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DIEGO PALLARA (*)

Some New Results on Functions of Bounded Variation (**) (***)

ABSTRACT. — The aim of this paper is to present some connections between the classical spaces of functions of bounded variation and other classes of functions whose variation is in some sense controlled, namely the GBV classes introduced by E. De Giorgi and L. Ambrosio, as well as the classes BBV, LBV, GBV^* defined by the author in his note [PD].

Alcuni nuovi risultati sulle funzioni a variazione limitata

RIASSUNTO. — Vengono presentate alcune connessioni tra gli spazi classici delle funzioni a variazione limitata ed altre classi di funzioni la cui variazione è opportunamente controllata, cioè le classi GBV introdotte da E. De Giorgi e L. Ambrosio, e le classi BBV, LBV, GBV^* introdotte dall'autore nella nota [PD].

INTRODUCTION

In the last few years a number of papers have been devoted to define some new functional classes related to the classical space BV of the functions of bounded variation and to study some variational problems which seem to be tractable when posed in such classes. In particular in [DGA] the spaces SBV and GBV of *special* and respectively *generalized* functions of bounded variation have been introduced, and some variational problems have been formulated (see also [DGCT]). Such classes have been subsequently studied in [AL1], [AL2], and weak solutions of many of the above-mentioned problems have been obtained (see also [CLPP]); meanwhile, the classes SBV have also proved useful to find classical solutions for some of these problems through suitable regularity results (see [DGCL], [CL], [CT]). We refer to [DG3] for a survey

(*) Indirizzo dell'Autore: Dipartimento di Matematica dell'Università della Basilicata, via N. Sauro 85, 85100 Potenza, Italy.
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of these results as well as for many conjectures concerning further develop ments and related open problems.

In this paper we first study a class of integral functionals strictly tied to the total variation measure of a (vector-valued) function of bounded variation and some related properties of *BV* functions (Section 2); then we prove some results concerning the functional classes *GBV* introduced in [DGA]; *GBV* classes were defined by means of a variational condition (cf. Definition 3.1): we show that they may be characterized by a distributional property (see Theorem 3.5) which could be useful in their handling.

Subsequently (Section 4), we introduce the classes *BBV*, whose members are bounded functions of bounded variation (on a bounded set with finite perimeter), and the related classes *LBV* and GBV^* and study some relations between them and the spaces *BV* and *GBV*; the most interesting result seems to be Theorem 4.16, where a connection between *LBV* and *BV* is shown.

The main results contained in this paper have been announced without proof in [PD]. I wish to thank E. De Giorgi and L. Ambrosio for many helpful conversations.

1. - Notations and preliminaries

In this paper Ω always indicate an open subset of $\mathbf{R}^n$, $n \geq 1$; moreover we adopt the following standard notation:

$B_\varrho(x) = \{y \in \mathbf{R}^n : |x-y| < \varrho\}$, $B_\varrho = B_\varrho(0)$, $S^{n-1} = \partial B_1$,

$\langle \cdot | \cdot \rangle$ the scalar product in an euclidean space,

(e_i) an orthonormal basis of $\mathbf{R}^n$,

$\mathscr{L}(\mathbf{R}^n, \mathbf{R}^k) = \{w : \mathbf{R}^n \to \mathbf{R}^k,\ w \text{ linear}\}$, endowed with the norm

$$|w| = \sqrt{\sum_{i=1}^{n} |w(e_i)|^2}\,,$$

$\mathscr{A}(\Omega)$ the class of all open sets $A \subseteq \Omega$,

$\mathscr{B}(\Omega)$ the class of all Borel sets $B \subseteq \Omega$,

$A \subset\subset \Omega$ means that $A \in \mathscr{A}(\Omega)$, $\bar{A}$ is compact and $\bar{A} \subset \Omega$,

χ_B the characteristic function of a set B,

Id the identity function, $\mathrm{Id}(x) = x\ \forall x$,

$|B|$ the Lebesgue measure of a measurable set $B \subseteq \mathbf{R}^n$,

$\mathscr{H}_k$ the k-dimensional Hausdorff measure in $\mathbf{R}^n$, $k \in [0, n]$,

$\tilde{\mathbf{R}}^k = \mathbf{R}^k \cup \{\infty\}$ the one-point compactification of $\mathbf{R}^k$,

$|\mu|$ the total variation measure of a measure μ.

DEFINITION 1.1 (cf. [DGA]): Let $u\colon \Omega \to \mathbf{R}^k$ be a Borel function. For $x \in \Omega$, $z \in \bar{\mathbf{R}}^k$, we set $z = \operatorname{aplim}_{y \to x} u(y)$ (approximate limit of u at x) if

$$g(z) = \lim_{\varrho \to 0} |B_\varrho|^{-1} \int_{B_\varrho} g(u(x+\xi))\, d\xi$$

for every continuous (hence bounded) function $g\colon \bar{\mathbf{R}}^k \to \mathbf{R}$.

If $z \in \mathbf{R}^k$ then this definition agrees with those given in [FH], [VA] (see [AL2]).

If $u(x) = \operatorname{aplim}_{y \to x} u(y)$ then u is said to be approximately continuous at x; moreover, the set

$$S_u = \left\{x \in \Omega\colon \operatorname{aplim}_{y \to x} u(y) \text{ does not exist}\right\}$$

is a Borel set, negligible with respect to the Lebesgue measure: one can thus always select a function equivalent to u which is approximately continuous almost everywhere: it will be denoted by $\tilde{u}$.

DEFINITION 1.2 (cf. [DGA]): Let $u\colon \Omega \to \mathbf{R}^k$ be a Borel function, and $\nu \in S^{n-1}$. For $x \in \Omega$, $z \in \bar{\mathbf{R}}^k$ we set $z = \operatorname{tr}^+(x, u, \nu)$ (outer trace in x along ν) if

$$g(z) = \lim_{\varrho \to 0} 2|B_\varrho|^{-1} \int_{B_\varrho \cap \{\xi\colon \langle \xi | \nu \rangle > 0\}} g(u(x+\xi))\, d\xi$$

for every continuous function $g\colon \bar{\mathbf{R}}^k \to \mathbf{R}$ (see [VA] for an equivalent definition); define also the inner trace as follows:

$$\operatorname{tr}^-(x, u, \nu) = \operatorname{tr}^+(x, u, -\nu).$$

Notice that the following statements are equivalent:

(i) $z = \operatorname{aplim}_{y \to x} u(y)$,

(ii) $\operatorname{tr}^-(x, u, \nu) = \operatorname{tr}^+(x, u, \nu) = z$.

Moreover, if both the inner and the outer trace exist in $x \in S_u$ along two directions ν, ν' then necessarily $\nu = \pm \nu'$.

DEFINITION 1.3 (cf. [DGA]): Let $u\colon \Omega \to \mathbf{R}^k$ be a Borel function. For $x \in \Omega \setminus S_u$ with $\tilde{u}(x) \in \mathbf{R}^k$, we say that u is approximately differentiable at x if there exists a (necessarily unique) linear operator $\nabla u(x) \in \mathscr{L}(\mathbf{R}^n, \mathbf{R}^k)$ (approximate differential of u at x) such that

$$\operatorname{aplim}_{y \to x} \frac{|u(y) - \tilde{u}(x) - [\nabla u(x)](y-x)|}{|y-x|} = 0.$$

If $u \in C^1$ then the approximate differential coincides with the classical differential.

2. - Some remarks on functions of bounded variation

This section is devoted to point out some properties of functions of bounded variation which are partially scattered in the literature (see *e.g.* [AMT], [DGA], [GE], [MM], [VA], [ZW]) and could hopefully pave the way to the subsequent generalizations.

Definition 2.1: We shall indicate by CN the class of all convex functions $\vartheta\colon \mathscr{L}(\mathbb{R}^n, \mathbb{R}^k) \to [0, +\infty)$ such that

$$\begin{aligned} \vartheta(\lambda w) &= \lambda\vartheta(w) && \forall\lambda > 0,\ w \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^k), \\ \vartheta(w) &= 0 && \text{iff } w = 0 . \end{aligned}$$

In particular, for every such function ϑ, there exist positive constants α, β such that

$$\alpha|w| < \vartheta(w) < \beta|w| \tag{2.1}$$

for every $w \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^k)$.

Throughout this section the letter u will always indicate an $\mathbb{R}^k$ valued Borel function defined in an open set $\Omega \subseteq \mathbb{R}^n$.

Definition 2.2: Let u be as above, and $\vartheta \in CN$; for every $A \in \mathscr{A}(\Omega)$ we set:

$$\int_A \vartheta(Du) = \inf\left\{\liminf_{h\to\infty}\int_A \vartheta(\nabla u_h)\,dx,\ (u_h) \subset C^1(A),\ u_h \to u \text{ a.e. in } A\right\}.$$

Remark 2.3: If $\int_\Omega \vartheta(Du) < +\infty$, then $u \in L^1_{loc}(\Omega)$, and we have:

$$\int_A \vartheta(Du) = \inf\left\{\liminf_{h\to\infty}\int_A \vartheta(\nabla u_h)\,dx,\ (u_h) \subset C^1(A),\ u_h \to u \text{ in } L^1_{loc}(A)\right\} \tag{2.2}$$

for every $A \in \mathscr{A}(\Omega)$.

In fact, let $B \subset\subset \Omega$ be a sphere; from the well-known Poincaré-Wirtinger inequality (see *e.g.* [ZW, 5.11.2]), we readily deduce (for suitable constants c and γ)

$$\int_B |u - c|\,dx < \gamma\int_B \vartheta(Du) , \tag{2.3}$$

whence $u \in L^1_{loc}(\Omega)$; equality (2.2) follows at once. ■

Notice that the functional $\int_\Omega \vartheta(Du)$ is L^1_{loc} lower semicontinuous; moreover if $\int_\Omega \vartheta(Du) < +\infty$ holds for some $\vartheta \in CN$, then from (2.1) it follows that it still holds for every function in CN.

LEMMA 2.4: *For every $\vartheta \in CN$ and every u we have:*

$$\int_\Omega \vartheta(Du) = \sup\left\{\int_A \vartheta(Du),\ A \subset\subset \Omega\right\}.$$

PROOF: The inequality

$$\sup\left\{\int_A \vartheta(Du),\ A \subset\subset \Omega\right\} \le \int_\Omega \vartheta(Du)$$

is trivial. To prove the other one, suppose first $\int_\Omega \vartheta(Du) < +\infty$. Arguing as in [AMT, Theorem 4.3], by (2.1) we get

$$\int_\Omega \vartheta(Du) \le \int_A \vartheta(Du) + \int_{\Omega\setminus\bar{B}} \vartheta(Du) \le \int_A \vartheta(Du) + \beta \int_{\Omega\setminus\bar{B}} |Du|$$

for $B \subset\subset A \subset\subset \Omega$. Given $\varepsilon > 0$, choose A, B in such a way that $\int_{\Omega\setminus\bar{B}} |Du| < \varepsilon/\beta$, and $B \subset\subset A \subset\subset \Omega$; we obtain

$$\int_\Omega \vartheta(Du) \le \int_A \vartheta(Du) + \varepsilon\,,$$

which yields $\int_\Omega \vartheta(Du) \le \sup\left\{\int_A \vartheta(Du),\ A \subset\subset \Omega\right\}$.

If $\int_\Omega \vartheta(Du) = +\infty$, then $\int_\Omega |Du| = +\infty$, too, so that for every $M > 0$ there exists $A \subset\subset \Omega$ such that $\int_A |Du| > M/\alpha$, whence $\int_A \vartheta(Du) > M$; this completes the proof. ■

Taking into account Proposition 5.5 and Théorème 5.6 of [DGL] we can deduce the following result.

PROPOSITION 2.5: *If $\int_\Omega \vartheta(Du) < +\infty$, then the set function*

$$A \mapsto \int_A \vartheta(Du)\,, \qquad A \in \mathscr{A}(\Omega)$$

is the trace of a Borel measure. Moreover, it can be uniquely extended to a regular Borel measure setting:

$$\int_B \vartheta(Du) = \inf\left\{\int_A \vartheta(Du),\ B \subset A,\ A \in \mathscr{A}(\Omega)\right\}.$$

Such measure will be denoted $\mu_{\vartheta,u}$.

REMARK 2.6: For a given function u, the measures defined in Proposition 2.5 for every $\vartheta \in CN$ are all mutually absolutely continuous, in view of our coercivity assumption (2.1).

The next results provide some other characterizations of the functional we are dealing with.

THEOREM 2.7: *Let $u \in L^1_{\mathrm{loc}}(\Omega)$ and $\vartheta \in CN$; then the following equalities hold:*

(i) $$\int_\Omega \vartheta(Du) = \sup\left\{\int_\Omega \left\langle u \,\Big|\, \sum_{j=1}^n \partial_{x_j} w_j \right\rangle dx,\ w \in C^1_0(\Omega, \mathscr{L}(\mathbf{R}^n, \mathbf{R}^k)),\ \vartheta^*(w) = 0\right\},$$

where $\vartheta^*(\zeta) = \sup\{\langle y \,|\, \zeta\rangle - \vartheta(y),\ y \in \mathscr{L}(\mathbf{R}^n, \mathbf{R}^k)\}$, and we have set $w_j = w(e_j)$ with (e_j) the canonical basis of $\mathbf{R}^n$;

(ii) *for every $\varphi \in C^\infty_0(\mathbf{R}^n)$ such that $\varphi \geq 0$, $\int_{\mathbf{R}^n} \varphi = 1$, set $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$, we have*

$$\int_\Omega \vartheta(Du) = \sup\left\{\limsup_{\varepsilon \to 0} \int_A \vartheta(\nabla(u * \varphi_\varepsilon))\, dx,\ A \subset\subset \Omega\right\}.$$

PROOF: Set for every $v \in L^1_{\mathrm{loc}}(\Omega)$

$$G(v, A) = \sup\left\{\int_A \left\langle v \,\Big|\, \sum_{j=1}^n \partial_{x_j} w_j \right\rangle dx,\ w \in C^1_0(A, \mathscr{L}(\mathbf{R}^n, \mathbf{R}^k)),\ \vartheta^*(w) = 0\right\};$$

let us check that $G(v, A) = \int_A \vartheta(\nabla v)\, dx$ for every $v \in C^1(A)$. To this end, we first remark that ϑ^* is the indicator function of the bounded set

$$B = \{\zeta \in \mathscr{L}(\mathbf{R}^n, \mathbf{R}^k) : \langle y \,|\, \zeta\rangle \leq 1 \ \ \forall y \text{ such that } \vartheta(y) \leq 1\}, \quad \text{and that } \vartheta^{**} = \vartheta.$$

Then, by well-known measurable selection theorems (see *e.g.* [RT, Corollary 1C]) for every $v \in C^1(A)$ there exists a measurable function $w: A \to \mathscr{L}(\mathbf{R}^n, \mathbf{R}^k)$ such that:

$$\int_A \vartheta(\nabla v)\, dx = \int_A \sup_\zeta \{\langle \nabla v(x) \,|\, \zeta\rangle - \vartheta^*(\zeta)\}\, dx =$$
$$= \int_A \sup_\zeta \{\langle \nabla v(x) \,|\, \zeta\rangle,\ \zeta \in B\}\, dx = \int_A \langle \nabla v(x) \,|\, w(x)\rangle\, dx.$$

On the other hand, $w(x) \in B$ for a.a. $x \in A$, hence $w \in L^\infty(A, \mathscr{L}(\mathbf{R}^n, \mathbf{R}^k))$. A standard approximation argument and integration by parts thus give $G(v, A) = \int_A \vartheta(\nabla v)\, dx$ for every $v \in C^1(A)$, as claimed. Moreover, the functional G is L^1_{loc} lower semi-continuous, hence $G(u, A) \leq \int_A \vartheta(Du)$. Arguing as in [AG] (see also the Appendix in [AMT]) we obtain for every $A \subset\subset \Omega$:

$$G(u, A) \geq \limsup_{\varepsilon \to 0} \int_A \theta(\nabla(u * \varphi_\varepsilon))\, dx$$

and also, since $u * \varphi_\varepsilon \to u$ in L^1_{loc}:

$$\limsup_{\varepsilon \to 0} \int_A \theta(\nabla(u * \varphi_\varepsilon))\, dx \geq \int_A \theta(Du)\,.$$

Taking the supremum for $A \subset\subset \Omega$ the thesis follows. ■

Remark 2.8: If $\Omega = \mathbf{R}^n$, then another characterization of $\int_\Omega \theta(Du)$ similar to (ii) in Theorem 2.7 can be given. Indeed, let $u \in L^p(\mathbf{R}^n)$, $1 < p < \infty$, $\theta \in CN$; then, set for $\lambda > 0$:

$$u_\lambda(x) = (\lambda/\pi)^{n/2} \int_{\mathbf{R}^n} u(y) \exp\left[-\lambda |x-y|^2\right] dy\,,$$

we have:

$$\int_{\mathbf{R}^n} \theta(Du) = \lim_{\lambda \to +\infty} \int_{\mathbf{R}^n} \theta(\nabla u_\lambda)\, dx\,.$$

This result is well-known when $\theta(v) = |v|$ and $p = 1$ (see [DG1]). Our general statement may be checked by adapting the classical proof.

Proposition 2.9: *Let* (v_h) *be a sequence of* $C^1(\Omega)$ *functions such that*:

$v_h \to u$ *in* $\mathscr{D}'$ (i.e. *weakly in the sense of distributions*),

$$\limsup_{h \to \infty} \int_\Omega \theta(Dv_h) < +\infty\,;$$

then $u \in L^1_{loc}(\Omega)$ *and moreover*

$$\int_\Omega \theta(Du) = \inf\left\{\liminf_{h \to \infty} \int_\Omega \theta(\nabla u_h)\, dx,\ (u_h) \subset C^1(A),\ u_h \to u \text{ in } \mathscr{D}'(\Omega)\right\}. \tag{2.4}$$

Proof: In view of Poincaré-Wirtinger inequality (cf. (2.3)) the sequence (v_h) is $L^1_{loc}(\Omega)$ relatively compact, whence $u \in L^1_{loc}(\Omega)$. Furthermore, the same argument applied to a sequence minimizing the right-hand side of (2.4) shows that

$$\int_\Omega \theta(Du) \leq \inf\left\{\liminf_{h \to \infty} \int_\Omega \theta(\nabla u_h)\, dx,\ (u_h) \subset C^1(A),\ u_h \to u \text{ in } \mathscr{D}'(\Omega)\right\}.$$

The opposite inequality is trivial. ■

From our Proposition 2.5 and Theorem 3 in [RY] we can deduce

THEOREM 2.10: *Let $\vartheta \in CN$ and let ϑ^2 be strictly convex (i.e. the level sets of ϑ don't contain any segment); let (u_h) be a sequence in $C^1(\Omega)$ converging to u a.e.; if*

$$\lim_{h\to\infty} \int_\Omega \vartheta(\nabla u_h)\,dx = \int_\Omega \vartheta(Du)$$

then

$$\lim_{h\to\infty} \int_\Omega \tau(\nabla u_h)\,dx = \int_\Omega \tau(Du) \qquad \blacksquare$$

for every $\tau \in CN$.

COROLLARY 2.11: *Let ϑ_1, ϑ_2 be in CN, and let $\vartheta = \vartheta_1 + \vartheta_2$; then for every $A \in \mathscr{A}(\Omega)$, and for every u:*

$$\int_A \vartheta(Du) = \int_A \vartheta_1(Du) + \int_A \vartheta_2(Du)\,.$$

The following definitions contain some generalizations of the measures $\int_B \vartheta(Du)$, $\vartheta \in CN$, which, in view of the results just presented, seem to be very natural.

DEFINITION 2.12: Let φ be a non-negative Borel function defined in Ω, and let $\mu_{\vartheta,u}$ be the measure defined in Proposition 2.5. We set:

$$\int_B \varphi\vartheta(Du) = \int_B \varphi(x)\,d\mu_{\vartheta,u}(x) \qquad \text{for every } B \in \mathscr{B}(\Omega)\,.$$

DEFINITION 2.13: Let $\varphi: \Omega \to \mathbf{R}$ be a Borel function. If $\int_B |\varphi|\vartheta(Du) < +\infty$, we set:

$$\int_B \varphi\vartheta(Du) = \int_B (\varphi \vee 0)\vartheta(Du) - \int_B [(-\varphi)\vee 0]\vartheta(Du)$$

for every $B \in \mathscr{B}(\Omega)$.

DEFINITION 2.14: Let ϑ_1, ϑ_2 be in CN, and $\vartheta = \vartheta_1 - \vartheta_2$; for every $B \in \mathscr{B}(\Omega)$ such that $\int_B \vartheta_i(Du) < +\infty$, $i = 1, 2$, we set:

$$\int_B \vartheta(Du) = \int_B \vartheta_1(Du) - \int_B \vartheta_2(Du)\,.$$

Notice that this definition, according to Corollary 2.11, does not depend on the decomposition of ϑ.

As particular cases in Definition 2.14 we may take the functions $\tau_{ij}(w) = \langle w(e_j) | e'_i \rangle$, where (e_j) (resp. (e'_i)) is an orthonormal basis in $\mathbf{R}^n$ (resp. $\mathbf{R}^k$); for a given function u set

$$\sigma_{ij}(B) = \int_B \tau_{ij}(Du) .$$

For a given $\vartheta \in CN$, the Radon-Nikodym derivative of σ_{ij} with respect to the measure $\mu_{\vartheta,u}$ (the notation is the same as in Definition 2.12) is then a well defined Borel function which will be denoted $v_{ij} = d\sigma_{ij}/(\vartheta(Du))$. The matrix whose entries are the functions v_{ij} $(i = 1, \ldots, k; j = 1, \ldots, n)$ will be denoted $\big(Du/(\vartheta(Du))\big)$.

We are now ready to define integrals involving Du in a quite general context.

Definition 2.15: Let $\varphi\colon \Omega \times \mathscr{L}(\mathbf{R}^n, \mathbf{R}^k) \to \mathbf{R}$ be a Borel function such that $\varphi(x, \lambda w) = \lambda\varphi(x, w)$, $\forall \lambda > 0$, $x \in \Omega$, $w \in \mathscr{L}(\mathbf{R}^n, \mathbf{R}^k)$. For every $B \in \mathscr{B}(\Omega)$ we set:

$$\int_B \varphi(x, Du) = \int_B \varphi\left(x, \frac{Du}{\vartheta(Du)}\right) \vartheta(Du) , \tag{2.5}$$

where the last integral is independent of the particular choice of $\vartheta \in CN$, according to Remark 2.6 and the homogeneity of φ.

In order to go into some details concerning integrals of the kind (2.5) we need to recall the main properties of functions of bounded variation. For more information see [FH], [GE], [MM], [VA], [VH], [ZW].

We will say that $u \in L^1(\Omega)$ is a function of bounded variation in Ω ($u \in BV(\Omega, \mathbf{R}^k)$) if

$$\int_\Omega |Du| < +\infty ;$$

furthermore, we will say that $u \in BV_{\mathrm{loc}}(\Omega, \mathbf{R}^k)$ if $u \in BV(A, \mathbf{R}^k)$ for every $A \subset\subset \Omega$, and set $BV(\Omega) = BV(\Omega, \mathbf{R})$. The space $BV(\Omega, \mathbf{R}^k)$ of all functions of bounded variation, endowed with the norm $\|u\|_{BV(\Omega)} = \int_\Omega |u|\, dx + \int_\Omega |Du|$, turns out to be a Banach space. According to Proposition 2.5, $u \in BV(\Omega, \mathbf{R}^k)$ if and only if its distributional derivative Du is a ($\mathscr{L}(\mathbf{R}^n, \mathbf{R}^k)$-valued) Radon measure. We emphasize that $u \in BV(\Omega, \mathbf{R}^k)$ if and only if its components are real valued BV functions.

If $E \subset \mathbf{R}^n$ is a Borel set we define its perimeter relative to Ω as follows: $P(E, \Omega) = \int_\Omega |D\chi_E|$, and we say that E has finite perimeter in Ω if $P(E, \Omega) < +\infty$. Notice that if $u \in BV(\Omega)$ then for a.a. $t \in \mathbf{R}$ the set $\{x \in \Omega\colon u(x) < t\}$ has finite perimeter in Ω ([FH, 4.5.9(12)]); furthermore, if $u \in BV(\Omega, \mathbf{R}^k) \cap L^\infty(\Omega, \mathbf{R}^k)$ and $P(E, \Omega) < \infty$ then $\chi_E u \in BV(\Omega, \mathbf{R}^k)$. If $P(E, \Omega) < +\infty$, then

$P(E, \Omega) = \mathscr{H}_{n-1}(\partial^* E \cap \Omega)$, where $\partial^* E$ is the reduced boundary of E, *i.e.* the set of the points $x \in \mathbf{R}^n$ such that:

$$\int_{B_\varrho(x)} |D\chi_E| > 0 \qquad \text{for all } \varrho > 0\,,$$

$$\exists \lim_{\varrho \to 0} \left\{ \Bigl(\int_{B_\varrho(x)} |D\chi_E| \Bigr)^{-1} \int_{B_\varrho(x)} D\chi_E \right\} = \nu(x) \quad \text{and} \quad |\nu(x)| = 1\,.$$

Let us recall also the *coarea formula* (see *e.g.* [GE, 1.23], [FH, 4.5.9(13)], [ZW, 5.4.4]):

$$\int_B |Du| = \int_{-\infty}^{+\infty} \int_B |D\chi_{U_t}|\, dt = \int_{-\infty}^{+\infty} \mathscr{H}_{n-1}(\partial^* U_t \cap B)\, dt \tag{2.6}$$

where $U_t = \{x \in \Omega : u(x) < t\}$ and $B \in \mathscr{B}(\Omega)$.

REMARK 2.16: If $u \in BV(\Omega, \mathbf{R}^k)$ then its behaviour near the singular set S_u can be well precised; in particular:

(i) the set S_u is countably $(\mathscr{H}_{n-1}, n-1)$-rectifiable, *i.e.* it can be covered (up to an $\mathscr{H}_{n-1}$-negligible set) by a sequence of C^1 hypersurfaces (cf. [DG2], [FH, 4.5.9(16)], [VA]);

(ii) $\mathscr{H}_{n-1}\Bigl(\Bigl\{x : \operatorname*{aplim}_{y \to x} u(y) = \infty\Bigr\}\Bigr) = 0$ (cf. [FH, 4.5.9(3)]);

(iii) for $\mathscr{H}_{n-1}$-a.e. $x \in S_u$ there exists $\nu \in S^{n-1}$ such that both the inner and the outer trace along ν exist ([FH, 4.5.9(17), (22)]); henceforth we shall write $u^+(x)$ (resp. $u^-(x)$) instead of $\operatorname{tr}^+(x, u, \nu)$ (resp. $\operatorname{tr}^-(x, u, \nu)$) whenever no confusion occurs.

The following theorem yields a slight generalization of the *coarea formula* (cf. *e.g.* [DM, Lemma 2.4]).

THEOREM 2.17: *Let* $u : \Omega \to \mathbf{R}$ *and* $\int_\Omega \theta(Du) < +\infty$. *Then for every* $B \in \mathscr{B}(\Omega)$ *the following generalized* coarea formula *holds*:

$$\int_B \theta(Du) = \int_{-\infty}^{+\infty} \int_B \theta(D\chi_{U_t})\, dt$$

where $U_t = \{x \in \Omega : u(x) < t\}$.

Taking into account Theorem 2.17 and Remark 2.6, some of the results proved in [AL1, Section 3] concerning the decomposition of the distributional derivative of a BV function can be generalized as follows, by repeating essentially the same proof.

THEOREM 2.18: *Let $\vartheta \in CN$, and $\int_\Omega \vartheta(Du) < +\infty$. Then, there exists a Borel set $C_u \subset \Omega$ with $|C_u| = 0$ such that for every $B \in \mathscr{B}(\Omega)$:*

$$\mathscr{H}_{n-1}(B) < +\infty \quad \textit{implies} \quad \int_{C_u \cap B} \vartheta(Du) = 0$$

and

$$(2.7) \qquad \int_B \vartheta(Du) = \int_B \vartheta(\nabla u)\, dx + \int_{S_u \cap B} \vartheta(\nu \otimes (u^+ - u^-))\, d\mathscr{H}_{n-1} + \int_{C_u \cap B} \vartheta(Du)\,.$$

Moreover, for every $B \in \mathscr{B}(\Omega)$ the following equality holds:

$$\int_B \vartheta(\nabla u)\, dx = \inf\left\{ \int_{B \setminus K} \vartheta(Du) \colon K \text{ compact, } |K| = 0 \right\}.$$

Theorem 2.18 shows that the measure $\mu_{\vartheta,u}$ splits into three parts: the first term in (2.7) is absolutely continuous with respect to the Lebesgue measure and is completely identified by the approximate differential of u; the second one $\left[\text{where the tensor product is the linear map defined by } (\nu \otimes (u^+ - u^-))(\xi) = \langle \nu | \xi \rangle (u^+ - u^-) \text{ for every } \xi \in \mathbf{R}^n\right]$ is absolutely continuous with respect to the $(n-1)$-dimensional Hausdorff measure; the third term might be called the *Cantor part* of the measure $\mu_{\vartheta,u}$ (this terminology has already been introduced in [DGA] in a slightly different fashion): in fact if u is the well-known Cantor-Vitali function, then C_u is the Cantor's middle third set and both the first and the second term in the right-hand side of (2.7) vanish.

We are now in a position to define integrals with respect to the *Cantor part* of the derivative of a BV function.

DEFINITION 2.19: Let φ be as in Definition 2.15, $\vartheta \in CN$, $\int_\Omega \vartheta(Du) < +\infty$; we set:

$$\int_B \varphi(x, CDu) = \int_{C_u \cap B} \varphi(x, Du) \qquad \forall B \in \mathscr{B}(\Omega)\,,$$

where the set C_u is defined in Theorem 2.18.

We end this section by recalling the classical definition of function of bounded variation on an open subset of the real line, in order to show how can one characterize functions in $BV(\Omega)$, $\Omega \subset \mathbf{R}^n$, through their one-dimensional slices (see *e.g.* [ZW]).

If $\Omega \in A(\mathbf{R})$ and J is a connected component of Ω, we define the *total variation* and the *essential variation* of u in J respectively as follows

$$V_J(u) = \sup\left\{ \sum_{i=1}^{k-1} |u(t_{i+1}) - u(t_i)|,\ \inf J < t_1 < \ldots < t_k < \sup J \right\};$$

$$\text{ess-}V_J(u) = \inf\{V_J(v) \colon v = u \text{ a.e. in } J\}\,.$$

Set

$$\text{ess-}V_\Omega(u) = \sum\{V_J(u),\ J \text{ connected component of } \Omega\},$$

one has the equality:

$$\int_\Omega |Du| = \text{ess-}V_\Omega(u),$$

hence $u \in BV(\Omega)$ if and only if $u \in L^1(\Omega)$ and $\text{ess-}V_\Omega(u) < +\infty$.

THEOREM 2.20: *Let $\Omega \subset \mathbf{R}^n$ be an open set, and $u \in L^1(\Omega)$. Set for any $\nu \in S^{n-1}$*

$$\pi_\nu = \{x \in \mathbf{R}^n : \langle x|\nu\rangle = 0\}$$

and for $x \in \pi_\nu$

$$\Omega_x = \{t \in \mathbf{R} : x + t\nu \in \Omega\},$$

$$u_x(t) = u(x + t\nu).$$

The following conditions are equivalent:

(i) *$u \in BV(\Omega)$,*

(ii) *for every $\nu \in S^{n-1}$ it holds:*

$$\int_{\pi_\nu} \text{ess-}V_{\Omega_x}(u_x)\, d\mathscr{H}_{n-1} < +\infty.$$

Moreover, we have the following inequalities:

$$\int_{\pi_\nu} \text{ess-}V_{\Omega_x}(u_x)\, d\mathscr{H}_{n-1} \le \int_\Omega |Du| \qquad \forall \nu \in S^{n-1}$$

and

$$\int_\Omega |Du| \le \sum_{i=1}^n \int_{\pi_{e_i}} \text{ess-}V_{\Omega_x}(u_x)\, d\mathscr{H}_{n-1}$$

for every orthonormal basis (e_i). Finally, one can compute the total variation of u from the variation of the slices as follows:

$$\int_\Omega |Du| = \sup\left\{\sum_{i=1}^\infty \int_{\pi_{\nu_i}} \text{ess-}V_{(E_i)_x}(u_x)\, d\mathscr{H}_{n-1}\right\},$$

where the supremum is taken among all the countable Borel partitions (E_i) of Ω and among all the choices of (ν_i) in S^{n-1}.

3. - On generalized functions of bounded variation

In this section, after recalling the definition of the classes GBV given in [DGA] and making some comments, we prove the announced distributional characterization of GBV functions.

Let $g\colon \Omega\times\mathbf{R}^k \to [0,+\infty)$ be a continuous function; following [DGA], we set

$$F_g(v,\Omega)=\int_\Omega g(x,v)|\nabla v|\,dx$$

for every $v\in C^1(\Omega,\mathbf{R}^k)$, and

$$F_g(u,\Omega)=\inf\left\{\liminf_{h\to\infty} F_g(v_h,\Omega),\ (v_h)\subset C^1(\Omega,\mathbf{R}^k),\ v_h\to u \text{ a.e. in } \Omega\right\}$$

for every measurable function $u\colon \Omega\to\mathbf{R}^k$.

Definition 3.1 (cf. [DGA]): Let u be as above, and $A\in\mathscr{A}(\Omega\times\mathbf{R}^k)$; we say that u is a *generalized function of bounded variation in* A ($u\in GBV(\Omega,A)$) if $F_g(u,\Omega)<+\infty$ for every non negative continuous function g with compact support in A.

Notice that the set A in Definition 3.1 is not necessarily a cartesian product. On the other hand, as products are easier to handle, we point out the following equivalence, which will be useful in the sequel.

Lemma 3.2: *Let $u\colon \Omega\to\mathbf{R}^k$ be measurable, and let $A\in\mathscr{A}(\Omega\times\mathbf{R}^k)$; then*

$$u\in GBV(\Omega,A) \quad \textit{iff} \quad (\mathrm{Id},u)\in GBV(\Omega,\Omega\times A).$$

Proof: Suppose $u\in GBV(\Omega,A)$, and take a continuous function $g\colon \Omega\times\mathbf{R}^n\times\mathbf{R}^k$ with compact support contained in $\Omega\times A$; set $\varphi(x,y)=g(x,x,y)$, φ has compact support contained in A, then there exists a sequence $(u_h)\subset C^1$ such that $u_h\to u$ a.e. and $F_\varphi(u_h,\Omega)\to F_\varphi(u,\Omega)$; set $v_h=(\mathrm{Id},u_h)$, it holds

$$F_g(v_h,\Omega)\le n^{\frac12}\int_\Omega \varphi(x,u_h(x))\,dx+\int_\Omega \varphi(x,u_h(x))|\nabla u_h|\,dx\le \text{const}<+\infty,$$

hence $F_g((\mathrm{Id},u),\Omega)<+\infty$.

Conversely, let $(\mathrm{Id},u)\in GBV(\Omega,\Omega\times A)$, take a continuous function $\varphi\colon \Omega\times\mathbf{R}^k\to\mathbf{R}$, with compact support in A, and let $\chi\colon \Omega\to\mathbf{R}$ be a cut-off function such that $\chi=1$ in a neighbourhood of $\{x\in\Omega\colon \varphi(x,y)\neq 0$ for some $y\in\mathbf{R}^k\}$; set $g(x,z,y)=\chi(z)\varphi(x,y)$, and take a sequence $(v_h)\subset C^1$ such that $F_g(v_h,\Omega)\to F_g((\mathrm{Id},u),\Omega;)$ set $(u_h)_i=(v_h)_{n+i}$, one has

$$F_\varphi(u_h,\Omega)\le F_g(v_h,\Omega)\le\text{const}<+\infty, \qquad \text{hence } F_\varphi(u,\Omega)<+\infty. \qquad \blacksquare$$

REMARK 3.3: It can be easily seen that if A is a cartesian product, say $A = \Omega \times E$, for some $E \in \mathscr{A}(\mathrm{R}^k)$, Definition 3.1 can be rephrased as follows:

$u \in GBV(\Omega, \Omega \times E)$ iff $F_g(u, \Omega') < +\infty$ for every continuous function $g: \mathrm{R}^k \to [0, +\infty)$ with compact support contained in E and for every $\Omega' \subset\subset \Omega$.

Most of the properties of $GBV(\Omega, A)$ we are going to present rely on the chain rule for composition of BV functions with Lipschitz continuous functions as recently established by L. Ambrosio and G. Dal Maso (see [ADM]). For reader's convenience, we recall a statement contained in [ADM] which is sufficient for our purposes.

THEOREM 3.4: *Let $u \in BV(\Omega, \mathrm{R}^k)$ and $\varphi: \mathrm{R}^k \to \mathrm{R}$ be Lipschitz continuous; set $v = \varphi \circ u$.* Then *$v \in BV_{loc}(\Omega)$, $\mathscr{H}_{n-1}(S_v \setminus S_u) = 0$, and for every $B \in \mathscr{B}(\Omega)$ it holds*

$$\int_B |Dv| \le \mathrm{Lip}(\varphi) \int_B |Du|,$$

$$\int_{B \cap S_v} |Dv| = \int_{B \cap S_u} |\varphi(u^+) - \varphi(u^-)| \, d\mathscr{H}_{n-1};$$

hence

$$|\tilde{D}v|(B) \le \mathrm{Lip}(\varphi) |\tilde{D}u|(B),$$

where we have set $|\tilde{D}v|(B) = |Dv|(B \setminus S_v)$, $|\tilde{D}u|(B) = |Du|(B \setminus S_u)$.

We are now ready to prove the main result of this section.

THEOREM 3.5: *Let $u: \Omega \to \mathrm{R}^k$ measurable, and $A \in \mathscr{A}(\Omega \times \mathrm{R}^k)$. For $\varphi: \Omega \times \times \mathrm{R}^k \to \mathrm{R}$ Lipschitz continuous, denote by $\nabla_y \varphi$ the gradient with respect to the last k variables; then $u \in GBV(\Omega, A)$ iff for every $\varphi \in \mathrm{Lip}(\Omega \times \mathrm{R}^k)$, with* spt $(\nabla_y \varphi)$ *compact contained in A, set $w = \varphi \circ (\mathrm{Id}, u)$, one has $w \in BV_{loc}(\Omega)$.*

PROOF: In view of Lemma 3.2, we can suppose without loss of generality that A is a cartesian product, say $A = \Omega \times E$ with $E \in \mathscr{A}(\mathrm{R}^k)$.

We prove first the *only if* part.

Let $u \in GBV(\Omega, \Omega \times E)$ and fix $\Omega' \subset\subset \Omega$; take

$$\varphi \in \mathrm{Lip}(\Omega \times \mathrm{R}^k) \quad \text{with} \quad \mathrm{spt}(\nabla_y \varphi) \text{ compact contained in } \Omega \times E$$

and set $g(x, y) = |\nabla_y \varphi(x, y)|$. We have $F_g(u, \Omega) < +\infty$, hence there exists a sequence of C^1 functions converging a.e. to u such that $F_g(u_h, \Omega) \le \text{const} < < +\infty$; since

$$\int_{\Omega'} |D(\varphi(x, u_h(x)))| \le \int_{\Omega'} |\nabla_x \varphi(x, u_h(x))| \, dx + F_g(u_h, \Omega) \le \text{const},$$

by the lower semicontinuity of the left-hand side we infer $\int_{\Omega'} |D(\varphi \circ u)| < +\infty$; since $\varphi \circ u$ is bounded the thesis follows.

Let us now prove the *if* part.

Taking into account Remark 3.3, it will be sufficient to prove the following statement:

> *if for every* $\varphi \in C^1(\mathbf{R}^k)$ *s.t.* $\nabla\varphi$ *has compact support in* E $\varphi \circ u \in BV_{loc}(\Omega)$, *then* $F_g(u, \Omega') < +\infty$ *for every non negative continuous function* $g: \mathbf{R}^k \to [0, +\infty)$ *with compact support contained in* E *and for every* $\Omega' \subset\subset \Omega$.

Let then a function g as above be given, and fix $\Omega' \subset\subset \Omega$.

A crucial role in the proof will be played by the following estimate, proved in [AMT, (4.2)], which holds for every $w \in BV(\Omega', \mathbf{R}^k)$:

$$F_g(w, \Omega') \le \int_{\Omega' \setminus S_w} g(\tilde{w}(x))\,|\bar{D}w(x)| + \int_{\Omega' \cap S_w} \delta_g(w^+(x), w^-(x))\, d\mathcal{H}_{n-1}(x)\,; \tag{3.1}$$

the function δ_g in (3.1) is defined for every $p, q \in \mathbf{R}^k$ as follows:

$$\delta_g(p, q) = \inf\left\{\int_0^1 g(\gamma(t))\,|\gamma'(t)|\,dt;\ \gamma \in C^1([0, 1], \mathbf{R}^k),\ \gamma(0) = p,\ \gamma(1) = q\right\}.$$

Notice that, since the support of g is compact, $\sup\{\delta_g(p, q); p, q \in \mathbf{R}^k\} < +\infty$.

Our aim is now to construct a sequence (w_h) of BV functions which converges a.e. to u, and such that $F_g(w_h, \Omega') \le \text{const} < +\infty$; the thesis will follow via an obvious diagonal argument.

Let K be the support of g, $K \subset E$, and let U be an open neighbourhood of K with boundary of class C^∞ such that $\mathbf{R}^k \setminus \bar{U}$ has a finite number of connected components. To construct such a set U, let $d = \operatorname{dist}(K, \mathbf{R}^k \setminus E)$; since K is totally bounded, there exist a finite number of points $y_i \in K$ such that $K \subset \bigcup_i B_{d/2}(y_i)$. The boundary of $\bigcup_i B_{d/2}(y_i)$ can be smoothed without intersecting $\mathbf{R}^k \setminus E$, thus getting U as required. Let then be C_0 the only unbounded connected component of $\mathbf{R}^k \setminus \bar{U}$ (if $k > 2$; the union of the two unbounded connected components of $\mathbf{R} \setminus \bar{U}$ if $k = 1$), and let $C_1, \ldots, C_r$ the other ones. Notice that, due to the smoothness of ∂U $(= \bigcup_i \partial C_i)$, $\bar{C}_i \cap \bar{C}_j = \emptyset$. The next step is the proof of the following

Claim: *There exist open sets* $B_0, \ldots, B_r \subset \mathbf{R}^k$ *such that* $\forall i$ $C_i \subset B_i$, $B_i \cap B_j = \emptyset$ $\forall j \neq i$, $K \cap B_i = \emptyset$, $P(u^{-1}(B_i), \Omega') < +\infty$. *Moreover, set* $F = \Omega' \setminus \bigcup_{i=0}^{r} u^{-1}(B_i)$, *the function* $u_F = \chi_F u \in BV(\Omega', \mathbf{R}^k)$.

In fact, let ψ_i $(i = 0, \ldots, r)$ be C^1 functions on $\mathbf{R}^k$ such that

$$\psi_i > 0\,, \qquad \operatorname{spt}(\nabla\psi_i) \text{ compact contained in } E \setminus K\,,$$

$\psi_i = 0$ on C_i, and

$$\psi_i^{-1}([0,1]) \cap \psi_j^{-1}([0,1]) = \emptyset \qquad \text{for } i \neq j .$$

For every i, $\psi_i \circ u \in BV(\Omega')$, hence there exist $t_i \in]0,1[$ such that $(\psi_i \circ u)^{-1}([0, t_i[)$ have finite perimeter in Ω'; the sets $B_i = \psi_i^{-1}([0, t_i[)$ have then the required properties.

Finally, let $\psi : \mathbf{R}^k \to \mathbf{R}^k$ be a C^1 function with $\operatorname{spt}(\nabla\psi)$ compact contained in E such that $\psi = \mathrm{Id}$ on $\mathbf{R}^k \setminus \bigcup_{i=0}^{\nu} B_i$; then $\psi \circ u \in BV(\Omega', \mathbf{R}^k) \cap L^\infty(\Omega', \mathbf{R}^k)$, whence, as $P(F, \Omega') < +\infty$, $\chi_F(\psi \circ u) \in BV(\Omega', \mathbf{R}^k)$; but $\chi_F(\psi \circ u) = u_F$, so the Claim is proved. Remark also that arguing as in Theorem 4.6 below it is possible to prove that if $x \in F$, then

$$\operatorname*{aplim}_{y \to x} \chi_F(y) = 1 . \tag{3.2}$$

To complete our construction, let us show that $(\nu + 1)$ sequences of piecewise constant BV functions (v_{ih}), $i = 1, \ldots, \nu$, $h \in \mathbf{N}$, exist such that $v_{ih} \to u$ a.e. in $\Omega' \cap u^{-1}(B_i)$ as $h \to +\infty$, and $\tilde{v}_{ih}(x) \in B_i$ for a.a. $x \in u^{-1}(B_i) \cap \Omega'$. To this end, recall that every open set in $\mathbf{R}^k$ can be written as a countable union of disjoint boxes with sides parallel to the coordinate axes; moreover, to simplify the notation, let B be one of the B_i's, and fix arbitrarily $c \in B$. Fix now a decomposition of B as above, and, using as usual diadic partitions of each box, build a sequence $\zeta_h = \sum_{l=1}^{l_h} c_{hl}\chi_{E_{hl}} + c\chi_{D_h}$ (with $c_{hl} \in B$ and $D_h = u^{-1}(B) \cap \cap\, \Omega' \setminus \bigcup_{l=1}^{l_h} E_{hl}$) of piecewise constant measurable functions converging to u a.e. in $\Omega' \cap u^{-1}(B)$. Then, let A_{hl} be open sets such that $E_{hl} \subset A_{hl}$ and $|A_{hl} \setminus E_{hl}| < (2hl_h)^{-1}$, and let A'_{hl} be finite unions of boxes deduced respectively from partitions of A_{hl} such that $|A'_{hl} \,\Delta\, E_{hl}| < (hl_h)^{-1}$. Finally, set

$$E'_{hl} = \Big\{A'_{hl} \setminus \Big[\bigcup_{j \neq l} A'_{hj}\Big]\Big\} \cap u^{-1}(B) , \qquad E''_h = u^{-1}(B) \setminus \Big[\bigcup_{l=1}^{l_h} E'_{hl}\Big] ,$$

and $v_h = \sum_{l=1}^{l_h} c_{hl}\chi_{E'_{hl}} + c\chi_{E''_h}$. Since $P(E'_{hl}, \Omega') < +\infty$, $P(E''_h, \Omega') < +\infty$, the functions v_h belong to $BV(\Omega', \mathbf{R}^k)$, and $\tilde{v}_h(x) \in B$ for a.a. $x \in \Omega' \cap u^{-1}(B)$. Moreover, $v_h = \zeta_h$ on $\bigcup_{l=1}^{l_h} (E'_{hl} \cap E_{hl})$. But

$$[\Omega' \cap u^{-1}(B)] \setminus \bigcup_{l=1}^{l_h} (E'_{hl} \cap E_{hl}) \subset \bigcup_{l=1}^{l_h} (A'_{hl} \,\Delta\, E_{hl}) \cup D_h ,$$

and the measure of this last set goes to 0 as $h \to +\infty$. Then, $v_h \to u$ a.e. in $\Omega' \cap u^{-1}(B)$. (The previous construction does not work—*verbatim*—if $B = B_0$

because $|B_0| = +\infty$; in such a case, a sequence of compact sets invading B_0 should be used, but we don't enter into the details.)

Let now denote by v_{ih} the sequences just constructed for $B = B_i$, and set $F_i = \Omega' \cap u^{-1}(B_i)$, and $w_h = u\chi_F + \sum_{i=0}^{\infty} v_{ih}\chi_{F_i}$; the last step will be the estimate of $F_g(w_h, \Omega')$. First of all, remark that $\delta_g(p, q) = 0$ if $p, q \in B_i$ (p and q can be joined by a path which lies entirely in B_i, hence does not intersect K); therefore, the integral over S_w in (3.1) (for $w = w_h$) can be estimated as follows:

$$\int_{S_{w_h} \cap \Omega'} \delta_g(w_h^+(x), w_h^-(x))\, d\mathscr{H}_{n-1}(x) < \int_{S_u \cap F} \delta_g(u^+(x), u^-(x))\, d\mathscr{H}_{n-1}(x) +$$
$$+ \sup_{p,q} \delta_g(p, q) \sum_{i=0}^{\infty} P(F_i, \Omega') < +\infty$$

(the integral over $S_u \cap F$ does make sense because of (3.2)). Moreover, since v_{ih} are piecewise constant, $|\tilde{D}w_h|(\Omega') = |\tilde{D}u_F|(\Omega') = |\tilde{D}u|(F)$ so that:

$$F_g(w_h, \Omega') \le (\sup g)|\tilde{D}u|(F) + \int_{S_u \cap F} \delta_g(u^+(x), u^-(x))\, d\mathscr{H}_{n-1}(x) +$$
$$+ \sup_{p,q} \delta_g(p, q) \sum_{i=0}^{\infty} P(F_i, \Omega') < +\infty$$

and the proof is complete. ■

REMARK 3.6: In the course of the proof of Theorem 3.5 the following result has been obtained as a particular case:

$$u \in GBV(\Omega, \Omega \times \mathbf{R}^k) \text{ iff } \forall \varphi \in \mathrm{Lip}(\mathbf{R}^k)$$
$$\text{s.t. } \mathrm{spt}(\nabla\varphi) \text{ is compact, it holds } \varphi \circ u \in BV_{\mathrm{loc}}(\Omega)\,.$$

In the scalar-valued case, the previous statement can be rephrased as follows:

$$u \in GBV(\Omega, \Omega \times \mathbf{R})$$
$$\textit{iff for every } a > 0 \textit{ the function } u^a = (-a) \vee (u \wedge a) \textit{ is in } BV_{\mathrm{loc}}(\Omega)\,.$$

Let us show that functions which are smooth outside a small closed set fall in the class GBV.

PROPOSITION 3.7: *Let $K \subseteq \mathbf{R}^n$ be a closed set with $\mathscr{H}_{n-1}(K \cap \Omega) < +\infty$, and let $u \in C^1(\Omega \setminus K)$ such that*

$$\int_{A \setminus K} |\nabla u| < +\infty \qquad \textit{for every } A \subset\subset \Omega\,;$$

then $u \in GBV(\Omega, \Omega \times \mathbf{R}^k)$.

Proof: We adapt the proof of Lemma 2.3 in [DGCL].

For every $h \in \mathbb{N}$ there exists a locally finite covering of K with balls $B^h_{\varrho_i}(x_i)$ such that $\varrho_i < 1/h$ and $\sum_{i=1}^{\infty} \mathscr{H}_{n-1}(\partial B^h_{\varrho_i}(x_i)) < c(\mathscr{H}_{n-1}(K \cap \Omega) + 1)$, where c depends only on the space dimension n. Fix ψ as in Remark 3.6, and set $v = \psi(u)$,

$$v_h(x) = \begin{cases} v(x) & \text{for } x \in \Omega \setminus \bigcup B^h_{\varrho_i}(x_i)\,, \\ 0 & \text{for } x \in \bigcup B^h_{\varrho_i}(x_i)\,. \end{cases}$$

Since ψ is locally bounded, v_h and v are in $L^1_{loc}(\Omega)$, and furthermore using the (classical) chain rule

$$\int_A |Dv_h| < \mathrm{Lip}\,(\psi) \int_{A \setminus K} |\nabla u|\, dx + c(\sup \psi - \inf \psi)(\mathscr{H}_{n-1}(K \cap \Omega) + 1)$$

for every $A \subset\subset \Omega$; hence $v_h \in BV_{loc}(\Omega)$ and, from the lower semicontinuity of the total variation with respect to the $L^1_{loc}(\Omega)$ convergence, it follows that $v \in BV_{loc}(\Omega)$. ∎

Proposition 3.8: *If* $\arctan(u) \in BV_{loc}(\Omega)$, *then* $u \in GBV(\Omega, \Omega \times \mathbb{R})$.

Proof: Set $u^a = (-a) \vee (u \wedge a)$, $0 < a < +\infty$, and $v = \arctan(u)$; an application of the chain rule gives

$$\int_A |Du^a| < (1 + a^2) \int_A |Dv| < +\infty \qquad \forall A \subset\subset \Omega\,,$$

whence $u \in BV_{loc}(\Omega)$ for every a, and then $u \in GBV(\Omega, \Omega \times \mathbb{R})$, according to Remark 3.6. ∎

4. - The classes BBV, LBV, GBV^*

This section is devoted to define some new functional classes introduced in [PD] and to study their relationships with the classes BV and GBV. The main results proved here have been announced without proof in [PD].

Definition 4.1: Let $u: \Omega \to \mathbb{R}^k$, and let $E \subset \Omega$ be such that

(i) $\operatorname{diam}(E) < +\infty$,

(ii) $P(E, \mathbb{R}^n) < +\infty$,

(iii) $x \in E \Rightarrow \operatorname{aplim}_{y \to x} \chi_E(y) = 1$;

we set $u \in BBV(E)$, or equivalently $E \in BBV(u)$, if, set

$$u_E(x) = \begin{cases} u(x) & \text{if } x \in E, \\ 0 & \text{if } x \in \mathbf{R}^n \setminus E, \end{cases}$$

it holds $u_E \in BV(\Omega, \mathbf{R}^k) \cap L^\infty(\Omega, \mathbf{R}^k)$.

The previous definition enables us to generalize Definition 2.19.

DEFINITION 4.2: Let $u: \Omega \to \mathbf{R}^k$ and let $\varphi: \Omega \times \mathscr{L}(\mathbf{R}^n, \mathbf{R}^k) \to [0, +\infty)$ be a Borel function positively 1-homogeneous with respect to the variable in $\mathscr{L}(\mathbf{R}^n, \mathbf{R}^k)$. For $E \in \mathscr{B}(\mathbf{R}^n)$ we set

$$\int_E \varphi(x, CDu) = \sup\left\{ \int_{B \cap E} \varphi(x, CDu),\ B \in BBV(u) \right\}$$

where the integrals in the right-hand side have been defined in Definition 2.19.

Given a bounded set E with finite perimeter, the class $BBV(E)$ has nice stability properties which can be easily deduced by the definition. We collect them in the following

REMARK 4.3: If E satisfies (i), (ii), (iii) in Definition 4.1, the class $BBV(E)$ is trivially a vector space, and if $u, v \in BBV(E)$ then uv and $u \vee v$, $u \wedge v$ are in $BBV(E)$, too. Moreover, given $u \in BBV(E)$, for every $\varphi: \mathbf{R}^k \to \mathbf{R}^m$ locally Lipschitz continuous we have $\varphi \circ u \in BBV(E)$, and for every B with $P(B, \mathbf{R}^n) < < +\infty$ we have $u_B \in BBV(E)$.

DEFINITION 4.4: Let $u: \Omega \to \mathbf{R}^k$ and $E \subset \Omega$; we set $E \in LBV(u)$ if there exists a sequence (E_h), $E_h \subset \Omega$, such that

$$\mathscr{H}_{n-1}\left(E \setminus \bigcup_{h=1}^{\infty} E_h\right) = 0,$$

$$\sum_{h=1}^{\infty} P(E_h, \mathbf{R}^n) < +\infty,$$

$$E_h \in BBV(u) \qquad \forall h \in \mathbf{N}.$$

If $E \in LBV(u)$ we shall also write $u \in LBV(E)$.

REMARK 4.5: From Definition 4.4 it follows that if $E \in LBV(u)$ then every $F \subset E$ belongs to $LBV(u)$, too. Moreover, as regards the behaviour near the singular set, LBV functions heir the properties of BV functions (i), (ii), (iii) in Remark 2.16.

The forthcoming result exhibites a less immediate connection between BV and LBV.

THEOREM 4.6: *If* $u \in BV_{\mathrm{loc}}(\Omega, \mathbf{R}^k)$ *and* $K \subset \Omega$ *is compact, then* $K \in LBV(u)$.

PROOF: As remarked in the preceding Section, $u \in BV(\Omega, \mathbf{R}^k)$ iff its components are in $BV(\Omega)$, thus we may suppose $k = 1$. To simplify the notation, choose a normal field ν to S_u in such a way that $u^+ > u^-$ (whenever they exist: see Remark 2.16); then u^+ and u^- are characterized by

$$u^+(x) = \inf\left\{\tau \in \mathbf{R} : \limsup_{\varrho \to 0} \varrho^{-n} |B_\varrho(x) \cap \{y : u(y) > \tau\}| = 0\right\}$$

and

$$u^-(x) = \sup\left\{\tau \in \mathbf{R} : \limsup_{\varrho \to 0} \varrho^{-n} |B_\varrho(x) \cap \{y : u(y) < \tau\}| = 0\right\}.$$

For every $t, s \in \mathbf{R}$ set $A_{t,s} = \{x \in \Omega : s < u^-(x) \le u^+(x) < t\}$; for every $x \in A_{t,s}$ we may select τ such that $u^+(x) < \tau < t$, obtaining:

$$\limsup_{\varrho \to 0} \varrho^{-n} |B_\varrho(x) \cap \{y : u(y) > t\}| \le \limsup_{\varrho \to 0} \varrho^{-n} |B_\varrho(x) \cap \{y : u(y) > \tau\}| = 0\,;$$

the same argument can be used to check that

$$\limsup_{\varrho \to 0} \varrho^{-n} |B_\varrho(x) \cap \{y : u(y) < s\}| = 0\,,$$

and therefore $\operatorname{aplim}_{y \to x} \chi_{A_{t,s}}(y) = 1$ for every $x \in A_{t,s}$. Let now be $K \subset \Omega$ compact and $A \in \mathscr{A}(\Omega)$ such that $K \subset A \subset\subset \Omega$ and $P(A, \mathbf{R}^n) < +\infty$; in view of Remark 4.5 it is enough to prove that $A \in LBV(u)$. Since $u \in BV(A)$, from the coarea formula (2.6) it follows that there exists a sequence t_h $(h \in \mathbf{Z})$, $t_h \to +\infty$ as $h \to +\infty$ and $t_h \to -\infty$ as $h \to -\infty$, such that, set $F_h = \{x \in \Omega : t_{h-1} < u^-(x) \le u^+(x) < t_{h+1}\}$, $\sum_h P(F_h, A) < +\infty$. Set $E_h = A \cap F_h$, $E_h \in BBV(u)$ and moreover

$$P(E_h, \mathbf{R}^n) \le P(F_h, A) + \mathscr{H}_{n-1}(\partial^* A \cap F_h)\,,$$

therefore $\sum_h P(E_h, \mathbf{R}^n) < +\infty$; from Remark 2.16(ii) we can infer that $\mathscr{H}_{n-1}(A \setminus \bigcup_h E_h) = 0$, then the theorem is proved. ■

The following definition identifies another class of functions whose variation is in some sense controlled.

DEFINITION 4.7: Let $u : \Omega \to \mathbf{R}^k$ measurable, and $A \in \mathscr{A}(\Omega \times \mathbf{R}^k)$. We say that $u \in GBV^*(\Omega, A)$ iff for every Lipschitz continuous function $\varphi : A \to \mathbf{R}$ with compact support the function $v = \varphi \circ (\mathrm{Id}, u)$ belongs to $BV(\Omega)$.

Notice that if $A = \Omega \times \mathbf{R}^k$ then Definition 4.7 may be rephrased as follows (cf. Remark 3.6):

$u \in GBV^*(\Omega, \Omega \times \mathbf{R}^k)$ iff for every Lipschitz continuous function $\psi : \mathbf{R}^k \to \mathbf{R}$ with compact support the function $v = \psi \circ u$ belongs to $BV_{\mathrm{loc}}(\Omega)$.

REMARK 4.8: As in the GBV case, the following equivalence holds:

$$u \in GBV^*(\Omega, A) \quad \text{iff } (\mathrm{Id}, u) \in GBV^*(\Omega, \Omega \times A).$$

In fact, let $u \in GBV^*(\Omega, A)$, and let $\varphi: \Omega \times \mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}$ be a Lipschitz continuous function with compact support in $\Omega \times A$; set $\psi(x, y) = \varphi(x, x, y)$, ψ has compact support in A and obviously satisfies $\varphi \circ (\mathrm{Id}, \mathrm{Id}, u) = \psi \circ (\mathrm{Id}, u)$ in Ω, hence $\varphi \circ (\mathrm{Id}, \mathrm{Id}, u) \in BV_{loc}(\Omega)$.

Conversely, let $(\mathrm{Id}, u) \in GBV^*(\Omega, \Omega \times A)$, let $\psi: \Omega \times \mathbf{R}^k \to \mathbf{R}$ be a Lipschitz continuous function with compact support in $\Omega \times A$, and let $B \subset\subset \Omega$; set $\varphi(x, \xi, y) = \chi(\xi)\,\psi(x, y)$, where χ is a cut off function such that $\chi = 1$ in a neighbourhood of B, φ has compact support in $\Omega \times A$ and $\varphi \circ (\mathrm{Id}, \mathrm{Id}, u) = \psi(\mathrm{Id}, u)$ in B so that $\varphi(\mathrm{Id}, \mathrm{Id}, u) \in BV(B)$.

A comparison between the classes GBV and GBV^* is in order.

REMARK 4.9: Since for $k > 1$ and $K \subset \mathbf{R}^k$ compact the set $\mathbf{R}^k \setminus K$ has only one unbounded connected component, Theorem 3.5 implies that in the vector-valued case the classes $GBV(\Omega, \Omega \times \mathbf{R}^k)$ and $GBV^*(\Omega, \Omega \times \mathbf{R}^k)$ coincide. On the contrary, in the scalar-valued case they are different. In fact, let $\Omega = (-1, 1)$, and

$$(4.1) \qquad u(x) = \frac{1}{x} \frac{\sin(1/x)}{|\sin(1/x)|} \qquad \text{for } 0 \neq x \in \Omega.$$

It is easy to check that $u \in GBV^*(\Omega, \Omega \times \mathbf{R})$ but $u \notin GBV(\Omega, \Omega \times \mathbf{R})$. Conversely, the inclusion $GBV(\Omega, \Omega \times \mathbf{R}) \subset GBV^*(\Omega, \Omega \times \mathbf{R})$ is trivial.

As regards the relations between the classes $GBV(\Omega, \Omega \times \mathbf{R})$ and $GBV^*(\Omega, \Omega \times \mathbf{R})$, a sufficient condition for membership in GBV will be given later (see Theorem 4.15); for the present time, we will limit ourselves to point out the following property:

if $u \in GBV^*(\Omega, \Omega \times \mathbf{R})$ and u is bounded from below or from above then $u \in GBV(\Omega, \Omega \times \mathbf{R})$; in particular if $u \in GBV^*(\Omega, \Omega \times \mathbf{R})$ then

$$|u| \in GBV(\Omega, \Omega \times \mathbf{R}).$$

In fact, given $u \in GBV^*(\Omega, \Omega \times \mathbf{R})$ with, say, $u > a$, it suffices to show that $u^b = u \wedge b \in BV_{loc}(\Omega)$ for every $b > a$. Fix b, and let $\psi \in \mathrm{Lip}(\mathbf{R})$ be such that

$$\psi(t) = \begin{cases} 0 & \text{if } t < a - 1, \\ t - b & \text{if } a < t < b, \\ 0 & \text{if } t > b, \end{cases}$$

ψ has compact support, hence $u \in GBV^*(\Omega, \Omega \times \mathbf{R})$ implies $\psi \circ u \in BV_{loc}(\Omega)$; but $Du^b = D(\psi \circ u)$, therefore $u^b \in BV_{loc}(\Omega)$, too.

In connection with example (4.1), we point out that if $u = (u_1, \ldots, u_k) \in GBV^*(\Omega, \Omega \times \mathbf{R}^k)$ and $u_i \in L^\infty(\Omega) \cap BV(\Omega)$ for $i = 1, \ldots, k-1$, then

$u_k \in GBV^*(\Omega, \Omega\times\mathbf{R})$, whereas the corrisponding result is not true in $GBV(\Omega, \Omega\times\mathbf{R}^k)$, even though one supposes $u_i = \text{const}$ for $i = 1, \dots, k-1$. Notice that $u = (u_1, \dots, u_k) \in GBV^*(\Omega, \Omega\times\mathbf{R}^k)$ does not imply $u_i \in GBV^*(\Omega, \Omega\times\mathbf{R})$ without further assumptions; in fact, set $\Omega = (-1, 1)$, and

$$f(x) = \begin{cases} \sin(1/x) & \text{for } -1 < x < 0, \\ 1/x & \text{for } 0 < x < 1, \end{cases}$$

the function $u(x) = (f(x), f(-x))$ is of class $GBV(\Omega, \Omega\times\mathbf{R}^2) = GBV^*(\Omega, \Omega\times\mathbf{R}^2))$ but its components are neither in $GBV(\Omega, \Omega\times\mathbf{R})$ nor in $GBV^*(\Omega, \Omega\times\mathbf{R})$. Conversely, one trivially has $[GBV^*(\Omega, \Omega\times\mathbf{R})]^k \subset GBV^*(\Omega, \Omega\times\mathbf{R}^k)$.

The following lemma partially extends to GBV^* functions some results which are well-known for BV functions (cf. Remark 2.16), and has been proved in [AL2] in the GBV setting (see Propositions 1.3 and 1.4 therein). The same arguments work also in our case.

Lemma 4.10: *If $u \in GBV^*(\Omega, \Omega\times\mathbf{R}^k)$ then:*

(i) *u is approximately differentiable a.e. in Ω;*

(ii) *for $\mathscr{H}_{n-1}$-a.e. $x \in S_u$ there exist $\nu \in S^{n-1}$, $z, z' \in \overline{\mathbf{R}}^k$ such that $z = \operatorname{tr}^+(x, u, \nu)$ and $z' = \operatorname{tr}^-(x, u, \nu)$;*

(iii) *S_u is countably $(\mathscr{H}_{n-1}, n-1)$ rectifiable.*

In [DGA] for every $u\colon \Omega \to \mathbf{R}^k$ the sets $GBV \operatorname{amb}(u)$ and $GBV \operatorname{dom}(u)$ have been defined; $GBV \operatorname{amb}(u)$ is the union of the open subsets A of $\Omega\times\mathbf{R}^k$ such that $u \in GBV(\Omega, A)$; the set $GBV \operatorname{dom}(u)$ is defined by

$$GBV \operatorname{dom}(u) = \left\{x \in \Omega \setminus S_u \colon (x, \operatorname*{aplim}_{y\to x} u(y)) \in GBV \operatorname{amb}(u)\right\}.$$

In the same vein, we set

$$GBV^* \operatorname{amb}(u) = \bigcup \{A \subset \Omega\times\mathbf{R}^k \colon u \in GBV^*(\Omega, A)\}.$$

Notice that in general $u \notin GBV(\Omega, GBV \operatorname{amb}(u))$, whereas $u \in GBV^*(\Omega, GBV^* \operatorname{amb}(u))$ always holds.

To define the set $GBV^* \operatorname{dom}(u)$ we must give the notion of *closed approximate limit.*

Definition 4.11: Let $u\colon \Omega \to \mathbf{R}^k$ be a measurable function, and let $x \in \Omega$; we set

$$C\text{-}\operatorname*{aplim}_{y\to x} u(y) = \bigcap \{K \subset \widetilde{\mathbf{R}}^k,\ K \in \mathscr{K}(u)\},$$

where $K \in \mathscr{K}(u)$ iff K is closed and moreover for every $\varphi \in C(\widetilde{\mathbf{R}}^k)$ with compact support in $\widetilde{\mathbf{R}}^k \setminus K$ it holds $\operatorname*{aplim}_{y\to x} \varphi(u(y)) = 0$.

Notice that $C\text{-}\operatorname{aplim}_{y\to x} u(y) = \{\operatorname{aplim}_{y\to x} u(y)\}$ whenever this latter exists; in view of Lemma 4.10 (ii), if $u \in GBV^*(\Omega, \Omega\times\mathbf{R}^k)$ then for $\mathscr{H}_{n-1}$-a.a. $x \in S_u$ we have $C\text{-}\operatorname{aplim}_{y\to x} u(y) = \{u^+(x), u^-(x)\}$.

We are now ready to define $GBV^*\operatorname{dom}(u)$.

DEFINITION 4.12: Let $u : \to \tilde{\mathbf{R}}^k$ measurable; we set

$$GBV^*\operatorname{dom}(u) = \left\{x \in \Omega:\ C\text{-}\operatorname{aplim}_{y\to x}(y, u(y)) \subset GBV^*\operatorname{amb}(u)\right\}.$$

THEOREM 4.13. *If $u \in GBV^*(\Omega, A)$ then for every compact set $K \subset \Omega$ we have*

$$K \cap GBV^*\operatorname{dom}(u) \in LBV(u).$$

PROOF. We can suppose without loss of generality that $A = GBV^*\operatorname{amb}(u) = \Omega\times B$ for some $B \in \mathscr{A}(\mathbf{R}^k)$ (cf. Remark 4.8). Take a compact set $K \subset \Omega$, an increasing sequence (B_h) of bounded open sets such that $\bigcup B_h = B$, and Lipschitz continuous functions $\psi_h : \mathbf{R}^k \to \mathbf{R}$ with compact support contained in B such that $\psi_h(y) = y$ for $y \in B_h$. At this point an argument similar to that one in the proof of Theorem 4.6 can be used to achieve the result. ∎

Notice that if $u \in GBV^*(\Omega, \Omega\times\mathbf{R}^k)$ then

$$\Omega \setminus GBV^*\operatorname{dom}(u) = \left\{x \in \Omega:\ \infty \in C\text{-}\operatorname{aplim}_{y\to x} u(y)\right\}.$$

The following theorem shows that, under suitable conditions,

$$\mathscr{H}_{n-1}(\Omega \setminus GBV^*\operatorname{dom}(u)) = 0.$$

THEOREM 4.14: *If $u \in GBV^*(\Omega, \Omega\times\mathbf{R}^k)$ and*

$$\int_\Omega |\nabla u|\,dx + \int_\Omega |CDu| + \mathscr{H}_{n-1}(S_u) < +\infty \tag{4.2}$$

then $\mathscr{H}_{n-1}\left(\left\{x \in \Omega:\ \infty \in C\text{-}\operatorname{aplim}_{y\to x} u(y)\right\}\right) = 0.$

In particular, under the previous hypotheses, $\Omega \in LBV(u)$.

PROOF: Set $\Omega_\infty(u) = \left\{x \in \Omega:\ \infty \in C\text{-}\operatorname{aplim}_{y\to x} u(y)\right\}$; since $|u| \in GBV(\Omega, \Omega\times\mathbf{R})$ (cf. Remark 4.9) and $\Omega_\infty(u) = \Omega_\infty(|u|)$, we can suppose $u \in GBV(\Omega, \Omega\times\mathbf{R})$, and $u > 0$. We will divide the proof in two parts.

1st *part.* $\mathscr{H}_{n-1}(\Omega_\infty(u) \cap S_u) = 0$.

To prove this statement, we use a slicing argument (cf. Theorems 2.20 and 4.6 for the notation). The result is obvious if $n = 1$ (S_u is a finite set, and $u \in BV(J) \cap C(J)$ for every connected component $J \subseteq \Omega \setminus S_u$). Thus, for any $\nu \in S^{n-1}$ and for $\mathscr{H}_{n-1}$-a.a. $x \in \pi$, the set $\Omega_\infty(u_x)$ is empty, whence, in view of Lemma 4.10(iii), and [FH, 3.3.13 and 3.2.26], $\mathscr{H}_{n-1}(\Omega_\infty(u) \cap S_u) = 0$.

2^{nd} part. $\mathscr{H}_{n-1}(\Omega_\infty(u)\setminus S_u) = 0$.

Set $B = \Omega\setminus S_u$, and, for any $h \in \mathbf{N}$, $B_h = \{x \in B : h-1 < \tilde{u}(x) < h\}$; from the coarea formula (2.6), it follows that for any $h \in \mathbf{N}$ there exists $t_h \in (h-1, h)$ such that $\mathscr{H}_{n-1}(\partial^*\{u < t_h\} \cap B) < |\bar{D}u|(B_h)$. Since $|\bar{D}u|(B) = \sum_{h=0}^{\infty} |\bar{D}u|(B_h)$, we infer

$$\lim_{h\to\infty} \mathscr{H}_{n-1}(\partial^*\{u > t_h\} \cap \Omega) < \mathscr{H}_{n-1}(S_u)\,.$$

Arguing as in [FH, 4.5.9(3)] and localizing we obtain that for every $A \in \mathscr{A}(\Omega)$ $\mathscr{H}_{n-1}(\Omega_\infty \cap A) \leq c\mathscr{H}_{n-1}(S_u \cap A)$, where c depends only on the space dimension n; therefore $\mathscr{H}_{n-1}(\Omega_\infty\setminus S_u) = 0$ and the theorem is proved. ■

The previous result enables us to exhibite some further connections between the functional classes we are dealing with.

THEOREM 4.15: *If* $u \in GBV^*(\Omega, \Omega\times\mathbf{R})$ *and*

$$\int_\Omega |\nabla u|\,dx + \int_\Omega |CDu| + \mathscr{H}_{n-1}(S_u) < +\infty\,,$$

then $u \in GBV(\Omega, \Omega\times\mathbf{R})$.

PROOF. From Theorem 4.14 it follows that

$$\mathscr{H}_{n-1}\Big(\Big\{x \in \Omega : \infty \in C\text{-}\operatorname*{aplim}_{y\to x} u(y)\Big\}\Big) = 0\,,$$

hence for every $a > 0$, set $u^a = (-a)\vee(u\wedge a)$, $\mathscr{H}_{n-1}[S_{u^a}\setminus S_u] = 0$. Fix $a > 0$ and $A \subset\subset \Omega$; let φ be a Lipschitz continuous function with compact support contained in $\Omega\times\mathbf{R}$ and $\varphi(x, y) = y$ in a neighbourhood of $A\times[-a, a]$. Let u^a be as above, and $w = \varphi\circ(\mathrm{Id}, u)$; we have $w \in BV(\Omega)$ and

$$\int_A |Du^a| < \int_A |Dw| + 2a\mathscr{H}_{n-1}(S_u) < +\infty\,,$$

whence $u^a \in BV(A)$; in view of Remark 3.6 the proof is complete. ■

The following theorems contain interesting links between the classes LBV, GBV^* and the classical spaces BV.

THEOREM 4.16: *Let* $u : \Omega \to \mathbf{R}^k$, $u \in LBV(\Omega)$; *if*

$$|\Omega| + \int_\Omega |\nabla u|\,dx + \mathscr{H}_{n-1}(S_u) + \int_\Omega |CDu| < +\infty \tag{4.3}$$

then $\arctan(u) \in BV(\Omega)$.

Proof: The proof relies on Theorems 2.20 and 3.4, where the notation which we will use here has been introduced. For clarity's sake, we divide the proof in three steps.

Step 1. *Let* $\Omega \subset \mathbf{R}$ *and* $u \in LBV(\Omega)$; *if* (4.3) *holds then* $\arctan(u) \in BV(\Omega)$.

Since $\Omega \in LBV(u)$ we have $\Omega = \bigcup_{h=1}^{\infty} E_h$, where each E_h satisfies (i), (ii), (iii) of Definition 4.1, hence every E_h is a finite union of bounded open intervals (see *e.g.* [VA]); moreover for every h, set $u_{E_h}(x) = u(x)$ if $x \in E_h$, $u_{E_h}(x) = 0$ otherwise, we have $u_{E_h} \in BV(\Omega) \cap L^{\infty}(\Omega)$, hence, by Theorem 3.4, $\arctan(u_{E_h}) \in BV(\Omega)$; furthermore, $\arctan(u_{E_h}) \to \arctan(u)$ in $L^1(\Omega)$, and

$$\|\arctan(u_{E_h})\|_{BV(\Omega)} < \pi|\Omega| + \int_{\Omega} |\nabla u|\, dx + \pi \mathscr{H}_{n-1}(S_u) + \int_{\Omega} |CDu| ,$$

therefore $\arctan(u) \in BV(\Omega)$.

We now come back to the general case, $\Omega \subseteq \mathbf{R}^n$, $n \geq 2$.

Step 2. *Let* $u \in LBV(\Omega)$ *and verifies* (4.3). *Let* $\nu \in S^{n-1}$, *and* $\pi_\nu = \{y \in \mathbf{R}^n : \langle y|\nu\rangle = 0\}$; *then for* $\mathscr{H}_{n-1}$*-a.a.* $x \in \pi_\nu$ *the function* $u_x(t) = u(x + t\nu)$ *belongs to* $LBV(\Omega_x)$, *where* $\Omega_x = \{t \in \mathbf{R} : x + t\nu \in \Omega\}$.

Let $\Omega = \bigcup_{h=0}^{\infty} E_h$, with $\mathscr{H}_{n-1}(E_0) = 0$ and $E_h \in BBV(u)$ for every $h \geq 1$. Fix $\nu \in S^{n-1}$, and set $l_\nu(x) = \{y \in \mathbf{R}^n : y = x + t\nu,\ t \in \mathbf{R}\}$; then by (i), (ii), (iii) of Definition 4.1 and [VA, Section 6] we have that for $\mathscr{H}_{n-1}$-a.a. $x \in \pi_\nu$ the set $E_0 \cap l_\nu(x)$ is empty and for every $h \geq 1$ the set $E_h^x = E_h \cap l_\nu(x)$ is a finite union of intervals; hence for $\mathscr{H}_{n-1}$-a.a. $x \in \pi_\nu$ the sets E_h^x verify (i), (ii), (iii) of Definition 4.1 (with $n = 1$) for every $h \geq 1$ and they cover Ω_x. Then, it is enough to prove that $((u_x)_{E_h^x}) \in BV(\mathbf{R}) \cap L^{\infty}(\mathbf{R})$ for $\mathscr{H}_{n-1}$-a.a. $x \in \pi_\nu$. Since $u_{E_h} \in BV(\mathbf{R}^n) \cap L^{\infty}(\mathbf{R}^n)$, for every $h \geq 1$, this follows from Theorem 2.20 and the equality $(u_{E_h})_x = (u_x)_{E_h^x}$ ($\mathscr{H}_{n-1}$-a.a. $x \in \pi_\nu$).

Step 3. *Conclusion.*

For every $\nu \in S^{n-1}$ and for $\mathscr{H}_{n-1}$-a.a. $x \in \pi_\nu$, by step 2, $u_x \in LBV(\Omega_x)$ and, by step 1, $\arctan(u_x) \in BV(\Omega_x)$; since $\arctan(u)_x = \arctan(u_x)$, by Theorem 2.20

$$\int_{\Omega} |D(\arctan(u))| < \sum_{i=1}^{n} \int_{\pi_{e_i}} \text{ess-}V_{\Omega_x}(\arctan(u_x)) <$$

$$< \sum_{i=1}^{n} \int_{\pi_{e_i}} \left\{ \int_{\Omega_x} |\nabla u_x|\, dt + \pi \mathscr{H}_0(\Omega_x \cap S_{u_x}) + \int_{\Omega_x} |CDu_x| \right\} d\mathscr{H}_{n-1}(x) <$$

$$< n \left\{ \int_{\Omega} |\nabla u|\, dx + \pi \mathscr{H}_{n-1}(S_u) + \int_{\Omega} |CDu| \right\} < +\infty$$

(as usual, (e_i) is an orthonormal basis of $\mathbf{R}^n$). Therefore since $|\Omega| < +\infty$ and $\arctan(u)$ is bounded, the thesis follows. ∎

It is easy to adapt the previous proof to get the following result (cf. also Theorem 4.6).

THEOREM 4.17: *Let* $u\colon \Omega \to \mathbf{R}^k$. *Then* $u \in BV(\Omega, \mathbf{R}^k)$ *iff* $u \in LBV(\Omega)$ *and*

$$\int_\Omega (|u| + |\nabla u|)\, dx + \int_{S_u} |u^+ - u^-|\, d\mathscr{H}_{n-1} + \int_\Omega |CDu| < +\infty .$$

The following result appears to be a partial converse of Proposition 3.8, and follows immediately from Theorems 4.14 and 4.16.

PROPOSITION 4.18. *If* $u \in GBV^*(\Omega, \Omega \times \mathbf{R})$ *and*

$$\int_\Omega |\nabla u|\, dx + \mathscr{H}_{n-1}(S_u) + \int_\Omega |CDu| < +\infty$$

then $\arctan(u) \in BV_{loc}(\Omega)$.

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