Rendiconti
Accademia Nazionale delle Scienze detta dei XL
Memorie di Matematica
108° (1990), Vol. XIV, fasc. 18, pagg. 341-360

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A Note on Composition Operators in Sobolev Spaces and an Extension of the Chain Rule (**)

ABSTRACT. — We study the operator T defined by $T[f, g](x) = f(g(x))$ from special subsets of $W^{m,p}(\Omega_1) \times (W^{m,q}(\Omega))^n$ to $W^{m,r}(\Omega)$. Conditions are found in order to ensure continuity and boundedness of T on bounded sets.

Sull'operatore di composizione negli spazi di Sobolev

SUNTO. — Si studia l'operatore T definito da $T[f, g](x) = f(g(x))$ su speciali sottoinsiemi di $W^{m,p}(\Omega_1) \times (W^{m,q}(\Omega))^n$ in $W^{m,r}(\Omega)$. Si danno condizioni esplicite affinchè T sia continuo e limitato sui limitati.

1. - INTRODUCTION

In this note we study the composition operator defined by

$$T[f, g](x) = f(g(x)), \qquad f \in W^{m,p}(\Omega_1),\ g \in (W^{m,q}(\Omega))^n, \tag{1.1}$$

where Ω, Ω_1 are open subsets of $\mathbb{R}^n$, $g(\Omega) \subseteq \Omega_1$ and where $W^{m,p}(\Omega_1)$ and $W^{m,q}(\Omega)$ denote Sobolev Spaces of exponents m, p and m, q respectively (cf. Adams (1975)). Operators such as T occur in the study of nonlinear differential equations and are well-known in the literature. We mention the work of Marcus and Mizel (1972-79), Adams (1976), Szigeti (1983, 1985), Valent (1982, 1985). For extensive references we refer to Appel (1987) and to Appel and Zabrejko (1990). The author was motivated to prove a part of the statements contained in this work to support the analysis of Lanza (1991). In the present paper we make no continuity assumption on the function f, as normally done in the literature. This of course requires some explanation concerning what is meant by the composition of an equivalence class of functions of $W^{m,p}(\Omega_1)$

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(**) Memoria presentata il 30 ottobre 1990 da Giuseppe Scorza Dragoni, uno dei XL.

ISSN-0392-4106

with an equivalence class of functions of $(W^{m,q}(\Omega))^n$. Our methods are novel, but apply only when $mq > n$, g is one-to-one, and the determinant of the gradient matrix of g is a.e. different from zero and has a reciprocal in $L^\gamma(\Omega)$ for some $0 < \gamma < \infty$. We consider the operator T from special subsets of $W^{m,p}(\Omega_1) \times (W^{m,q}(\Omega))^n$ to $W^{m,r}(\Omega)$ (cf. (2.9), (2.12)) and give explicit conditions on $m, p, q, \gamma, r, n, \Omega, \Omega_1$ in order to T maps bounded sets into bounded sets and is continuous. We also produce counterexamples to show how some of these conditions are sharp.

2. - Preliminaries and notation

We denote the norm on a Banach space $\mathfrak{X}$, by $\|\cdot : \mathfrak{X}\|$. Let $\mathfrak{X}$, $\mathfrak{Y}$ be Banach spaces. We equip the product space $\mathfrak{X} \times \mathfrak{Y}$ with the norm $\|\cdot : \mathfrak{X} \times \mathfrak{Y}\| = \|\cdot : \mathfrak{X}\| + \|\cdot : \mathfrak{Y}\|$. We say that $\mathfrak{X}$ is embedded in $\mathfrak{Y}$ provided that there exists a continuous injective map of $\mathfrak{X}$ into $\mathfrak{Y}$. The inverse function of a function g is indicated $g^{(-1)}$ as opposed to the reciprocal of a real valued function f, which is denoted f^{-1}. Let $u \in L^\gamma(\Omega)$. Although $(\int_\Omega |u(\mathbf{x})|^\gamma d\mathbf{x})^{1/\gamma}$ is not a norm on $L^\gamma(\Omega)$ when $0 < \gamma < 1$, we write $\|u : L^\gamma(\Omega)\| = (\int_\Omega |u(\mathbf{x})|^\gamma d\mathbf{x})^{1/\gamma}$. Let Ω be an open subset of $\mathbf{R}^n$. The space of m-times continuously differentiable functions on Ω, is denoted with $C^m(\Omega)$. The space of those functions of $C^\infty(\Omega)$ which have compact support contained in Ω is denoted $\mathfrak{D}(\Omega)$. Let $1 < p \leq \infty$, $\beta = (\beta_1, \dots, \beta_n) \in \mathbf{N}^n$ and $|\beta| = \beta_1 + \dots + \beta_n$. $W^{m,p}(\Omega)$ denotes the Sobolev space of all equivalence classes modulus the relation of equality a.e. of real-valued functions in $L^p(\Omega)$, all of whose distributional derivatives up to order m are in $L^p(\Omega)$. The space $W^{m,p}(\Omega)$ is equipped with the norm $\|u : W^{m,p}(\Omega)\| = \sum_{|\beta| \leq m} \|D^\beta u : L^p(\Omega)\|$. Let $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n) \in \mathbf{R}^n$. We set $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^n x_i y_i$. Throughout the paper, we agree that $1/r = 0$, if $r = \infty$.

Let Ω be an open subset of $\mathbf{R}^n$. We adopt the convention that whenever g belongs to $(W^{m,q}(\Omega))^n$, $mq > n$, it is taken to be the continuous representative of its equivalence class. The existence of such a representative is ensured by the Sobolev Imbedding Theorem.

Our first goal is to clarify the meaning of the composition in (1.1) and our starting point is the following simplified version of a Theorem due to Marcus and Mizel (1973*a*, pp. 791-792).

Theorem 2.1: *Let Ω be a bounded open subset of $\mathbf{R}^n$. Let $\infty \geq q > n$. Let $g \in (W^{1,q}(\Omega))^n$ be injective and satisfy $g(\Omega) \subseteq \mathbf{R}^n$. Let $f: g(\Omega) \to \mathbf{R}$ be measurable and A be a measurable subset of Ω. If $f \in L^1(g(A))$, then $f(g(\cdot))|\det Dg(\cdot)| \in L^1(A)$ and*

$$\int_{g(A)} f(\mathbf{y})\, d\mathbf{y} = \int_A f(g(\mathbf{x}))\, |\det Dg(\mathbf{x})|\, d\mathbf{x}. \tag{2.2}$$

Conversely, if $f(g(\cdot))|\det Dg(\cdot)| \in L^1(A)$, *then* $f \in L^1(g(A))$ *and* (2.2) *holds. Furthermore* g *maps sets of measure zero into sets of measure zero.*

We note that in general, even though $g \in (W^{m,q}(\Omega))^n$, $mq > n$, $g(\Omega) \subseteq \Omega_1$, $f, f_1 \in L^p(\Omega_1)$, $f = f_1$ a.e. in Ω_1, we cannot conclude $f(g(\cdot)) = f_1(g(\cdot))$ a.e. in Ω. However, we have the following

LEMMA 2.3: *Let* $\mathrm{N} \ni n > 1$, $q \in (1, \infty)$, $m > n/q$. *Let* Ω *be a bounded open subset of* R^n. *Let* $g \in (W^{m,q}(\Omega))^n$ *be injective*, $\det Dg(\cdot) \neq 0$ *a.e. in* Ω. *Then the following hold*

(i) $g(\Omega)$ *is open and* $g^{(-1)}: g(\Omega) \to \Omega$ *is continuous.*

(ii) *The inverse function* $g^{(-1)}: g(\Omega) \to \Omega$ *maps sets of measure zero into sets of measure zero.*

(iii) *If* $f, f_1 \in L^p(\Omega)$, $f = f_1$ *a.e. in* Ω, *then* $f(g(\cdot)) = f_1(g(\cdot))$ *a.e. in* Ω.

(iv) *A subset* A *of* Ω *is measurable if and only if* $g(A)$ *is measurable.*

(v) *A function* $f: g(\Omega) \to \mathrm{R}$ *is measurable if and only if* $f \circ g: \Omega \to \mathrm{R}$ *is measurable.*

PROOF: (i) is a well known fact, cf. e.g. Deimling (1980, p. 23). Now, let H be a subset of $g(\Omega)$ of measure zero. Let B be a Borel subset of $g(\Omega)$ of measure zero such that $H \subseteq B$. Then Theorem 2.1 implies that $\int_B dy = \int_{g^{(-1)}(B)} |\det Dg(\mathbf{x})|\, d\mathbf{x} = 0$. Since $|\det Dg(\mathbf{x})| > 0$ a.e. in Ω, we have

$$0 \leq \operatorname{meas}(g^{(-1)}(H)) \leq \operatorname{meas}(g^{(-1)}(B)) = 0 .$$

Statement (iii) is a trivial consequence of (ii). Since every measurable set can be represented as the union of a Borel set with a set of measure zero, and since g is a homeomorphism of Ω onto $g(\Omega)$, statement (iv) follows from (ii) and from the last part of Theorem 2.1. Statement (v) follows trivially from (iv). ■

We can now introduce the following definition.

2.4. DEFINITION: *Let* Ω *and* Ω_1 *be open subsets of* R^n, $m \in \mathrm{N}$, $1 \leq p \leq \infty$. *Let* g *be as in Lemma* 2.3, $f \in W^{m,p}(\Omega_1)$. *We indicate by* $f \circ g$ *or* $T[f, g]$ *the class of functions of* Ω *into* R, *which are almost everywhere equal to a representative of* f *composed with the continuous representative of* g. *We say that* $f \circ g$ *satisfies the chain rule, provided that for all* $1 \leq i \leq n$, $\sum_{l=1,\dots,n} (\partial f/\partial y_l)(g(\cdot))(\partial g_l/\partial x_i)$ *is in* $L^1_{loc}(\Omega)$ *and is the* $(\partial/\partial x_i)$*-derivative of* $f \circ g$ *in the sense of distributions.*

To shorten notation, we write

$$(2.5) \quad T^1[f, g](\mathbf{x}) = \left(T\left[\frac{\partial f}{\partial y_l}, g\right](\mathbf{x})\right)_{l=1,\dots,n}, \quad T^{\cdot}[f, g](\mathbf{x}) = T^1[f, g](\mathbf{x}) \cdot Dg(\mathbf{x}),$$
$$\det Dg(\mathbf{x}) = G(\mathbf{x}) .$$

The following Theorem due to Valent (1985, p. 64) is also important in our analysis. (For similar results cf. Grisvard (1985, p. 28).)

2.6. Theorem: *Let $p>1$, $q>1$, $r>1$. Assume that Ω has the cone property, and that $p>r$, $q>r$,*

$$\frac{m}{n} > \frac{1}{p} + \frac{1}{q} - \frac{1}{r}. \tag{2.7}$$

Then, if $u \in W^{m,p}(\Omega)$ and $v \in W^{m,q}(\Omega)$, we have $uv \in W^{m,r}(\Omega)$ and there exists a positive number $c>0$ independent of u and v such that

$$\| uv : W^{m,r}(\Omega) \| < c \| u : W^{m,p}(\Omega) \| \, \| v : W^{m,q}(\Omega) \| . \tag{2.8}$$

Since $W^{m,\infty}(\Omega)$ is imbedded in $W^{m,q}(\Omega)$ it is easily seen that in the Theorem above, we can choose $p=r$, $q=\infty$.

We now introduce the following notation. Let Ω, Ω_1 be open subsets of $\mathbb{R}^n$, Ω bounded. Let $f \in W^{m,p}(\Omega_1)$, $1<p<\infty$, g as in Lemma 2.3, $0<c<\infty$, $0<\gamma<\infty$. Under these conditions g is continuous, injective and open. Then we can define the following

$$J_{m,p,q,\gamma,c}(\Omega) = \{(f,g) : g \in (W^{m,q}(\Omega))^n,\ g \text{ is injective},\ G(x) \neq 0, \text{ a.e. in } \Omega,$$
$$G^{-1}(\mathbf{x}) \in L^{\gamma}(\Omega),\ \| G(\mathbf{x})^{-1} : L^{\gamma}(\Omega) \| < c,\ f \in W^{m,p}(g(\Omega))\} . \tag{2.9}$$

$J_{m,p,q,\gamma,c}(\Omega)$ is clearly a subset of $\Big(\bigcup_{\mathbb{R}^n \supseteq O \text{ open}} W^{m,p}(O) \Big) \times W^{m,q}(\Omega, \mathbb{R}^n)$, which is not a Banach space. We then introduce the notion of boundedness in $J_{m,p,q,\gamma,c}(\Omega)$.

2.10. Definition: *A subset $\mathfrak{E}$ is said to be bounded in $J_{m,p,q,\gamma,c}(\Omega)$ provided that*

$$\sup_{(f,g) \in \mathfrak{E}} \{ \| g : (W^{m,q}(\Omega))^n \| ,\ \| f : W^{m,p}(g(\Omega)) \| \} < \infty . \tag{2.11}$$

Let $\mathfrak{U} \subset J_{m,p,q,\gamma,c}(\Omega)$, let $\mathfrak{Y}$ be a Banach space and let $Q : \mathfrak{U} \to \mathfrak{Y}$. We say that Q is bounded if Q maps bounded subsets of $\mathfrak{U}$ into bounded subsets of $\mathfrak{Y}$. Let

$$Y_{m,p,q,\gamma,c}(\Omega, \Omega_1) = \{(f,g) \in W^{m,p}(\Omega_1) \times (W^{m,q}(\Omega))^n : g \text{ is injective},$$
$$g(\Omega) \subseteq \Omega_1,\ G(\mathbf{x}) \neq 0, \text{ a.e. in } \Omega,\ G(\mathbf{x})^{-1} \in L^{\gamma}(\Omega),\ \| G(\mathbf{x})^{-1} : L^{\gamma}(\Omega) \| < c\} , \tag{2.12a}$$

$$X_{m,p,q,\gamma,c}(\Omega, \Omega_1) = \{(f,g) \in Y_{m,p,q,\gamma,c}(\Omega, \Omega_1) : g(\Omega) = \Omega_1\} . \tag{2.12b}$$

3. - The composition theorems for the operator T

The following imbedding will be considered in the sequel

(3.1) $C^1(\mathrm{cl}\,\Omega_1)$ is densely included in $W^{1,p}(\Omega_1)$, $\quad p\in(1,\infty)$.

It is known that the above imbedding holds if Ω_1 has the segment property (cf. Adams (1975, p. 54)).

3.2. Theorem: *Let Ω, Ω_1 be open subsets of $\mathbb{R}^n$. Let Ω be bounded and have the cone property. Let $1<p<\infty$, $1\le m\in\mathbb{N}$, $1<q<\infty$, $0<\gamma<\infty$, $0<\varepsilon<\infty$, $m>n/q$. Let*

$$(3.3)\quad \tilde{r}(m,p,q,\gamma,n)=\begin{cases}\dfrac{pqn}{pm[n-(m-1)q]+nq+n(q/\gamma)}, & \text{if } (m-1)<n/q,\\[2ex] \dfrac{p}{1+(1/\gamma)} & \text{if } (m-1)>n/q,\end{cases}$$

and $\tilde{r}(m,p,q,\gamma,n)\in(1,q]$. Then

(i) *If $(m-1)<n/q$, then T maps $J_{m,p,q,\gamma,\infty}(\Omega)$ and $Y_{m,p,q,\gamma,\infty}(\Omega,\Omega_1)$ to $W^{m,r}(\Omega)$, for all $1<r<\tilde{r}(m,p,q,\gamma,n)$. If $m=1$, we can choose $r=\tilde{r}(m,p,q,\gamma,n)$.*

(ii) *If $(m-1)>n/q$, then T maps $J_{m,p,q,\gamma,\infty}(\Omega)$ and $Y_{m,p,q,\gamma,\infty}(\Omega,\Omega_1)$ to $W^{m,\tilde{r}(m,p,q,\gamma,n)}(\Omega)$.*

Both in cases (i) and (ii), T is bounded on $J_{m,p,q,\gamma,\varepsilon}(\Omega)$, $Y_{m,p,q,\gamma,\varepsilon}(\Omega,\Omega_1)$ for all real numbers $\varepsilon>0$, and the elements in the range of T satisfy the chain rule. Furthermore, if assumption (3.1) holds, then T is continuous on $Y_{m,p,q,\gamma,\varepsilon}(\Omega,\Omega_1)$ for all $\varepsilon>0$ both in cases (i) and (ii). In case (ii), if $\gamma=\infty$, and (3.1) holds, T is continuous even if $\varepsilon=\infty$.

Proof: We notice that if $(f,g)\in Y_{m,p,q,\gamma,\varepsilon}(\Omega,\Omega_1)$, then

$$\|f|_{g(\Omega)}: W^{m,p}(g(\Omega))\|<\|f: W^{m,p}(\Omega_1)\| \quad\text{and}\quad (f,g)\in J_{m,p,q,\gamma,\varepsilon}(\Omega).$$

Then the boundedness of T on $J_{m,p,q,\gamma,\varepsilon}(\Omega)$ implies the boundedness of T on $Y_{m,p,q,\gamma,\varepsilon}(\Omega,\Omega_1)$. Moreover, it is clear that if T maps $J_{m,p,q,\gamma,\varepsilon}(\Omega)$ into $W^{m,r}(\Omega)$ for all $\varepsilon>0$, then T maps $J_{m,p,q,\gamma,\infty}(\Omega)$ into $W^{m,r}(\Omega)$. Finally, the inductive arguments used in this proof to prove the continuity of T when (3.1) holds, are the same as those used to prove the boundedness of T and are accordingly omitted. We split the proof in parts A)-E).

A) We first consider case $(m-1)<n/q$. Clearly, in this case $1<q<\infty$, $0<\gamma\le\infty$. We will only consider $\gamma<\infty$. Case $\gamma=\infty$ can be handled similarly. We proceed by induction on m. Let $m=1$. By Theorem 2.1, Lemma 2.3, and Hölder inequality, we deduce that

$$(3.4)\quad \Big(\int_{\Omega}|f(g(\mathbf{x}))|^{pq\gamma/(\gamma(p+q)+q)}d\mathbf{x}\Big)^{(\gamma(p+q)+q)/pq\gamma} =$$

$$= \Big(\int_{g(\Omega)}|f(\mathbf{y})|^{pq\gamma/(\gamma(p+q)+q)}|G^{-1}(g^{(-1)}(\mathbf{y}))|\,d\mathbf{y}\Big)^{(\gamma(p+q)+q)/pq\gamma} <$$

$$< \Big(\int_{g(\Omega)}|f(\mathbf{y})|^{p}\,d\mathbf{y}\Big)^{1/p}\Big(\int_{g(\Omega)}|G^{-1}(g^{(-1)}(\mathbf{y}))|^{(\gamma(p+q)+q)/(p\gamma+q)}d\mathbf{y}\Big)^{(p\gamma+q)/\gamma pq} =$$

$$= \|f : L^p(g(\Omega))\|\Big(\int_{\Omega}|G^{-1}(\mathbf{x})|^{\gamma q/(\gamma p+q)}d\mathbf{x}\Big)^{(\gamma p+q)/\gamma pq} <$$

$$< \|f : L^p(g(\Omega))\|\,\|G^{-1} : L^{\gamma}(\Omega)\|^{1/p}(\text{meas}\,(\Omega))^{1/q}.$$

Similarly, we can show that for each $i = 1, \ldots, n$,

$$(3.5)\quad \|T_i'[f,g] : L^{\gamma pq/(\gamma(p+q)+q)}(\Omega)\| <$$

$$< \sum_{l=1}^{n}\Big\|\frac{\partial f}{\partial y_l} : L^p(g(\Omega))\Big\|\,\|G^{-1} : L^{\gamma}(\Omega)\|^{1/p}\Big\|\frac{\partial g_l}{\partial x_i} : L^q(\Omega)\Big\|.$$

Next we show that $T_i'[f,g]$ is actually the $(\partial/\partial x_i)$-derivative of $T[f,g]$. The following argument applies when $\gamma < \infty$. (If $\gamma = \infty$, we just consider the imbedding $L^\infty(\Omega) \subseteq L^t(\Omega)$ with t large enough.) Since $W^{1,p}(g(\Omega)) \cap C^\infty(g(\Omega))$ is dense in $W^{1,p}(g(\Omega))$ there exists a sequence $\{f_j\}$ in $W^{1,p}(g(\Omega))$ which converges to f. By assumption $r(1,p,q,\gamma,n) > 1$ we have $p\gamma/(\gamma+1) > 1$. Then $\{f_j\}$ converges to f in $L^{1+(1/\gamma)}(g(\Omega))$ by boundedness of $g(\Omega)$. Furthermore, if $f_j \in C^\infty(g(\Omega))$, $g \in W^{1,q}(\Omega)$, $q > n$, the chain rule holds (cf. Marcus and Mizel (1972, Th. 4.3)). By assumption $\int|G^{-1}(\mathbf{x})|^\gamma d\mathbf{x} < \infty$, and by Theorem 2.1, we have $|G^{-1}(g^{(-1)}(\cdot))| \in L^{1+\gamma}(g(\Omega))$. Furthermore $(\partial\phi/\partial x_i)(g^{(-1)}(\cdot)) \in L^\infty(g(\Omega))$. Then, by using repeatedly Theorem 2.1, we have

$$(3.6)\quad -\int_{\Omega} f(g(\mathbf{x}))\frac{\partial\phi}{\partial x_i}(\mathbf{x})\,d\mathbf{x} = -\int_{g(\Omega)} f(\mathbf{y})\frac{\partial\phi}{\partial x_i}(g^{(-1)}(\mathbf{y}))|G^{-1}(g^{(-1)}(\mathbf{y}))|\,d\mathbf{y} =$$

$$= -\lim_j \int_{g(\Omega)} f_j(\mathbf{y})\frac{\partial\phi}{\partial x_i}(g^{(-1)}(\mathbf{y}))|G^{-1}(g^{(-1)}(\mathbf{y}))|\,d\mathbf{y} =$$

$$= -\lim_j \int_{\Omega} f_j(g(\mathbf{x}))\frac{\partial\phi}{\partial x_i}(\mathbf{x})\,d\mathbf{x} =$$

$$= \lim_j \int_{\Omega}\sum_{l=1}^{n}\frac{\partial f_j}{\partial y_l}(g(\mathbf{x}))\frac{\partial g_l}{\partial x_i}(\mathbf{x})\phi(\mathbf{x})\,d\mathbf{x} =$$

$$= \lim_j \int_{\Omega}\sum_{l=1}^{n}\frac{\partial f_j}{\partial y_l}(\mathbf{y})\frac{\partial g_l}{\partial x_i}(g^{(-1)}(\mathbf{y}))\,\phi(g^{(-1)}(\mathbf{y}))|G^{-1}(g^{(-1)}(\mathbf{y}))|\,d\mathbf{y}.$$

We now observe that $(\partial\phi/\partial x_i)(g^{(-1)}(\cdot)) \in L^\infty(g(\Omega))$, $\partial g_k/\partial x_i \in L^q(\Omega)$, and that

$$(3.7)\qquad \int_{g(\Omega)} \left|\frac{\partial g_k}{\partial x_i}(g^{(-1)}(y))\right|^{p/(p-1)} |G^{-1}(g^{(-1)}(y))|^{p/(p-1)}\,dy <$$

$$< \left(\int_\Omega \left|\frac{\partial g_k}{\partial x_i}(x)\right|^q dx\right)^{p/(q(p-1))} \left(\int_\Omega |G^{-1}(x)|^{q/(pq-(q+p))}\,dx\right)^{(pq-(q+p))/(q(p-1))}.$$

Then, by assumption $\bar p(1, p, q, \gamma, n) > 1$, we have $G^{-1} \in L^\gamma(\Omega) \subset L^{q/(pq-(p+q))}(\Omega)$ and the last limit in (3.6) equals

$$(3.8)\qquad \int_{g(\Omega)} \sum_{i=1}^{n} \frac{\partial f}{\partial y_i}(y) \frac{\partial g_i}{\partial x_k}(g^{-1}(y))\,\phi(g^{-1}(y))\,|G^{-1}(g^{-1}(y))|\,dy =$$

$$= \int_\Omega \sum_{i=1}^{n} \frac{\partial f}{\partial y_i}(g(x)) \frac{\partial g_i}{\partial x_k}(x)\,\phi(x)\,dx.$$

We note that by (3.4) and (3.5), T is bounded from $J_{1,p,q,\gamma,c}(\Omega)$ to $W^{1,\bar p(1,p,q,\gamma,n)}(\Omega)$ if $c < \infty$. We now show the continuity of T in case (3.1) holds and $c < \infty$. Let $\{(f_j, g_j)\}$ be a sequence in $Y_{1,p,q,\gamma,c}(\Omega, \Omega_1)$ converging to

$$(f, g) \in Y_{1,p,q,\gamma,c}(\Omega, \Omega_1).$$

Let $\varepsilon > 0$ and $\eta \in C^1(\mathrm{cl}\,\Omega_1)$ be such that $\|f - \eta : W^{1,p}(\Omega_1)\| < \varepsilon$. By the same argument used to prove inequality (3.5), we have

$$(3.9)\qquad \left\|\frac{\partial f_j}{\partial y_i}(g_j(x)) - \frac{\partial f}{\partial y_i}(g(x)) : L^{p\gamma/(\gamma+1)}(\Omega)\right\| <$$

$$< \left\|\frac{\partial f_j}{\partial y_i} - \frac{\partial f}{\partial y_i} : L^p(\Omega_1)\right\| \|G_j^{-1} : L^\gamma(\Omega)\|^{1/p} + \left\|\frac{\partial f}{\partial y_i} - \frac{\partial \eta}{\partial y_i} : L^p(\Omega_1)\right\| \|G_j^{-1} : L^\gamma(\Omega)\|^{1/p} +$$

$$+ \left(\int_\Omega \left|\frac{\partial \eta}{\partial y_i}(g_j(x)) - \frac{\partial \eta}{\partial y_i}(g(x))\right|^{p\gamma/(\gamma+1)} dx\right)^{(\gamma+1)/p\gamma} +$$

$$+ \left\|\frac{\partial \eta}{\partial y_i} - \frac{\partial f}{\partial y_i} : L^p(\Omega_1)\right\| \|G^{-1} : L^\gamma(\Omega)\|^{1/p}.$$

Now note that $\sup_j \{\|G_j^{-1}\, L^\gamma(\Omega)\|, \|G^{-1} : L^\gamma(\Omega)\|\} < c$, $\lim_j \partial f_j/\partial y_i = \partial f/\partial y_i$ in $L^p(\Omega_1)$, and that $\lim_j g_j = g$ in $L^\infty(\Omega)$ by Sobolev imbedding and by assumption $m > n/q$. Moreover, $\partial\eta/\partial y_i$ is uniformly continuous on the compact set $\mathrm{cl}\left\{\bigcup_j g_j(\Omega)\right\}$ and meas $(\Omega) < \infty$. It follows that the $\limsup_{j\to\infty}$ of left hand-side of (3.9) is less or equal than $2\varepsilon c^{1/p}$. By arbitrariness of ε, and by the convergence of $\partial g_{jk}/\partial x_i$ to $\partial g_k/\partial x_i$ in $L^q(\Omega)$ we conclude that the sequence $\{T'[f_j, g_j]\}$ converges to $T'[f, g]$ in $L^{\bar p(1,p,q,\gamma,n)}(\Omega)$. Similarly, we can show that the sequence $\{T[f_j, g_j]\}$ converges to $T[f, g]$ in $L^{\bar p(1,p,q,\gamma,n)}(\Omega)$. We now prove the statement for $m > 1$. By Sobolev imbedding, we have $W^{m,q}(\Omega) \subseteq W^{1,(nq/(n-(m-1)q))}(\Omega)$. Since $m > n/q$, we have $1 \cdot nq/(n-(m-1)q) > n$. Moreover, a simple computation shows that the assumptions $\gamma > 0$, $\bar{r}(m, p, q, \gamma, n) \in (1, q]$ imply

$$\bar{r}\left(1, p, \frac{nq}{n-(m-1)q}, \gamma, n\right) \in \left(1, \frac{nq}{n-(m-1)q}\right].$$

Hence, by case $m = 1$, and inequality

$$\bar{r}\left(1, p, \frac{nq}{n-(m-1)q}, \gamma, n\right) > \bar{r}(m, p, q, \gamma, n),$$

we conclude that $T[f, g]$ satisfies the chain rule and that T is bounded from $J_{m,p,q,\gamma,c}(\Omega)$ into $W^{1,\bar{r}(m,p,q,\gamma,n)}(\Omega)$ if $c < \infty$. We now consider the operator T'. Clearly, we have

$$W^{m,q}(\Omega) \subseteq W^{m-1,(nq/(n-q))}(\Omega), \qquad [(m-1)-1]\frac{qn}{n-q} < n, \qquad (m-1)\frac{qn}{n-q} > n,$$

and

$$\bar{r}\left(m-1, p, \frac{nq}{n-q}, \gamma, n\right) > \bar{r}(m, p, q, \gamma, n) > 1.$$

Furthermore, assumptions $\bar{r}(m, p, q, \gamma, n) < q$ and $m > n/q$ imply that

$$\bar{r}\left(m-1, p, \frac{nq}{n-q}, \gamma, n\right) < \frac{nq}{n-q}.$$

Hence, by the inductive hypothesis, we can conclude that for all

$$s \in \left(1, \bar{r}\left(m-1, p, \frac{nq}{n-q}, \gamma, n\right)\right),$$

the nonlinear operator T' is bounded from $J_{m,p,q,\gamma,c}(\Omega)$ to $W^{m-1,s}(\Omega)$ if $c < \infty$. Now let $r \in (1, \bar{r}(m, p, q, \gamma, n))$. By Lemma 2.6, the proof is complete if the following inequality holds

$$\text{(3.10)} \qquad \frac{m-1}{n} > \frac{1}{q} + \frac{1}{s} - \frac{1}{r} \qquad \text{for some } s \in \left(1, \bar{r}\left(m-1, p, \frac{nq}{n-q}, \gamma, n\right)\right).$$

It is easy to check that

$$\frac{m-1}{n} = q^{-1} + \bar{r}\left(m-1, p, \frac{nq}{n-q}, \gamma, n\right)^{-1} - \bar{r}(m, p, q, \gamma, n)^{-1}.$$

Then

$$\frac{m-1}{n} > q^{-1} + \bar{r}\left(m-1, p, \frac{nq}{n-q}, \gamma, n\right)^{-1} - r^{-1},$$

and consequently (3.10) holds for some

$$s \in \left(1, \bar{r}\left(m-1, p, \frac{nq}{n-q}, \gamma, n\right)\right)$$

sufficiently close to $\bar{r}(m-1, p, nq/(n-q), \gamma, n)$.

B) We consider case $q=\infty$, $1<\gamma<\infty$, $1<p<\infty$. The corresponding case with $\gamma=\infty$ can be treated similarly. We proceed by induction on m. Case $m=1$ can be treated as in the proof of (i). We now assume $m>1$. Since the imbedding $W^{m,p}(\Omega_1)\times W^{m,\infty}(\Omega)\subset W^{1,p}(\Omega_1)\times W^{1,\infty}(\Omega)$ holds, the chain rule applies to $f\circ g$ and T is bounded from $J_{m,p,q,\gamma,\varepsilon}(\Omega)$ to $W^{1,(p\gamma/(\gamma+1))}(\Omega)$ if $0<\varepsilon<\infty$. By inductive hypothesis, T^* is bounded from $J_{m,p,q,\gamma,\varepsilon}(\Omega)$ to $W^{m-1,(p\gamma/(\gamma+1))}(\Omega)$ if $0<\varepsilon<\infty$. By continuity and boundedness of the multiplication from $W^{m-1,(p\gamma/(\gamma+1))}(\Omega)\times W^{m-1,\infty}(\Omega)$ to $W^{m-1,(p\gamma/(\gamma+1))}(\Omega)$ (cf. Lemma 2.6), we conclude that T' is bounded from $J_{m,p,q,\gamma,\varepsilon}(\Omega)$ to $W^{m-1,(p\gamma/(\gamma+1))}(\Omega)$ if $0<\varepsilon<\infty$. This concludes the proof of *B*).

C) We now consider case $1<q<\infty$, $0<\gamma<\infty$, $1<p<\infty$, $q>n$, $(m-1)q>n$, $q>n/m$. The corresponding case with $\gamma=\infty$ can be treated similarly. We note that the statement is meaningful only when $n>2$. We proceed as above by induction on m. Let $m=2$ and $p\gamma/(\gamma+1)\in(1,q]$. Since the imbedding $W^{2,p}(\Omega_1)\times W^{2,q}(\Omega)\subset W^{1,p}(\Omega_1)\times W^{1,\infty}(\Omega)$ holds, we can apply case $q=\infty$, $m=1$ to conclude that T is bounded from $J_{2,p,q,\gamma,\varepsilon}(\Omega)$ to $W^{1,(p\gamma/(\gamma+1))}(\Omega)$ if $0<\varepsilon<\infty$. We now note again that the imbedding $W^{2,p}(\Omega_1)\times W^{2,q}(\Omega)\subseteq W^{2,p}(\Omega_1)\times W^{1,\infty}(\Omega)$ holds. Hence, by case $q=\infty$, $m=1$, the operator T^* is bounded from $J_{2,p,q,\gamma,\varepsilon}(\Omega)$ to $W^{1,(p\gamma/(\gamma+1))}(\Omega)$ if $0<\varepsilon<\infty$. Then, by Lemma 2.6, the proof of case $m=2$ is complete. We now assume that the statement is true for $m-1$ and prove it for $m>2$. Since the imbedding $W^{m,p}(\Omega_1)\times W^{m,q}(\Omega)\subseteq W^{2,p}(\Omega_1)\times W^{2,q}(\Omega)$ holds, case $m=2$ implies that the operator T is bounded from $J_{m,p,q,\gamma,\varepsilon}(\Omega)$ to $W^{2,(p\gamma/(\gamma+1))}(\Omega)$ if $0<\varepsilon<\infty$. We now note that the imbedding $W^{m,q}(\Omega)\subset W^{m-1,t}(\Omega)$, $\forall t>q$ holds. We choose t such that

$$t>q\,,\qquad (m-1)t>n\,,\qquad (m-2)t>n\,,\qquad \frac{p\gamma}{\gamma+1}\in(1,t]\,. \tag{3.11}$$

Then T^* is bounded from $J_{m,p,t,\gamma,\varepsilon}(\Omega)$ to $W^{m-1,(p\gamma/(\gamma+1))}(\Omega)$ if $0<\varepsilon<\infty$. Then, by Lemma 2.6, the proof of *C*) is complete.

D) We now consider the case $1<q<\infty$, $1<\gamma<\infty$, $1<p<\infty$, $q<n$, $(m-1)q>n$, $mq>n$. The corresponding case with $\gamma=\infty$ can be treated similarly. Since the statement requires $m>3$, we start by examing $m=3$.

By assumption $2q > n$ we have $W^{2,q}(\Omega) \subset W^{1,\infty}(\Omega)$. Then, by case $m = 1$, $q = \infty$, $\gamma < \infty$, T^{ε} is bounded from $J_{2,p,q,\gamma,\varepsilon}(\Omega)$ to $W^{1,(p\gamma/(\gamma+1))}(\Omega)$ if $0 < \varepsilon < \infty$. We now note that

$$(3.12a) \qquad W^{2,q}(\Omega) \subset W^{1,(nq/(n-q))}(\Omega) \qquad \text{if } q < n\,;$$

$$(3.12b) \qquad W^{2,q}(\Omega) \subset W^{1,t}(\Omega)\,, \qquad \forall t > q\,, \text{ if } n = q\,.$$

In case (3.12a) holds, we have $2(nq/(n-q)) > n$, $1(nq/(n-q)) > n$. Hence, by case $m = 2$, $mq > n$, $(m-1)q > n$, the operator T^{ε} is bounded from $J_{2,p,q,\gamma,\varepsilon}(\Omega)$ to $W^{1,(p\gamma/(\gamma+1))}(\Omega)$ if $0 < \varepsilon < \infty$. Similarly we conclude if (3.12b) holds, as long as we choose $t > n$. Since $2/n > q^{-1}$, $q > p\gamma/(\gamma+1)$, Lemma 2.6 implies that T^{ε} is bounded from $J_{2,p,q,\gamma,\varepsilon}(\Omega)$ to $W^{2,(p\gamma/(\gamma+1))}(\Omega)$ if $0 < \varepsilon < \infty$. We now assume that the statement is true for $m-1$ and prove it for m. By the imbedding $W^{m,p}(\Omega_1) \times W^{m,q}(\Omega) \subset W^{1,p}(\Omega_1) \times W^{1,\infty}(\Omega)$ and by case $m = 1$, $q = \infty$, T is bounded from $J_{m,p,q,\gamma,\varepsilon}(\Omega)$ to $W^{1,(p\gamma/(\gamma+1))}(\Omega)$ if $0 < \varepsilon < \infty$. Now, let $q < n$. Then $W^{m,q}(\Omega) \subset W^{m-1,(nq/(n-q))}(\Omega)$ and $(nq/(n-q))(m-1) > n$, $(nq/(n-q))(m-2) > n$ and at least one of the following is inequalities is true

$$(3.13a) \qquad \frac{nq}{n-q} < n\,,$$

$$(3.13b) \qquad \frac{nq}{n-q} > n\,.$$

If $nq/(n-q) < n$, then we can use the inductive hypothesis conclude that T^{ε} is bounded from $J_{m,p,q,\gamma,\varepsilon}(\Omega)$ to $W^{m-1,(p\gamma/(\gamma+1))}(\Omega)$ if $0 < \varepsilon < \infty$. In case (3.13b) holds, then we can use the case discussed in *C*) to draw the same conclusion. If $q = n$, then $W^{m,q}(\Omega) \subset W^{m-1,t}(\Omega)$, $\forall t > q$. By choosing t such that $t(m-1) > n$, $t(m-2) > n$, $t > n$ and invoking again the case discussed in *C*), we conclude that T^{ε} is bounded from $J_{m,p,q,\gamma,\varepsilon}(\Omega)$ to $W^{m-1,(p\gamma/(\gamma+1))}(\Omega)$ if $0 < \varepsilon < \infty$. Since $(m-1)/n > q^{-1}$, $p > p\gamma/(\gamma+1)$ hold, the proof of case *D*) is complete by Lemma 2.6.

E) We now consider the case $mq > n$, $(m-1)q = n$, $1 < \gamma < \infty$, $q < \infty$, $p\gamma/(\gamma+1) > 1$. The corresponding case with $\gamma = \infty$ can be treated similarly. Since $n > 1$, $m > 1$, we must have $m > 2$, $n > q$. Clearly

$$(3.14) \qquad W^{m,q}(\Omega) \subseteq W^{1+(m-1),q}(\Omega) \subseteq W^{1,t}(\Omega)\,, \qquad \forall t > q\,.$$

By choosing t such that

$$(3.15) \qquad t > n\,, \qquad \frac{p\gamma}{\gamma+1} > \frac{\gamma p t}{\gamma(p+t)+t} > r > 1$$

and using case (i) with $m = 1$, we conclude that T is bounded from $J_{m,p,q,\gamma,\varepsilon}(\Omega)$ to $W^{1,r}(\Omega)$ if $0 < \varepsilon < \infty$. We now consider separately cases $m = 2$ and $m > 2$. Let $r < s < p\gamma/(\gamma+1)$, $m = 2$. Then $n = q$ and the imbedding $W^{1,p}(\Omega_1) \times$

$\times W^{1,s}(\Omega) \subset W^{1,s}(\Omega_1) \times W^{1,t}(\Omega)$, $\forall t > n = q$ holds. By arguing as above, T^* is bounded from $J_{2,p,q,\gamma,\varepsilon}(\Omega)$ to $W^{1,s}(\Omega)$ if $0 < \varepsilon < \infty$. Since $n^{-1} > q^{-1} + s^{-1} - r^{-1}$, Lemma 2.6 implies that T is bounded from $J_{2,p,q,\gamma,\varepsilon}(\Omega)$ to $W^{1,r}(\Omega)$ if $0 < \varepsilon < \infty$. This completes the proof in case $m = 2$. Now let $m > 2$. We choose t in (3.14) such that $t(m-1) > n$, $t(m-2) > n$, $t > n$. Then we invoke case $mq > n$, $(m-1)q > n$, $q > n$ to conclude that T^* is bounded from $J_{m,p,q,\gamma,\varepsilon}(\Omega)$ to $W^{m-1,s}(\Omega)$ if $0 < \varepsilon < \infty$. We observe that $(m-1)/n > q^{-1} + s^{-1} - r^{-1}$ and use Lemma 2.6 to conclude the proof of E). To prove the last statement let $(m-1) > n/q$, $\lim_j (f_j, g_j) = (f, g)$ in $Y_{m,p,q,\infty,\infty}(\Omega, \Omega_1)$. By Sobolev imbedding, $\lim G_j = G$ uniformly in Ω. Since $G^{-1}, G_j^{-1} \in L^\infty(\Omega)$, there exists $\varepsilon > 0$ such that (f, g) and the whole sequence $\{(f_j, g_j)\}$ is in $Y_{m,p,q,\infty,\varepsilon}(\Omega, \Omega_1)$.
■

Remark: It is interesting to note that if we take $\varepsilon = \infty$, $\gamma < \infty$ in Theorem 3.2, T might be neither bounded nor continuous. In other words, we could have

$$\lim_j (f_j, g_j) = (f, g) \text{ in } Y_{1,p,q,\gamma,\infty}(\Omega, \Omega_1), \quad \sup_j \| T[f_j, g_j] : W^{1,r}(\Omega) \| = \infty,$$

even though $T(Y_{1,p,q,\gamma,\infty}(\Omega, \Omega_1)) \subset W^{1,r}(\Omega)$, as the following example shows.

Example: Let $\Omega = \Omega_1 = (0, 1)^2$. Let

$$\alpha \in (0, 1), \quad q > 4/\alpha, \quad \frac{2}{\alpha} > p > \frac{4q}{3q\alpha - 4}, \quad \gamma > \frac{4q}{3pq\alpha - 4(p+q)}.$$

Let

$$f(y_1, y_2) = (y_1^2 + y_2^2)^{(1-\alpha)/2}, \quad g(x_1, x_2) = (x_1, x_2), \quad g_j(x_1, x_2) = \left(\int_0^{x_1} b_j(\xi)\, d\xi, x_2 \right),$$

where b_j is defined by

$$\begin{aligned}
b_j(\xi) &= \frac{\xi}{j} + \frac{1}{j^2}, && \text{if } \xi \in [0, 1/j]; \\
b_j(\xi) &= \frac{j^2 - 2}{j}\xi + \frac{4 - j^2}{j^2}, && \text{if } \xi \in [1/j, 2/j], \\
b_j(\xi) &= 1, && \text{if } \xi \in \left[2/j, 1 - \frac{2}{j}\right], \\
b_j(\xi) &= \xi \frac{j^2 - 2}{j} + \frac{3j^2 - j^3 - 4 + 2j}{j^2}, && \text{if } \xi \in [1 - 2/j, 1 - 1/j]; \\
b_j(\xi) &= \frac{\xi}{j} + \frac{2j^2 - 1 - j}{j^3}, && \text{if } \xi \in [1 - 1/j, 1].
\end{aligned}$$

It can be readily checked that $\lim_j (f, g_j) = (f, g)$ in $Y_{1,p,q,\gamma,\infty}(\Omega, \Omega_1)$ and that $T[f, g], T[f, g_j] \in W^{1,\tilde{r}(1,p,q,\gamma,\infty)}(\Omega)$. However, $\{T[f, g_j]\}$ does not converge to $T[f, g]$. Indeed $\sup_j \| T[f, g_j] : W^{1,\tilde{r}(1,p,q,\gamma,\infty)}(\Omega) \| = \infty$.

REMARK: We now consider the special case in which $n = 1$. Let $p, q > n$, $(f, g) \in J_{1,p,q,\infty,\infty}(\Omega)$, Ω an open interval of R, $r = pq/(p+q-1)$. (Note that $r > \bar{r}(1, p, q, \infty, n) = pq/(p+q)$.) By Hölder inequality, we have

$$\int_{\Omega} |f \circ g(\mathbf{x})|^r \, d\mathbf{x} \leq \| f : L^p(\Omega_1)\|^r \| G^{(-1)} : L^\infty(\Omega)\| \{\text{meas}\,(\Omega)\}^{(p-1)/(p+q-1)}.$$

Furthermore, by using the same arguments of the proof of (i) of Theorem 3.2, we can prove that $|df/dy|^r \in L^{p/r}(g(\Omega))$, and that

$$\left|\frac{dg}{dx}(g^{(-1)}(y))\right|^r |G^{-1}(g^{(-1)}(y))| \in L^{(p+q-1)/(p-1)}(g(\Omega)).$$

Indeed $(dg/dx)(x)G^{-1}(x) = 1 \in L^\infty(\Omega)$. Hence, we conclude that $T[f, g] \in W^{1,r}(\Omega)$. This fact has been observed for $f \in W^{s,p}(g(\Omega))$, $p(s-1) \geq 1$ (in which case $f \in W^{1,\infty}(g(\Omega))$), g monotone, by Szigeti (1985), who employed a completely different argument.

We observe that in Theorem 3.2, the continuity of T was obtained by using (3.1). We now show that something can still be said about the continuity of T if (3.1) is not assumed to hold. To do so, we introduce the following three lemmas.

3.16. LEMMA: *Let Ω, Ω_1 be open subsets of $\mathbf{R}^n$, $n > 1$, Ω bounded. Let ϕ, ϕ_i be continuous and one to one mappings of Ω onto Ω_1. Let $\phi, \phi_i \in C^0(\Omega)$. If $\{\phi_i\}$ converges uniformly to ϕ in Ω, then $\{\phi_i^{(-1)}\}$ converges to $\phi^{(-1)}$ pointwise in Ω_1.*

PROOF: Let $\mathbf{y} \in \Omega_1$ and assume by contradiction that the sequence $\{\phi_i^{(-1)}(\mathbf{y})\}$ does not converge to $\phi^{(-1)}(\mathbf{y})$. Let $\{\phi_{i_k}^{(-1)}(\mathbf{y})\}$ be a subsequence of $\{\phi_i^{(-1)}(\mathbf{y})\}$ converging to $\xi \in \text{cl}\,\Omega$, $\xi \neq \phi^{(-1)}(\mathbf{y})$. Let $\delta > 0$ be such that

$$B(\xi, \delta) \cap B(\phi^{(-1)}(\mathbf{y}), \delta) = \emptyset, \qquad B(\phi^{(-1)}(\mathbf{y}), \delta) \subseteq \Omega.$$

By injectivity of ϕ, we have

$$\phi(B(\xi, \delta) \cap \Omega) \cap \phi\big(B(\phi^{(-1)}(\mathbf{y}), \delta)\big) = \emptyset.$$

By domain invariance, both $\phi(B(\xi, \delta) \cap \Omega)$ and $\phi\big(B(\phi^{(-1)}(\mathbf{y}), \delta)\big)$ are open. Then

$$\phi\big(B(\phi^{(-1)}(\mathbf{y}), \delta)\big) \cap \text{cl}\,\phi(B(\xi, \delta) \cap \Omega) = \emptyset.$$

Let $\eta > 0$ be such that $B(\mathbf{y}, \eta) \subseteq \phi\big(B(\phi^{(-1)}(\mathbf{y}), \delta)\big)$. Hence,

(3.17) $\quad \bar{\mathbf{y}} \notin B(\mathbf{y}, \eta/2)$

for all $\bar{\mathbf{y}}$ such that $|\phi(\mathbf{x}) - \bar{\mathbf{y}}| < \eta/4$ for some $\mathbf{x} \in B(\xi, \delta) \cap \Omega$.

Now, let $j_0 \in \mathrm{N}$ be such that $|\phi_{j_0}^{(-1)}(y) - \xi| < \delta$, $\sup_{x \in D} |\phi_{j_0}(x) - \phi(x)| < \eta/4$. Then $|\phi(\phi_{j_0}^{(-1)}(y)) - y| = |\phi(\phi_{j_0}^{(-1)}(y)) - \phi_{j_0}(\phi_{j_0}^{(-1)}(y))| < \eta/4$ and consequently $y \notin B(y, \eta/2)$. A contradiction. ■

3.18. Lemma: *Let $\{x_j\}$ be a bounded sequence in a reflexive Banach space $\mathfrak{X}$, $x \in \mathfrak{X}$. Let $\mathfrak{Y}$ be a weakly dense subspace of the dual $\mathfrak{X}'$ of $\mathfrak{X}$. If*

$$\lim_j y(x_j) = y(x), \qquad \forall y \in \mathfrak{Y}, \tag{3.19}$$

then $\lim_j x'(x_j) = x'(x)$, $\forall x' \in \mathfrak{X}'$.

Proof: Assume by contradiction that there exists $u \in \mathfrak{X}'$, $\varepsilon > 0$ and a subsequence $\{x_{j_h}\}$ of $\{x_j\}$ such that

$$|u(x_{j_h}) - u(x)| > \varepsilon, \qquad \forall h \in \mathrm{N}. \tag{3.20}$$

Since $\{x_{j_h}\}$ is bounded and $\mathfrak{X}$ is reflexive, there exists a subsequence $\{x_{k_i}\}$ of $\{x_{j_h}\}$ and an element $\bar{x} \in \mathfrak{X}$ such that

$$\lim_i x'(x_{k_i}) = x'(\bar{x}), \qquad \forall x' \in \mathfrak{X}'. \tag{3.21}$$

In particular, $\lim_i y(x_{k_i}) = y(\bar{x})$, $\forall y \in \mathfrak{Y}$. Then assumption (3.19) implies that $y(\bar{x} - x) = 0$, $\forall y \in \mathfrak{Y}$. Since $\mathfrak{Y}$ is dense in $\mathfrak{X}'$, we conclude that $x = \bar{x}$. Hence, condition (3.21) contradicts (3.20). ■

From Lemma 3.18 and from a well-known result in functional analysis, (cf. e.g. Deimling (1980, Proposition 12.1, p. 112)), we deduce the following.

3.22. Proposition: *Let $\mathfrak{X}$ be a reflexive and locally uniformly convex Banach space. Let $\mathfrak{Y}$ be a weakly dense subspace of the dual $\mathfrak{X}'$ of $\mathfrak{X}$. Let $\{x_j\}$ be a sequence of $\mathfrak{X}$, $x \in \mathfrak{X}$. If*

$$\lim_j \|x_j : \mathfrak{X}\| = \|x : \mathfrak{X}\| ; \tag{3.23a}$$

$$\lim_j y(x_j) = y(x), \qquad \forall y \in \mathfrak{Y}. \tag{3.23b}$$

hold, then the sequence $\{x_j\}$ converges to x in $\mathfrak{X}$.

Note that Proposition 12.1 of Deimling (1980) also proves that a uniformly convex Banach space is locally uniformly convex and reflexive. Finally, it is well-known that the space $L^p(\omega)$, ω open subset of $\mathbf{R}^n$, $1 < p < \infty$, is uniformly convex (cf. Adams (1975, p. 38, Cor. 2.29)) and that $\mathfrak{D}(\omega)$ is dense in $L^p(\omega)$ and its strong dual, which can be identified with $L^{p/(p-1)}(\omega)$.

3.24. THEOREM: *Let Ω, Ω_1, be open subsets of* $\mathbb{R}^n$. *Let Ω be bounded and have the cone property. Let* $1<p<\infty$, $m\in\mathbb{N}$, $1<q<\infty$, $0<\gamma<\infty$, $\varepsilon>0$, $m>n/q$. *Let* $\{(f_i, g_i)\}$ *be a sequence converging to* (f, g) *in* $X_{m,p,q,\gamma,\varepsilon}(\Omega, \Omega_1)$ *and satisfying condition*

$$\lim_i \int_\Omega |G_i^{-1}(\mathbf{x})|^\gamma d\mathbf{x} = \int_\Omega |G^{-1}(\mathbf{x})|^\gamma d\mathbf{x} \qquad \text{if } 0<\gamma<\infty, \tag{3.25a}$$

$$G\in C^0(\Omega), \quad \lim_i G_i = G \quad \text{in } L^\infty(\Omega) \qquad \text{if } \gamma=\infty, \tag{3.25b, c}$$

and let $\tilde{r}(m, p, q, \gamma, n)\in(1, q]$. *Then, the following hold.*

(i) *If* $m-1<n/q$, *then* $\lim_i T[f_i, g_i] = T[f, g]$ *in* $W^{m,r}(\Omega)$ *for all* $1<r<\tilde{r}(m, p, q, \gamma, n)$. *If* $m=1$, *we can choose* $r=\tilde{r}(1, p, q, \gamma, n)$.

(ii) *If* $m-1>n/q$, *then* $\lim_i T[f_i, g_i] = T[f, g]$ *in* $W^{m,\tilde{r}(m,p,q,\gamma,n)}(\Omega)$.

PROOF: We only consider case $q<\infty$. Case $q=\infty$ can be treated similarly. We proceed by induction on m to prove (i). Let $m=1$. We must show that

$$\lim_i f_i\circ g_i = f\circ g, \qquad \text{in } L^{\tilde{r}(1,p,q,\gamma,n)}(\Omega), \tag{3.26a}$$

$$\lim_i T_k^1[f_i, g_i] = T_k^1[f, g] \qquad \text{in } L^{\tilde{r}(1,p,q,\gamma,n)}(\Omega). \tag{3.26b}$$

We only consider (3.26*b*). Indeed (3.26*a*) follows by the same argument. Since $\{\partial g_{ki}/\partial x_k\}_{i\in\mathbb{N}}$ converges to $\partial g_k/\partial x_k$ in $L^q(\Omega)$, and $p\gamma/(\gamma+1)\geq\tilde{r}(1, p, q, \gamma, n)>1$, Hölder inequality and Proposition 3.22 imply that (3.26*b*) can be deduced from

$$\lim_i \| T^k[f_i, g_i] : L^{p\gamma/(\gamma+1)}(\Omega)\| = \| T^k[f, g] : L^{p\gamma/(\gamma+1)}(\Omega)\|, \tag{3.27a}$$

$$\lim_i \int_\Omega T^k[f_i, g_i]\varphi(x)\, d\mathbf{x} = \int_\Omega T^k[f, g]\varphi(\mathbf{x})\, d\mathbf{x}, \qquad \forall\varphi\in\mathcal{D}(\Omega). \tag{3.27b}$$

We first consider $0<\gamma<\infty$. By the same argument used in the proof of Theorem 3.2, we deduce that

$$\left|\left\|\frac{\partial f_i}{\partial y_k}(g_i(\cdot)) : L^{p\gamma/(\gamma+1)}(\Omega)\right\| - \left\|\frac{\partial f}{\partial y_k}(g(\cdot)) : L^{p\gamma/(\gamma+1)}(\Omega)\right\|\right| \leq \tag{3.28}$$

$$\leq \left\|\frac{\partial f_i}{\partial y_k} - \frac{\partial f}{\partial y_k} : L^p(\Omega_1)\right\| \| G_i^{-1} : L^\gamma(\Omega)\|^{1/p} +$$

$$+ \left\|\frac{\partial f}{\partial y_k} : L^p(\Omega_1)\right\| \left\{\int_{\Omega_1} \left||G_i^{-1}(g_i^{(-1)}(\mathbf{y}))|^{-(\gamma+1)/p\gamma} - |G^{-1}(g^{(-1)}(\mathbf{y}))|^{-(\gamma+1)/p\gamma}\right|^{p\gamma} d\mathbf{y}\right\}^{1/p\gamma}.$$

(note that $p\gamma > p\gamma/(\gamma+1) > r(1, p, q, \gamma, n) > 1$.) Since the sequence $\{\partial f_i/\partial y_i\}_{i\in N}$ converges to $\partial f/\partial y_i$ in $L^p(\Omega_1)$, and $\sup_i \|G_i^{-1} L^\gamma(\Omega)\| < \infty$, condition (3.27$a$) follows from

$$(3.29) \qquad \lim_i |G_i^{-1}(g_i^{(-1)}(y))|^{(\gamma+1)/p\gamma} = |G^{-1}(g^{(-1)}(y))|^{(\gamma+1)/p\gamma} \qquad \text{in } L^{p\gamma}(\Omega_1)\,,$$

which is clearly equivalent to

$$(3.30) \qquad \lim_i |G_i^{-1}(g_i^{(-1)}(y))| = |G^{-1}(g^{(-1)}(y))| \qquad \text{in } L^{\gamma+1}(\Omega_1)\,,$$

which we now turn to prove. (Note that the corresponding argument does not apply if $\gamma = \infty$.) By Proposition 3.22 and condition $\gamma + 1 > 1$, it suffices to show that

$$(3.31a) \qquad \lim_i \int_{\Omega_1} |G_i^{-1}(g_i^{(-1)}(y))|^{\gamma+1} dy = \int_{\Omega_1} |G^{-1}(g^{(-1)}(y))|^{\gamma+1} dy\,,$$

$$(3.31b) \qquad \lim_i \int_{\Omega_1} |G_i^{-1}(g_i^{(-1)}(y))|\psi(y)\, dy = \int_{\Omega_1} |G^{-1}(g^{(-1)}(y))|\psi(y)\, dy\,, \qquad \forall \psi \in \mathfrak{D}(\Omega_1)\,.$$

Since

$$\int_{\Omega_1} |G_i^{-1}(g_i^{(-1)}(y))|^{\gamma+1} dy = \int_{\Omega} |G_i^{-1}|\, dx\,,$$

condition (3.31a) follows from assumption (3.25a). Now, note that

$$\int_{\Omega_1} |G_i^{-1}(g_i^{(-1)}(y))|\psi(y)\, dy = \int_{\Omega} \psi(g_i(x))\, dx\,.$$

Since $\{g_i\}$ converges to g in $W^{1,q}(\Omega)$, $q > n$, then $\{g_i\}$ converges to g uniformly in Ω and consequently the membership of $\psi \in \mathfrak{D}(\Omega)$ implies that (3.31b) holds. This concludes the proof of (3.27a). We now consider (3.27b). By Theorem 2.1, (3.27b) follows from the following condition

$$(3.32) \qquad \lim_i \frac{\partial f_i}{\partial y_i}(\cdot)\phi(g_i^{(-1)}(\cdot))|G_i^{-1}(g_i^{(-1)}(\cdot))| = \\ = \frac{\partial f}{\partial y_i}(\cdot)\phi(g^{(-1)}(\cdot))|G^{-1}(g^{(-1)}(\cdot))| \qquad \text{in } L^1(\Omega_1)\,.$$

Since pointwise multiplication is continuous from $L^p(\Omega_1) \times L^{p(\gamma+1)/(\gamma(p-1)-1)}(\Omega_1)$ to $L^{(\gamma+1)/\gamma}(\Omega_1)$ and from $L^{(\gamma+1)/\gamma}(\Omega_1) \times L^{\gamma+1}(\Omega_1)$ to $L^1(\Omega)$, and since

$$\lim_i \partial f_i/\partial y_i = \partial f/\partial y_i \qquad \text{in } L^p(\Omega_1)\,,$$

$$\lim_i |G_i^{-1}(g_i^{(-1)}(\cdot))| = |G^{-1}(g^{(-1)}(\cdot))| \qquad \text{in } L^{\gamma+1}(\Omega_1)\,,$$

all we need to show is that

$$\lim_j \phi(g_j^{(-1)}(\cdot)) = \phi(g^{(-1)}(\cdot)) \qquad \text{in } L^{(p(\gamma+1))/(\gamma(p-1)-1)}(\Omega_1). \tag{3.33}$$

By Proposition 3.22, condition (3.33) follows from

$$\lim_j \int_{\Omega_1} |\phi(g_j^{(-1)}(y))|^{(p(\gamma+1))/(\gamma(p-1)-1)}\,dy = \int_{\Omega_1} |\phi(g^{(-1)}(y))|^{(p(\gamma+1))/(\gamma(p-1)-1)}\,dy, \tag{3.34a}$$

$$\lim_j \int_{\Omega_1} \phi(g_j^{(-1)}(y))\,\psi(y)\,dy = \int_{\Omega_1} \phi(g^{(-1)}(y))\,\psi(y)\,dy \qquad \forall \psi \in \mathfrak{D}(\Omega_1). \tag{3.34b}$$

By Theorem 2.1, we have

$$\int_{\Omega_1} |\phi(g_j^{(-1)}(y))|^{(p(\gamma+1))/(\gamma(p-1)-1)}\,dy = \int_{\Omega} |\phi(x)|^{(p(\gamma+1))/(\gamma(p-1)-1)}\,|G_j(x)|\,dx.$$

Since $\lim_j \partial g_{ji}/\partial x_k = \partial g_i/\partial x_k$ in $L^n(\Omega)$, then $\lim_j G_j = G$ in $L^{q/n}(\Omega)$, which is imbedded in $L^1(\Omega)$. Then (3.34a) follows by $\psi \in \mathfrak{D}(\Omega_1)$. Since $\lim_j G_j = G$ in $L^1(\Omega)$, and $\lim_j g_j = g$ uniformly in Ω, and $\psi \in \mathfrak{D}(\Omega_1)$, we conclude that the sequence $\{\phi(x)\psi(g_j(x))|G_j(x)|\}_{j\in\mathbb{N}}$ converges to $\phi(x)\psi(g(x))|G(x)|$ in $L^1(\Omega)$. Then (3.34b) follows by Theorem 2.1 and the proof in case $m = 1$, $0 < \gamma < \infty$ is complete. We now consider $\gamma = \infty$. As above, we must prove (3.27a), (3.27b). To prove (3.27a) with $\gamma = \infty$, we obtain as above the following inequality

$$\begin{aligned} \left\| \left\| \frac{\partial f_j}{\partial y_i}(g_j(\cdot)) : L^p(\Omega) \right\| - \left\| \frac{\partial f}{\partial y_i}(g(\cdot)) : L^p(\Omega) \right\| \right\| &< \\ < \left\| \frac{\partial f_j}{\partial y_i} - \frac{\partial f}{\partial y_i} : L^p(\Omega_1) \right\| \, \|G_j^{-1} : L^\infty(\Omega)\|^{1/p} &+ \\ + \left\{ \int_{\Omega_1} \left| \frac{\partial f}{\partial y_i} \right|^p \left| |G^{-1}(g_j^{(-1)}(y))|^{1/p} - |G^{-1}(g^{(-1)}(y))|^{1/p} \right|^p dy \right\}^{1/p}&. \end{aligned} \tag{3.35}$$

Since $\lim_j \partial f_j/\partial y_i = \partial f/\partial y_i$ in $L^p(\Omega_1)$, $\sup_j \|G_j^{-1} : L^\infty(\Omega)\| < \infty$, (3.27a) can be deduced from the Lebesgue Dominated Convergence and from the following

$$\lim_j |G_j^{-1}(g_j^{(-1)}(y))| = |G^{-1}(g^{(-1)}(y))| \qquad \text{a.e. in } \Omega_1, \tag{3.36}$$

which we now turn to prove. Note that

$$\begin{aligned} \left| |G^{-1}(g^{(-1)}(y))| - G_j^{-1}(g_j^{(-1)}(y)) \right| &< \left| |G^{-1}(g^{(-1)}(y))| - G^{-1}(g_j^{(-1)}(y)) \right| + \\ &+ \left| |G^{-1}(g_j^{(-1)}(y))| - |G_j^{-1}(g_j^{(-1)}(y))| \right|. \end{aligned} \tag{3.37}$$

By assumption, there exists a subset $N \subset \Omega$ of measure zero such that $G(\mathbf{x}) \neq 0$, $\forall \mathbf{x} \in \Omega \setminus N$. By assumption (3.25*b*), we conclude that $G^{-1} \in C^0(\Omega \setminus N)$. By virtue of Marcus and Mizel (1973, Corollary 1, p. 791), the set $M = g(N) \cup \cup \left(\bigcup_i g_i(N)\right)$ has measure zero. Then we have

$$(3.38) \qquad \lim_i \left| |G^{-1}(g^{(-1)}(\mathbf{y}))| - |G^{-1}(g_i^{(-1)}(\mathbf{y}))| \right| = 0\,, \qquad \forall \mathbf{y} \in \Omega_1 \setminus M\,.$$

and we can deduce (3.36) by using condition (3.25*c*) and the following inequality.

$$(3.39) \qquad \limsup_i \operatorname*{ess\,sup}_{\mathbf{y} \in \Omega_1} \left| |G^{-1}(g_i^{(-1)}(\mathbf{y}))| - |G_i^{-1}(g_i^{(-1)}(\mathbf{y}))| \right| =$$
$$= \limsup_i \operatorname*{ess\,sup}_{\mathbf{x} \in \Omega} \left| |G_i^{-1}(\mathbf{x})| - |G^{-1}(\mathbf{x})| \right| <$$
$$< \limsup_i \left\{ \|G_i^{-1} : L^\infty(\Omega)\| \, \|G^{-1} : L^\infty(\Omega)\| \, \|G_i - G : L^\infty(\Omega)\| \right\} = 0\,.$$

We now examine (3.27*b*), which we reduce to (3.32) as above. Since $\lim_i \partial f_i / \partial y_1 = \partial f / \partial y_1$ in $L^p(\Omega_1)$, it suffices to prove that

$$(3.40) \quad \lim_i \phi(g_i^{(-1)}(\cdot))|G_i^{-1}(g_i^{(-1)}(\cdot))| = \phi(g^{(-1)}(\cdot))|G^{-1}(g^{(-1)}(\cdot)) \quad \text{in } L^{p/(p-1)}(\Omega_1).$$

By Proposition 3.22 the convergence in (3.40) is a consequence of the following two conditions

$$(3.41a) \qquad \lim_i \left(\int_{\Omega_1} |\phi(g_i^{(-1)}(\mathbf{y}))|^{p/(p-1)} |G_i^{-1}(g_i^{(-1)}(\mathbf{y}))|^{p/(p-1)} d\mathbf{y} \right)^{(p-1)/p} =$$
$$= \left(\int_{\Omega_1} |\phi(g^{(-1)}(\mathbf{y}))|^{p/(p-1)} |G^{-1}(g^{(-1)}(\mathbf{y}))|^{p/(p-1)} d\mathbf{y} \right)^{(p-1)/p},$$

$$(3.41b) \qquad \lim_i \int_{\Omega_1} \phi(g_i^{(-1)}(\mathbf{y})) |G_i^{-1}(g_i^{(-1)}(\mathbf{y}))| \eta(\mathbf{y})\, d\mathbf{y} =$$
$$= \int_{\Omega_1} \phi(g^{(-1)}(\mathbf{y})) |G^{-1}(g^{(-1)}(\mathbf{y}))| \eta(\mathbf{y})\, d\mathbf{y}\,, \qquad \forall \eta \in \mathfrak{D}(\Omega_1)\,.$$

To prove (3.41*a*) we observe that

$$(3.42) \qquad \Bigg| \left(\int_{\Omega_1} |\phi(g_i^{(-1)}(\mathbf{y}))|^{p/(p-1)} |G_i^{-1}(g_i^{(-1)}(\mathbf{y}))|^{p/(p-1)} d\mathbf{y} \right)^{(p-1)/p} -$$
$$- \left(\int_{\Omega_1} |\phi(g^{(-1)}(\mathbf{y}))|^{p/(p-1)} |G^{-1}(g^{(-1)}(\mathbf{y}))|^{p/(p-1)} d\mathbf{y} \right)^{(p-1)/p} \Bigg| <$$
$$< \left(\int_{\Omega} |\phi(\xi)|^{p/(p-1)} \left| |G_i^{-1}|^{1/p} - |G^{-1}|^{1/p} \right|^{p/(p-1)} d\xi \right)^{(p-1)/p} <$$
$$< \varepsilon^{2/p} \left(\int_{\Omega} |\phi(\xi)|^{p/(p-1)} \left| |G_i|^{1/p} - |G|^{1/p} \right|^{p/(p-1)} d\xi \right)^{(p-1)/p}.$$

Since $\varphi \in \mathfrak{D}(\Omega)$, $\lim_j G_j = G$ in $L^\infty(\Omega)$ it follows that (3.41*a*) holds. Since $\int_{\Omega_1} \phi(g_j^{(-1)}(\mathbf{y}))|G_j^{-1}(g_j^{(-1)}(\mathbf{y}))|\eta(\mathbf{y})\,d\mathbf{y} = \int_{\Omega} \phi(\xi)\eta(g_j(\xi))\,d\xi$, and $\lim_j g_j = g$ uniformly in Ω, we conclude that (3.41*b*) holds. The inductive arguments needed to complete the proof are completely analogous to those used to prove the boundedness of T in Theorem 3.2 and are accordingly omitted. ■

Remark: We note that in general, even in a one-dimensional setting, convergence of a sequence of functions $\{f_j\}$ in, say $L^p(0, 1)$ to f does not imply the convergence of the sequence $\{1/f_j\}$ of reciprocals to $1/f$, even though $\sup \|1/f_j : L^p(0, 1)\| < \infty$, as the following counterexample shows. Let $f_j(x) = (j+1)^{-1}$, if $x \in (0, 1/j^p)$, $f_j(x) = 1$ if $x \in (1/j^p, 1)$, $f = 1$ if $x \in (0, 1)$. Clearly $\lim_j f_j = f$ in $L^p(0, 1)$, $\sup \|1/f_j : L^p(0, 1)\| < \infty$, but $\|1/f_j - 1/f : L^p(0, 1)\| = 1$.

A straightforward application of Hölder inequality yields the following.

3.43. Lemma: *Let* $1 < s \leq \infty$, $0 < \gamma < \infty$. *Let* Ω *be an open subset of* $\mathbb{R}^n$. *Let* $f, f_j \in L^s(\Omega)$, $f, f_j \neq 0$, *a.e. in* Ω. *Then the following holds.*

$$(3.44) \qquad \int_\Omega |f_j^{-1}(\mathbf{x}) - f^{-1}(\mathbf{x})|^{\gamma/(2+(\gamma/s))}\,d\mathbf{x} \leq$$

$$\left(\int_\Omega |f_j^{-1}(\mathbf{x})|^\gamma\,d\mathbf{x}\right)^{1/(2+(\gamma/s))}\left(\int_\Omega |f^{-1}(\mathbf{x})|^\gamma\,d\mathbf{x}\right)^{1/(2+(\gamma/s))} \|f - f_j : L^s(\Omega)\|^{\gamma/(2+(\gamma/s))}.$$

We now deduce the following from Theorem 3.2 and Theorem 3.24.

3.45. Theorem: *Let* Ω, Ω_1 *be open subsets of* $\mathbb{R}^n$. *Let* Ω *be bounded and have the cone property. Let* $1 < p < \infty$, $m \in \mathbb{N}$, $1 < q < \infty$, $0 < \gamma < \infty$, $0 < \varepsilon < \infty$, $m > n/q$. *Let*

$$(3.46) \qquad \bar{r}(m, p, q, \gamma, n) = \begin{cases} \dfrac{pqn}{(pm+n)[n-(m-1)q] + nq + 2n(q/\gamma)} & \text{if } (m-1) < n/q, \\[2ex] \dfrac{p}{1+(2/\gamma)} & \text{if } (m-1) > n/q. \end{cases}$$

and let $\bar{r}(m, p, q, \gamma, n) \in (1, q]$. Then

(i) *If* $(m-1) < n/q$, *then* T *is continuous from* $X_{m,p,q,\gamma,\varepsilon}(\Omega, \Omega_1)$ *to* $W^{m,r}(\Omega)$, *for all* $1 < r < \bar{r}(m, p, q, \gamma, n)$. (*If* $m = 1$, *we can choose* $r = \bar{r}(1, p, q, \gamma, n)$.)

(ii) *If* $(m-1) > n/q$, *then* T *is continuous from* $X_{m,p,q,\gamma,\varepsilon}(\Omega, \Omega_1)$ *to* $W^{m,\bar{r}(m,p,q,\gamma,n)}(\Omega)$.

Proof: Let $\{(f_i, g_i)\}_{i\in\mathbb{N}}$ be a sequence in $X_{m,p,q,\gamma,c}(\Omega, \Omega_1)$. If $(m-1) < n/q$, then $W^{m,q}(\Omega) \subset W^{1,(nq/(n-(m-1)q))}(\Omega)$. Hence, by Hölder inequality,

$$\lim_i G_i = G \qquad \text{in } L^{nq/(n-(m-1)q)}(\Omega)\,.$$

Then, by applying Lemma 3.43, we deduce that

$$\lim_i \int_\Omega \big||G_i^{-1}| - |G^{-1}|\big|^{nq\gamma/(\gamma(n-(m-1)q)+2n)}\, dx = 0\,. \tag{3.47}$$

By considering separately case

$$0 < \frac{q\gamma}{\gamma\{n-(m-1)q\}+2q} < 1 \quad \text{and} \quad \frac{q\gamma}{\gamma\{n-(m-1)q\}+2q} > 1\,,$$

it is easy to see that (3.47) implies

$$\lim_i \int_\Omega |G_i^{-1}|^{nq\gamma/(\gamma(n-(m-1)q)+2n)}\, dx = \int_\Omega |G^{-1}|^{nq\gamma/(\gamma(n-(m-1)q)+2n)}\, dx\,. \tag{3.48}$$

and that

$$\frac{q\gamma}{\gamma\{n-(m-1)q\}+2q} < \gamma\,.$$

Hence, by Theorem 3.45 T is continuous from $X_{m,p,q,\gamma,c}(\Omega, \Omega_1)$ to $W^{m,r}(\Omega)$, for all

$$1 < r < \hat{r}\left(m, p, q, \frac{q\gamma}{\gamma\{n-(m-1)q\}+2q}, n\right) = \tilde{r}(m, p, q, \gamma, n)\,.$$

If $(m-1) > n/q$, then $W^{m,q}(\Omega) \subset W^{1,\infty}(\Omega)$ and similarly, we deduce that

$$\lim_i \int_\Omega |G_i^{-1}|^{\gamma/2}\, dx = \int_\Omega |G^{-1}|^{\gamma/2}\, dx\,. \tag{3.49}$$

Then by Theorem 3.45 T is continuous from $X_{m,p,q,\gamma,c}(\Omega, \Omega_1)$ to

$$W^{m,\hat{r}(m,p,q,\gamma/2,n)}(\Omega) = W^{m,\tilde{r}(m,p,q,\gamma,n)}(\Omega)\,.$$

If $(m-1) = n/q$, then $W^{m,q}(\Omega) \subset W^{1,s}(\Omega)$, $\forall s > q$ and we deduce

$$\lim_i \int_\Omega |G_i^{-1}|^{s\gamma/(\gamma n+2s)}\, dx = \int_\Omega |G^{-1}|^{s\gamma/(\gamma n+2s)}\, dx\,, \qquad \forall s > q\,. \tag{3.50}$$

Then by Theorem 3.45 T is continuous from $X_{m,p,q,\gamma,c}(\Omega, \Omega_1)$ to $W^{m,r}(\Omega)$ for all $1 < r < \hat{r}(m, p, q, \gamma s/(\gamma n + 2s), n)$. By arbitrariness of $s > q$, r can be taken in $(1, \hat{r}(m, p, q, \gamma/2, n))$. This completes the proof. ∎

3.51. THEOREM: *Let Ω, Ω_1 be open subsets of* $\mathbb{R}^n$. *Let Ω be bounded and have the cone property. Let* $1 < p < \infty$, $m \in \mathbb{N}$, $1 < q < \infty$, $m > n/q$. *If* $(m-1) > n/q$, *then T is continuous from* $X_{m,p,q,\infty,\infty}(\Omega, \Omega_1)$ *to* $W^{m,p}(\Omega)$. *In particular, it si continuous on* $X_{m,p,q,\infty,\varepsilon}(\Omega, \Omega_1)$ *for all* $\varepsilon > 0$.

PROOF: Observe that if $\lim_i (f_i, g_i) = (f, g)$ in $X_{m,p,q,\infty,\infty}(\Omega, \Omega_1)$, then $\lim_i G_i = G$ in $W^{m-1,q}(\Omega)$ and consequently (3.25*b*), (3.25*c*) hold. Moreover we notice that if $G^{-1} \in L^\infty(\Omega)$, $\lim_i G_i = G$ in $L^\infty(\Omega)$, then

$$\sup_i \| G_i^{-1} : L^\infty(\Omega) \| < \infty ,$$

and Theorem 3.24 applies. ■

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