < \delta < \varepsilon/2$, denote by $p_\varepsilon(t) = C^\varepsilon(t) \cap \{(x_1, x_2) : x_1 = -1 + \delta, x_2 < 0\}$, and let $f, b : [-1+\delta, 1-\delta] \times [a, +\infty[\to \mathbf{R}$ be the $\mathcal{C}^\infty$ functions defined as follows: f, b are the solutions of the nonlinear parabolic equation of the mean curvature motion in cartesian form, with

$$f(x_1, a) = f_0(x_1), \qquad b(x_1, a) = b_0(x_1) \qquad \forall x_1 \in [-1+\delta, 1-\delta],$$

$$f(-1+\delta, t) = p_\epsilon(t), \qquad b(-1+\delta, t) = p_{\epsilon/2}(t),$$

where $f_0, b_0: [-1+\delta, 1-\delta] \to R$ represent the graphs on $[-1+\delta, 1-\delta]$ of the lower part of ∂C^ϵ, $\partial C^{\epsilon/2}$, respectively, i.e.,

$$\{(x, f_0(x_1)) : |x_1| \leq 1-\delta\} = \{(x_1, x_2) : |x_1| \leq 1-\delta,\ x_2 < 0\} \cap \partial C^{\epsilon/2}(a),$$

$$\{(x, b_0(x_1)) : |x_1| \leq 1-\delta\} = \{(x_1, x_2) : |x_1| \leq 1-\delta,\ x_2 < 0\} \cap \partial C^{\epsilon}(a).$$

Then $f_0 \leq b_0$, $f_0(-1+\delta, a) < b_0(-1+\delta, a)$, $f_0(1-\delta, a) < b_0(1-\delta, a)$. Clearly in the interval $[a, b]$ the functions f and b represent the mean curvature evolution of the lower part of ∂C^ϵ, $\partial C^{\epsilon/2}$, respectively. Then (5.10) follows by the strong maximum principle. This proves the claim.

For any $0 < \lambda \leq 1$ let $F_\lambda(a)$ be the λ-homotetic of $C^{\epsilon/2}(a)$, i.e., $F_\lambda(a) = \lambda C^{\epsilon/2}(a)$. Clearly $F_\lambda(a) \subseteq F_1(a)$ and $F_\lambda(a) \subseteq \mathrm{int}(C)$ for any $0 < \lambda < 1$. Denote by $F_\lambda(t)$ the mean curvature evolution of $F_\lambda(a)$. Now $F_\lambda(\lambda^2 t)$ coincides with $\lambda F_1(t)$, and $F_1(t) = C^{\epsilon/2}(t)$ is smooth for $t \in [a, b]$, hence there exists $\lambda \in]0, 1[$ such that the Hausdorff distance between $\partial F_\lambda(t)$ and $\partial C^{\epsilon/2}(t)$, $t \in [a, b]$, is arbitrarily small. Hence by (5.10) there exists $\lambda \in]0, 1[$ such that

$$C^\epsilon(t) \subseteq F_\lambda(t) \qquad \forall t \in [a, b].$$

It follows that $C^\epsilon(b) \subseteq F_\lambda(b) \subseteq \mathrm{mibar}\,(\mathrm{int}\,(C), \mathcal{F}_0)(b)$. ∎

Concerning the fact that $\mathrm{mibar}\,(E, \mathcal{F}_g)$ is very sensible with respect to modifications of the set E on subsets with zero Lebesgue measure, we can prove the following.

EXAMPLE 5.1: Let $n = 2$, $g = 0$, and assume that $E = \{(x, y) \in R^2 : x^2 + y^2 \leq 1\}$. Denote by S a closed segment contained in the interior of E. Then

(5.11) $\quad \mathrm{mibar}\,(E \setminus \{(0, 0)\}, \mathcal{F}_0)(t) = \mathrm{mibar}\,(E, \mathcal{F}_0)(t) \qquad \forall t \in I,\ t > t_0,$

(5.12) $\quad \mathrm{mibar}\,(E \setminus S, \mathcal{F}_0)(t) = \mathrm{mibar}\,(E, \mathcal{F}_0)(t) \qquad \forall t \in I,\ t > t_0,$

and the same results hold when E is the open unit ball.

Equality (5.11) can be proved (if E is either closed or open) by taking a smooth set $f(t_0) \subseteq \mathrm{int}(E) \setminus \{(0, 0)\}$ of the form

$$f(t_0) = \{x = (x_1, x_2) \in E : |x| \leq 1 - \varepsilon, (x_1 - \varepsilon)^2 + x_2^2 \geq 4\varepsilon^2\}.$$

Indeed, given $\tau > 0$, we can find $\varepsilon > 0$ small enough so that the evolution $f(t_0 + \tau)$ of

$f(t_0)$ at time $t_0+\tau$ contains $\{(0,0)\}$, hence

$$\{(0,0)\}\in f(t_0+\tau)\subseteq \mathrm{mibar}(E,\mathcal{F}_0)(t_0+\tau).$$

Equality (5.12) can be proved (if E is either closed or open) by taking a smooth set $f(t_0)\subseteq \mathrm{int}(E)\setminus S$, such that $\partial f(t_0)=\{|x|=1-\varepsilon\}\cup\partial A_\varepsilon$, where A_ε is an ellipse, $\{x\in E: \mathrm{dist}(x,S)<\varepsilon\}\subseteq A_\varepsilon\subseteq \mathrm{int}(E)$, $|A_\varepsilon|=O(\varepsilon)$, and the two focuses of A_ε lie on a line l parallel to the line containing S, with $0<\mathrm{dist}(l,S)=O(\varepsilon)$. Then, recalling that the shrinking time of ∂A_ε is proportional to the area $|A_\varepsilon|$ of the ellipse, given $\tau>0$ we can find $f(t_0)$ with the properties listed above so that $S\subseteq f(t_0+\tau)\subseteq \mathrm{mibar}(E,\mathcal{F}_0)(t_0+\tau)$.

The next example can be proved arguing as in the proof of (5.7).

EXAMPLE 5.2: Let $n=2$, $g=0$, and assume that $E=\{(x_1,x_2)\in R^2: x_1^2+x_2^2\le 1\}$. Let p be a point of ∂E. Then

$$\mathrm{mibar}(E\setminus\{p\},\mathcal{F}_0)(t)=\mathrm{mibar}(\mathrm{int}(E),\mathcal{F}_0)(t)\qquad \forall t\in I,\ t>t_0.$$

PROPOSITION 5.1: *Let E be a subset of R^n. Then*

$$R^n\setminus \mathrm{mibar}(E,\mathcal{F}_g)\in \mathrm{Barr}(\mathcal{F}_{-g}). \tag{5.13}$$

In particular

$$\mathrm{mibar}(R^n\setminus E,\mathcal{F}_{-g})\subseteq R^n\setminus \mathrm{mibar}(E,\mathcal{F}_g), \tag{5.14}$$

and this inclusion can be strict. Finally

$$\mathrm{mibar}_*(R^n\setminus E,\mathcal{F}_{-g})=R^n\setminus \mathrm{mibar}^*(E,\mathcal{F}_g). \tag{5.15}$$

PROOF: Assume by contradiction that (5.13) is false. Then there exists a function $f:[a,b]\subseteq I\to\mathcal{P}(R^n)$, $f\in\mathcal{F}_{-g}$, with $f(a)\subseteq R^n\setminus \mathrm{mibar}(E,\mathcal{F}_g)(a)$, and with $f(b)\cap\mathrm{mibar}(E,\mathcal{F}_g)(b)\neq\emptyset$. Letting $f^c=R^n\setminus f$, we have $f^c\in\mathcal{F}_g$, $\mathrm{mibar}(E,\mathcal{F}_g)(a)\subseteq\mathrm{int}(f^c(a))$, and there exists $x\in\mathrm{mibar}(E,\mathcal{F}_g)(b)\setminus\mathrm{int}(f^c(b))$. Let us define

$$\phi(t)=\begin{cases}\mathrm{mibar}(E,\mathcal{F}_g)(t)\cap\mathrm{int}(f^c(t)) & \text{if } t\in[a,b],\\ \mathrm{mibar}(E,\mathcal{F}_g)(t) & \text{if } t\in I\setminus[a,b].\end{cases}$$

Since $\phi(b)$ is strictly contained in $\mathrm{mibar}(E,\mathcal{F}_g)(b)$, to have a contradiction it is enough to show that $\phi\in\mathrm{Barr}(\mathcal{F}_g)$. This is equivalent to say that if $b:[c,d]\subseteq I\to\mathcal{P}(R^n)$, $b\in\mathcal{F}_g$, with $b(c)\subseteq\phi(c)$, then $b(d)\subseteq\phi(d)$. If $[c,d]\cap[a,b]=\emptyset$, then $b(d)\subseteq\phi(d)$, since $\mathrm{mibar}(E,\mathcal{F}_g)\in\mathrm{Barr}(\mathcal{F}_g)$. We can assume that $c\ge a$ since, if $c<a<d$ we have $b(a)\subseteq\mathrm{mibar}(E,\mathcal{F}_g)(a)=\phi(a)$. Hence let $a\le c<d\le b$. Since $b(a)\subseteq\mathrm{mibar}(E,\mathcal{F}_g)(a)$ we have $b(d)\subseteq\mathrm{mibar}(E,\mathcal{F}_g)(d)$, and since $b(a)\subseteq\mathrm{int}(f^c(a))$, by

the comparison principle between smooth evolutions we have also $b(d) \subsetneq \mathrm{int}(f^*(d))$. Hence $b(d) \subsetneq \phi(d)$, a contradiction.

To prove that inclusion (5.14) can be strict, let $g = 0$, $n = 2$, and $E = \{(x_1, x_2) \in \mathbf{R}^2 : |x_1| \le 1, |x_2| \le 1\}$. Then $\mathrm{mibar}(E, \mathcal{F}_0)(t)$ is open for any $t > t_0$ (see (5.7) and (5.3)), hence $\mathbf{R}^n \setminus \mathrm{mibar}(E, \mathcal{F}_0)(t)$ is closed for any $t > t_0$. On the other hand $\mathbf{R}^n \setminus E$ is open, hence $\mathrm{mibar}(\mathbf{R}^n \setminus E, \mathcal{F}_0)(t)$ is open for any $t > t_0$ from (5.3).

It remains to show (5.15). Observe that $(\mathbf{R}^n \setminus E)^-_\varrho = \mathbf{R}^n \setminus E^+_\varrho$, so that, if $t \in I$, by (5.14)

$$\mathrm{mibar}_*(\mathbf{R}^n \setminus E, \mathcal{F}_{-g})(t) = \bigcup_{\varrho > 0} \mathrm{mibar}((\mathbf{R}^n \setminus E)^-_\varrho, \mathcal{F}_{-g})(t) =$$

$$= \bigcup_{\varrho > 0} \mathrm{mibar}(\mathbf{R}^n \setminus E^+_\varrho, \mathcal{F}_{-g})(t) \subseteq \bigcup_{\varrho > 0} [\mathbf{R}^n \setminus \mathrm{mibar}(E^+_\varrho, \mathcal{F}_g)(t)] =$$

$$= \mathbf{R}^n \setminus \bigcap_{\varrho > 0} \mathrm{mibar}(E^+_\varrho, \mathcal{F}_g)(t) = \mathbf{R}^n \setminus \mathrm{mibar}^*(E, \mathcal{F}_g)(t).$$

Let us prove that

$$\mathbf{R}^n \setminus \mathrm{mibar}^*(E, \mathcal{F}_g)(t) \subseteq \mathrm{mibar}_*(\mathbf{R}^n \setminus E, \mathcal{F}_{-g})(t) \quad \forall t \in I. \tag{5.16}$$

We claim that for any $\varrho, \varepsilon > 0$ we have

$$\mathrm{mibar}(E^+_{\varrho+\varepsilon}, \mathcal{F}_g)(t) \supseteq \mathbf{R}^n \setminus \mathrm{mibar}(\mathbf{R}^n \setminus E^+_\varrho, \mathcal{F}_{-g})(t) \qquad \forall t \in I. \tag{5.17}$$

Set $\phi(t) = \mathrm{mibar}(E^+_{\varrho+\varepsilon}, \mathcal{F}_g)(t)$ and $\psi(t) = \mathrm{mibar}(\mathbf{R}^n \setminus E^+_\varrho, \mathcal{F}_{-g})(t)$. Then $\phi \in \mathrm{Barr}(\mathcal{F}_g)$ and $\psi \in \mathrm{Barr}(\mathcal{F}_{-g})$. Let us apply Theorem 4.1: we have

$$\eta(t_0) = \mathrm{dist}(\mathbf{R}^n \setminus E^+_{\varrho+\varepsilon}, E^+_\varrho) = \varepsilon > 0,$$

so that $\eta(t) > 0$ for any $t \in I$. By (4.9) we then have

$$\mathrm{mibar}(E^+_{\varrho+\varepsilon}, \mathcal{F}_g)(t) \supseteq \mathbf{R}^n \setminus \mathrm{int}(\psi(t)) \supseteq \mathbf{R}^n \setminus \psi(t) = \mathbf{R}^n \setminus \mathrm{mibar}(\mathbf{R}^n \setminus E^+_\varrho, \mathcal{F}_{-g})(t),$$

which proves the claim.

By (5.17) we have

$$\mathrm{mibar}_*(\mathbf{R}^n \setminus E, \mathcal{F}_{-g})(t) = \bigcup_{\varrho > 0} \mathrm{mibar}(\mathbf{R}^n \setminus E^+_\varrho, \mathcal{F}_{-g})(t) \supseteq$$

$$\supseteq \bigcup_{\varrho, \varepsilon > 0} [\mathbf{R}^n \setminus \mathrm{mibar}(E^+_{\varrho+\varepsilon}, \mathcal{F}_g)(t)] = \mathbf{R}^n \setminus \bigcap_{\varrho, \varepsilon > 0} \mathrm{mibar}(E^+_{\varrho+\varepsilon}, \mathcal{F}_g)(t) =$$

$$= \mathbf{R}^n \setminus \mathrm{mibar}^*(E, \mathcal{F}_g)(t),$$

and this proves (5.16), and concludes the proof of (5.15). ∎

PROPOSITION 5.2: *Let E be a subset of R^n. The following properties hold:*

$$\text{mibar}_\star(E, \mathcal{F}_\xi) = \text{mibar}_\star(\text{int}(E), \mathcal{F}_\xi), \tag{5.18}$$

$$\text{mibar}^\star(E, \mathcal{F}_\xi) = \text{mibar}^\star(\overline{E}, \mathcal{F}_\xi). \tag{5.19}$$

Moreover $\text{mibar}_\star(E, \mathcal{F}_\xi)(t)$ *is open and* $\text{mibar}^\star(E, \mathcal{F}_\xi)(t)$ *is closed for any* $t \in I$.

PROOF: Equality (5.18) is a consequence of the fact that $\overline{R^n \setminus E} = R^n \setminus \text{int}(E)$, so that $\text{dist}(x, R^n \setminus E) = \text{dist}(x, R^n \setminus \text{int}(E))$ and $E_\varrho^- = (\text{int}(E))_\varrho^-$ (see (3.8)). Similarly, (5.19) follows since $E_\varrho^+ = (\overline{E})_\varrho^+$ (see (3.9)).

Let $t \in I$; $\text{mibar}_\star(E, \mathcal{F}_\xi)(t)$ is open, since $\text{mibar}(E_\varrho^-, \mathcal{F}_\xi)(t)$ is open by (5.3) for any $\varrho > 0$.

It remains to prove that $\text{mibar}^\star(E, \mathcal{F}_\xi)(t)$ is closed. Let us show that for any $\varrho, \varepsilon > 0$ we have

$$\overline{\text{mibar}(E_\varrho^+, \mathcal{F}_\xi)(t)} \subseteq \text{mibar}(E_{\varrho+\varepsilon}^+, \mathcal{F}_\xi)(t) \qquad \forall t \in I. \tag{5.20}$$

Set

$$\phi(t) = \text{mibar}(E_{\varrho+\varepsilon}^+, \mathcal{F}_\xi)(t), \qquad \psi(t) = R^n \setminus \text{mibar}(E_\varrho^+, \mathcal{F}_\xi)(t) \quad \forall t \in I.$$

Then $\phi \in \text{Barr}(\mathcal{F}_\xi)$ and by (5.13) we have $\psi \in \text{Barr}(\mathcal{F}_{-\xi})$. Let us apply Theorem 4.1: since

$$\eta(t_0) = \text{dist}(R^n \setminus \text{mibar}(E_{\varrho+\varepsilon}^+, \mathcal{F}_\xi)(t_0), \text{mibar}(E_\varrho^+, \mathcal{F}_\xi)(t_0)) =$$
$$= \text{dist}(R^n \setminus E_{\varrho+\varepsilon}^+, E_\varrho^+) = \varepsilon > 0,$$

it follows that $\eta(t) > 0$ for any $t \in I$. Hence by (4.9) we have

$$\text{mibar}(E_{\varrho+\varepsilon}^+, \mathcal{F}_\xi)(t) \supseteq R^n \setminus \text{int}(R^n \setminus \text{mibar}(E_\varrho^+, \mathcal{F}_\xi)(t)) = \overline{\text{mibar}(E_\varrho^+, \mathcal{F}_\xi)(t)},$$

i.e., (5.20).

Then by (5.20) we have, for any $t \in I$

$$\text{mibar}^\star(E, \mathcal{F}_\xi)(t) = \bigcap_{\varrho, \varepsilon > 0} \text{mibar}(E_{\varrho+\varepsilon}^+, \mathcal{F}_\xi)(t) \supseteq \bigcap_{\varrho > 0} \overline{\text{mibar}(E_\varrho^+, \mathcal{F}_\xi)(t)},$$

so that $\text{mibar}^\star(E, \mathcal{F}_\xi)(t) = \bigcap_{\varrho > 0} \overline{\text{mibar}(E_\varrho^+, \mathcal{F}_\xi)(t)}$, which is closed. This concludes the proof. ■

6. - COMPARISON BETWEEN THE BARRIERS AND AN ABSTRACT EVOLUTION

The main result of this section is Theorem 6.1, namely a comparison theorem between the evolutions $\text{mibar}(E, \mathcal{F}_\xi)$, $\text{mibar}_\star(E, \mathcal{F}_\xi)$, $\text{mibar}^\star(E, \mathcal{F}_\xi)$, and an abstract evolution law R satisfying suitable properties.

Let us give the definition of comparison flow; for simplicity of notation we shall drop the dependence on g in the notation of the comparison flow.

Definition 6.1: *Let $\mathcal{A}$ be a family of sets which contains the open sets and the closed sets of R^n. Let R be a function defined in $\mathcal{A} \times I$. We say that R is a comparison flow if the following holds. If $(E, \tau) \in \mathcal{A} \times I$, setting $\zeta = R(E, \tau)$, then $\zeta : [\tau, +\infty[\to \mathcal{P}(R^n)$, $\zeta(\tau) = E$, and the following properties hold:*

(i) *(semigroup property) for any $A \in \mathcal{A}$, A with compact boundary, any $t_0 \leq t_1 \leq t_2$, if we set $B = R(A, t_1)(t_2)$ we have*

$$R(A, t_1)(t) = R(B, t_2)(t) \qquad \forall t \in [t_2, +\infty[;$$

(ii) *(extension of smooth flows) if $C \subseteq R^n$ is a closed set with smooth compact boundary and if $t_1 \geq t_0$ then $R(C, t_1)(t)$ coincides with the smooth evolution of C by its mean curvature with forcing term g for all times $t \geq t_1$ for which such smooth evolution exists;*

(iii) *(comparison principle) if $A, B \in \mathcal{A}$ with $A \subseteq B$, and if $t_1 \in I$, then $R(A, t_1)(t) \subseteq R(B, t_1)(t)$ for any $t \in [t_1, +\infty[$.*

Lemma 6.1: *Let R be a comparison flow. Then*

$$R(E, t_0) \in \mathrm{Barr}(\mathcal{F}_g); \tag{6.1}$$

$$\mathrm{int}\,(R(E, t_0)) \in \mathrm{Barr}(\mathcal{F}_g); \tag{6.2}$$

$$R^n \setminus \overline{R(E, t_0)} \in \mathrm{Barr}(\mathcal{F}_{-g}). \tag{6.3}$$

Proof: Set as usual $I = [t_0, +\infty[$. Then $R(E, t_0) : I \to \mathcal{P}(R^n)$. Let us prove (6.1). Let $f : [a, b] \subseteq I \to \mathcal{P}(R^n)$, $f \in \mathcal{F}_g$, with $f(a) \subseteq R(E, t_0)(a)$; we have to show that $f(b) \subseteq R(E, t_0)(b)$. Since $f \in \mathcal{F}_g$, by (ii) of Definition 6.1 we have $f(t) = R(f(a), a)(t)$ for any $t \in [a, b]$. Therefore, using (iii) and (i) we get $f(b) = R(f(a), a)(b) \subseteq R(R(E, t_0)(a), a)(b) = R(E, t_0)(b)$, and (6.1) is proved.

Let us prove (6.2). Let $f : [a, b] \subseteq I \to \mathcal{P}(R^n)$, $f \in \mathcal{F}_g$, with $f(a) \subseteq \mathrm{int}\,(R(E, t_0)(a))$; we have to show that $f(b) \subseteq \mathrm{int}\,(R(E, t_0)(b))$. For any $t \in [a, b]$ set $\phi(t) = R(E, t_0)(t)$, $\psi(t) = R^n \setminus f(t)$, and $\eta(t) = \mathrm{dist}\,(R^n \setminus \phi(t), R^n \setminus \psi(t))$. Then $\phi \in \mathrm{Barr}(\mathcal{F}_g)$, $\psi \in \mathrm{Barr}(\mathcal{F}_{-g})$, either $R^n \setminus \mathrm{int}\,(\phi(a))$ or $R^n \setminus \mathrm{int}\,(\psi(a))$ is compact, and $\eta(a) > 0$. Applying Theorem 4.1 we obtain $\eta(b) > 0$, which implies $f(b) \subseteq \mathrm{int}\,(R(E, t_0)(b))$.

Let us prove (6.3). Let $f : [a, b] \subseteq I \to \mathcal{P}(R^n)$, $f \in \mathcal{F}_{-g}$, with $f(a) \subseteq R^n \setminus \overline{R(E, t_0)(a)} = \mathrm{int}\,(R^n \setminus R(E, t_0)(a))$; we have to show that $f(b) \subseteq R^n \setminus \overline{R(E, t_0)(b)}$. Since $\mathrm{dist}\,(f(a), \overline{R(E, t_0)(a)}) > 0$, we can find a smooth set $f_\varepsilon(a)$ with compact boundary such that $f(a) \subseteq \mathrm{int}(f_\varepsilon(a)) \subseteq f_\varepsilon(a) \subseteq \mathrm{int}\,(R^n \setminus R(E, t_0)(a))$ and, if $f_\varepsilon(t)$ denotes the mean curvature evolution of $f_\varepsilon(a)$ with forcing term $-g$, then $f_\varepsilon : [a, b] \to \mathcal{P}(R^n)$, $f_\varepsilon \in \mathcal{F}_{-g}$. Set $f'_\varepsilon = \overline{R^n \setminus f_\varepsilon}$; we have $f'_\varepsilon \in \mathcal{F}_g$, $f'_\varepsilon(a) \supseteq \overline{R(E, t_0)(a)}$, hence $f_\varepsilon^c(b) = \overline{\mathbf{R}^n \setminus f_\varepsilon(b)} \supseteq \overline{R(E, t_0)(b)}$. By Lemma 4.2 it follows that $\mathrm{dist}\,(f(b), \overline{\mathbf{R}^n \setminus f_\varepsilon(b)}) > 0$, and therefore $f(b) \subseteq \mathbf{R}^n \setminus \overline{R(E, t_0)(b)}$. ∎

Remark 6.1: *Let R be a comparison flow. Assume that*

$$\mathrm{int}\,(\overline{R(E, t_0)(t)}) \subseteq R(E, t_0)(t) \qquad \forall t \in I. \tag{6.4}$$

Then

$$\overline{R(E, t_0)} \in \mathrm{Barr}\,(\mathcal{F}_g).$$

Proof: Let $f : [a, b] \subseteq I = [t_0, +\infty[\to \mathcal{P}(\mathbf{R}^n)$, $f \in \mathcal{F}_g$, with $f(a) \subseteq \overline{R(E, t_0)(a)}$; we have to show that $f(b) \subseteq \overline{R(E, t_0)(b)}$. We can approximate $f(a)$ with a family $\{f_\varepsilon(a)\}$ of smooth sets with compact boundary so that $f_\varepsilon(a) \subseteq \mathrm{int}\,(\overline{R(E, t_0)(a)})$ and, if $f_\varepsilon(t)$ denotes the mean curvature evolution of $f_\varepsilon(a)$ with forcing term g, then $f_\varepsilon : [a, b] \to \mathcal{P}(\mathbf{R}^n)$, $f_\varepsilon \in \mathcal{F}_g$. Using (ii), (iii), (6.4), and (i) we have

$$f_\varepsilon(b) = R(f_\varepsilon(a), a)(b) \subseteq R(\mathrm{int}\,(\overline{R(E, t_0)(a)}, a)(b) \subseteq R(R(E, t_0)(a), a)(b) =$$

$$= R(E, t_0)(b) \subseteq \overline{R(E, t_0)(b)}.$$

Therefore $f(b) = \overline{\bigcup_{\varepsilon > 0} f_\varepsilon(b)} \subseteq \overline{R(E, t_0)(b)}$. ∎

The relations between the minimal barriers and a comparison flow read as follows.

Theorem 6.1: *Let $g \in \mathcal{C}^\infty(\mathbf{R}^n \times I) \cap L^\infty(\mathbf{R}^n \times I)$ be a function satisfying (2.1). Let $\mathfrak{C}$ be as in Definition 6.1, and let R be a comparison flow. Then, if $E \subseteq \mathfrak{C}$ has compact boundary, and if E_ϱ^-, E_ϱ^+ are defined as in (3.8) and (3.9) respectively, we have*

$$\mathrm{mibar}_\star(E, \mathcal{F}_g)(t) = \bigcup_{\varrho > 0} R(E_\varrho^-, t_0)(t) = \bigcup_{\varrho > 0} \overline{R(E_\varrho^-, t_0)(t)} \subseteq$$

$$\subseteq \mathrm{mibar}\,(E, \mathcal{F}_g)(t) \subseteq R(E, t_0)(t) \subseteq \overline{R(E, t_0)(t)} \subseteq \bigcap_{\varrho > 0} R(E_\varrho^+, t_0)(t) =$$

$$= \bigcap_{\varrho > 0} \overline{R(E_\varrho^+, t_0)(t)} = \mathrm{mibar}^\star(E, \mathcal{F}_g)(t),$$

for any $t \in I$.

Proof: Recalling $I = [t_0, +\infty[$, we have $R(E, t_0) : I \to \mathcal{P}(\mathbf{R}^n)$. The inclusion

$$\mathrm{mibar}\,(E, \mathcal{F}_g)(t) \subseteq R(E, t_0)(t) \qquad \forall t \in I \tag{6.5}$$

is an immediate consequence of (6.1) and the definition of $\mathrm{mibar}\,(E, \mathcal{F}_g)$.

We claim that

$$\overline{R(E_\varrho^-, t_0)(t)} \subseteq \operatorname{mibar}(E, \mathcal{F}_\varepsilon)(t) \qquad \forall t \in I, \ \forall \varrho > 0. \tag{6.6}$$

If $\varrho > 0$ we have $E_\varrho^- \in \mathcal{C}l$; for any $t \in I$ set $\phi(t) = \operatorname{mibar}(E, \mathcal{F}_\varepsilon)(t)$ and $\psi(t) = R^n \setminus \overline{R(E_\varrho^-, t_0)(t)}$. Then $\phi \in \operatorname{Barr}(\mathcal{F}_\varepsilon)$, $\psi \in \operatorname{Barr}(\mathcal{F}_{-\varepsilon})$ by (6.3), either $R^n \setminus \operatorname{int}(\phi(t_0))$ or $R^n \setminus \operatorname{int}(\psi(t_0))$ is compact, and

$$\eta(t_0) = \operatorname{dist}(\overline{R(E_\varrho^-, t_0)(t_0)}, R^n \setminus E) = \operatorname{dist}(E_\varrho^-, R^n \setminus E) = \varrho > 0.$$

It follows that $\eta(t) > 0$ for any $t \in I$, so that $R^n \setminus \psi(t) \subseteq \phi(t)$ (see (4.9)), and (6.6) is proved.

From (6.6) it follows that

$$\bigcup_{\varrho > 0} \overline{R(E_\varrho^-, t_0)(t)} \subseteq \operatorname{mibar}(E, \mathcal{F}_\varepsilon)(t) \qquad \forall t \in I. \tag{6.7}$$

Let us show that

$$\operatorname{mibar}_*(E, \mathcal{F}_\varepsilon)(t) = \bigcup_{\varrho > 0} R(E_\varrho^-, t_0)(t) = \bigcup_{\varrho > 0} \overline{R(E_\varrho^-, t_0)(t)} \qquad \forall t \in I. \tag{6.8}$$

By applying (6.5) with E replaced by E_ϱ^-, taking the union over $\varrho > 0$, and using the definition of $\operatorname{mibar}_*(E, \mathcal{F}_\varepsilon)$ we obtain

$$\operatorname{mibar}_*(E, \mathcal{F}_\varepsilon)(t) \subseteq \bigcup_{\varrho > 0} R(E_\varrho^-, t_0)(t) \qquad \forall t \in I. \tag{6.9}$$

In addition if $\varrho > 0$ and $t \geq t_0$, by (6.7) we have

$$\operatorname{mibar}(E_\varrho^-, \mathcal{F}_\varepsilon)(t) \supseteq \bigcup_{\delta > 0} \overline{R((E_\varrho^-)_\delta^-, t_0)(t)} \supseteq \overline{R(E_{2\varrho}^-, t_0)(t)} \qquad \forall t \geq t_0.$$

Hence

$$\operatorname{mibar}_*(E, \mathcal{F}_\varepsilon)(t) \supseteq \bigcup_{\varrho > 0} \overline{R(E_{2\varrho}^-, t_0)(t)} = \bigcup_{\varrho > 0} \overline{R(E_\varrho^-, t_0)(t)} \qquad \forall t \geq t_0,$$

and (6.8) follows.

Let us show that

$$\overline{R(E, t_0)(t)} \subseteq \bigcap_{\varrho > 0} R(E_\varrho^+, t_0)(t) \qquad \forall t \in I. \tag{6.10}$$

Let $\varrho > 0$ and for any $t \in I$ set $\phi(t) = R(E_\varrho^+, t_0)(t)$, $\psi(t) = R^n \setminus \overline{R(E, t_0)(t)}$. Then by (6.1) and (6.3) we have $\phi \in \operatorname{Barr}(\mathcal{F}_\varepsilon)$ and $\psi \in \operatorname{Barr}(\mathcal{F}_{-\varepsilon})$. Moreover $\eta(t_0) = \operatorname{dist}(R^n \setminus E_\varrho^+, \overline{E}) = \varrho > 0$. By Theorem 4.1 we obtain $\eta(t) > 0$ for any $t \in I$. Hence by (4.9) we get

$$R^n \setminus \psi(t) = \overline{R(E, t_0)(t)} \subseteq \phi(t) = R(E_\varrho^+, t_0)(t),$$

which implies (6.10).

It remains to show that

$$\bigcap_{\varrho>0} R(E_\varrho^+, t_0)(t) = \bigcap_{\varrho>0} \overline{R(E_\varrho^+, t_0)(t)} = \text{mibar}^\star(E, \mathcal{F}_\xi)(t) \qquad \forall t \in I. \tag{6.11}$$

The inclusion $\text{mibar}^\star(E, \mathcal{F}_\xi)(t) \subseteq \bigcap_{\varrho>0} R(E_\varrho^+, t_0)(t)$ follows by applying (6.5) with E replaced by E_ϱ^+, by taking the intersection over $\varrho > 0$, and recalling the definition of $\text{mibar}^\star(E, \mathcal{F}_\xi)$. Let us show that

$$\text{mibar}^\star(E, \mathcal{F}_\xi)(t) \supseteq \bigcap_{\varrho>0} \overline{R(E_\varrho^+, t_0)(t)} \qquad \forall t \in I. \tag{6.12}$$

If $\varrho > 0$ we have, by (6.7),

$$\text{mibar}(E_\varrho^+, \mathcal{F}_\xi)(t) \supseteq \bigcup_{\delta>0} \overline{R((E_\varrho^+)_\delta^-, t_0)(t)} \supseteq \overline{R(E_{\varrho/2}^+, t_0)(t)},$$

and (6.12) follows. The proof is complete. ■

7. - Comparison between the barriers and the viscosity evolution

In this section we compare the minimal barriers with the viscosity evolution. To this aim let us introduce some notation. Let E be a closed set with compact boundary; for any $t \in I$ we indicate by $V(E, g)(t)$ the mean curvature evolution of E with forcing term g in the viscosity sense [9, 10, 20, 25, 26]. This means that

$$V(E, g)(t) = \{v(\cdot, t) \leq 0\} \qquad \forall t \in I, \tag{7.1}$$

where $v \in \mathcal{C}(R^n \times I) \cap L^\infty(R^n \times I)$ is the unique viscosity solution of

$$\begin{cases} v_t - |\nabla v| \operatorname{div}\left(\dfrac{\nabla v}{|\nabla v|}\right) + |\nabla v| g = 0, \\ v(x, t_0) = v_0(x), \end{cases} \tag{7.2}$$

and the continuous function v_0 is constant outside some bounded set and is chosen so that $E = \{x \in R^n : v_0(x) \leq 0\}$ [20]. If E is an open set with compact boundary, we shall take $V(E, g)(t) = \{v(\cdot, t) < 0\}$, where v_0 is chosen so that $E = \{x \in R^n : v_0(x) < 0\}$.

Given the set E with compact boundary, when we write $V(E, g)$ we implicitly assume that E is either closed or open, and $V(E, g)$ is defined using the conventions described above. Observe that the connection between the case E closed and E open is given by the equality

$$R^n \setminus V(E, g)(t) = V(R^n \setminus E, -g)(t) \qquad \forall t \in I.$$

The following result follows directly from Theorem 6.1, since $V(E, g)$ satisfies properties (i)-(iii) of Definition 6.1, i.e., $V(E, g)$ is a comparison flow (in this case we choose $\mathcal{A}$ as the family of all open and closed subsets of R^n).

THEOREM 7.1: *Let v be the viscosity solution of (7.2). Then for any $t \in I$ we have*

$$\mathrm{mibar}_\star (E, \mathcal{F}_g)(t) = \{v(\cdot, t) < 0\} \subseteq \mathrm{mibar}\,(E, \mathcal{F}_g)(t) \subseteq \{v(\cdot, t) \leq 0\} = \mathrm{mibar}^\star (E, \mathcal{F}_g)(t).$$

In particular, if E is closed with compact boundary then

$$V(E, g)(t) = \mathrm{mibar}^\star (E, \mathcal{F}_g)(t) \qquad \forall t \in I,$$

and if E is open with compact boundary then

$$V(E, g)(t) = \mathrm{mibar}\,(E, \mathcal{F}_g)(t) = \mathrm{mibar}_\star (E, \mathcal{F}_g)(t) \qquad \forall t \in I.$$

Note that from Theorem 7.1 we deduce

$$\mathrm{mibar}^\star (E, \mathcal{F}_g)(t) \setminus \mathrm{mibar}_\star (E, \mathcal{F}_g)(t) = \{v(\cdot, t) = 0\} \qquad \forall t \in I.$$

The following result is related to the singularities of the mean curvature flow, and is a consequence of Remark 6.1.

THEOREM 7.2: *Let v be the viscosity solution of (7.2). If*

$$\text{(7.3)} \qquad \mathrm{int}\,(\overline{\{v(\cdot, t) < 0\}}) \subseteq \{v(\cdot, t) < 0\} \qquad \forall t \in I,$$

then

$$\overline{\{v < 0\}} \in \mathrm{Barr}\,(\mathcal{F}_g).$$

Assumption (7.3) is necessary; as a counterexample let $n = 2$, $g = 0$, and let us consider the viscosity evolution of the boundary of two equal squares in R^2 having a common edge, by taking v_0 negative inside the two squares. For any $t \in I$ the set $\overline{\{v(\cdot, t) < 0\}}$ corresponds to the union of the mean curvature evolution of each square separately, while $\overline{\{v_0 < 0\}}$ is a rectangle. Hence $\overline{\{v(\cdot, t) < 0\}} \notin \mathrm{Barr}\,(\mathcal{F}_g)$.

The result of Theorem 7.2 becomes interesting in connection with the presence of *fattening* in the viscosity evolution [3, 4, 17, 22, 28], since in such case the set $\overline{\{v(\cdot, t) < 0\}}$ is strictly smaller than $\{v(\cdot, t) \leq 0\}$. Accurate connections between $V(E, g)$ and mibar $(E, \mathcal{F}_g)$ seems however non trivial in presence of fattening due to property (7.3) and the counterexample above.

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