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MARINO DE LUCA (*)

# Stability Analysis for Variational Inequalities and Applications to Equilibrium Problems (**)

SUMMARY. — We perform stability analysis for a variational inequality characterizing the solution of the traffic equilibrium problem in the continuous case. In particular, we study continuous dependence of the optimal flow upon changes in costs and travel demands.

## Analisi di stabilità per diseguaglianze variazionali. Applicazioni a problemi di equilibrio

SOMMARIO. — Si effettua un'analisi di stabilità per una disequazione variazionale che caratterizza la soluzione del problema dell'equilibrio del traffico nel caso continuo. In particolare, si studia la dipendenza continua del flusso ottimale rispetto a variazioni nei costi e nelle domande di traffico.

## INTRODUCTION

In a recent paper, (see [1]), S. Dafermos and A. Nagurney perform stability and sensitivity analysis for the traffic equilibrium problem in the discrete case where the « user-optimizing » equilibrium pattern must be evaluated for a network with an assigned travel demand for every $O/D$ pair of nodes and travel costs which depend on the traffic flow.

Assuming a monotonicity condition on the cost function, they show that the equilibrium pattern depends continuously upon the assigned travel demand and travel costs and focus on the delicate question of the control and the forecast of the changes in the traffic pattern and the incurred travel costs, resulting from changes in the travel cost function and the travel demand. The stability analysis effected in [1] crucially depends on the well-known result that the equilibrium pattern can be expressed as solution of a variational inequality (see, e.g., [2], [3]).

(*) Indirizzo dell'Autore: Istituto di Tecnologia, Università, via Diana 4, 89100 Reggio Calabria.
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The purpose of this paper is to study the same question for a continuous model with a linear cost function where the equilibrium pattern is the « system-optimizing » pattern flow, expressed as the flow which minimize the global cost spent in the network. Also in this case, the stability analysis meets with success because the optimal flow can be expressed as solution of a variational inequality and, in some case, as solution of a non-standard Dirichlet problem (see [4] and [5]).

We think that the results of this work can be useful also for study the traffic paradoxes, like those exhibited by D. Braess, (see [6]), and by C. Fisk (see [7]), in the discrete case. We illustrate this situation by means of an example (see Sec. 2).

## 1. - Hypotheses and preliminaries

Let us denote by $K$ the convex:

$$K=\{u(x)=(u_1(x),u_2(x))\in H^1(\Omega,\mathbb{R}^2)|u(x)\geq 0,\ \forall x\in\Omega,\ u(x)|_{\partial\Omega}=\varphi(x),\ \operatorname{div}u+t(x)=0,\ \forall x\in\Omega\}, \tag{1.1}$$

and by $c(x,u(x))$ the « total cost » spent at the point $x\in\Omega$:

$$c(x,u(x))=(Au|u)+2(f|u)\ (^1). \tag{1.2}$$

Then, the system-optimizing pattern flow $u^0$, can be expressed as solution of the variational inequality:

$$\int_\Omega\{(Au^0|u-u^0)+(f|u-u^0)\}\,dx\geq 0, \qquad \forall u\in K^{L^2(\Omega)}. \tag{1.3}$$

In (1.1) $u_i$, $(i=1,2)$, denotes the traffic flow density in the $x_i$-direction, (for the sake of simplicity, we assume here that the traffic in the network flows only in the positive direction of two orthogonal axes), $\Omega$ is the bounded open set of $\mathbb{R}^2$ interested by the flow, $\varphi=(\varphi_1,\varphi_2)\in H^{\frac{1}{2}}(\partial\Omega,\mathbb{R}^2)$ is the prescribed travel demand on $\partial\Omega$ in the assigned directions, $t\in L^2(\Omega)$ denotes the density of the flow originating or terminating at $x\in\Omega$. In the following, we shall refer to the couple $(\varphi,t)$ as the « travel demand ».

In (1.2) $A$ represents the matrix $(a_{ij}(x))$, $(a_{ij}=a_{ji},\ i,j=1,2)$, $f$ is the vector $f_i(x)$, $(i=1,2)$ and $(\cdot|\cdot)$ is the scalar product in $\mathbb{R}^2$; finally, in (1.3) $K^{L^2(\Omega)}$ denotes the closure of the convex $K$ in $L^2(\Omega)$, (for more details on the relations and on the above mentioned results, see [4] and [5]; the set $K^{L^2(\Omega)}$

(1) We remember that the total cost (1.2) can be expressed by the « personal cost » $c_i(x,u(x))$ in the $x_i$-direction $(i=1,2)$, defined, in the linear case, as $c_i(x,u(x))=a_{i1}(x)u_1(x)+a_{i2}(x)u_2(x)+2f_i(x)$.

occurs in several questions related to decomposition problems for solenoidal vector fields, see e.g. [8], [9]).

In Section 2 we study the change in the cost functions and we show as the coerciveness hypothesis:

$$(1.4)\qquad (Au^1 - Au^2 | u^1 - u^2) \geq \nu \| u^1 - u^2 \|^2 , \qquad \forall u^1 , u^2 \in K^{L^2(\Omega)} , \ \nu \in \mathbf{R}^+ ,$$

guarantees the continuous dependence of the solution by the cost. In particular, by means of an example, we show that to an increase of the cost along one of two directions, corresponds a « weighted average increase » of opposite sign for the optimal flow in the same direction.

In Section 3, we are concerned with the stability analysis for changes in the travel demand; we use the results of [4] and we confine ourself to consider the particular case where $\Omega$ is the rectangle:

$$\Omega = ]0, a_1[ \times ]0, a_2[ .$$

Moreover, to avoid inessential mathematical complications, we assume $t(x)$ identically zero in $\Omega$.

As shown in [4], the system-optimizing equilibrium pattern flow, $u^0$, can be obtained, under suitable hypotheses, setting:

$$(1.5)\qquad u_1^0 = \frac{\partial U_0}{\partial x_2} , \qquad u_2^0 = -\frac{\partial U_0}{\partial x_1}$$

where $U^0 \in H^2(\Omega)$ is the solution of the Dirichlet problem:

$$(1.6)\qquad \left| \begin{array}{l} L(U^0) = a_{22} \dfrac{\partial^2 U^0}{\partial x_1^2} + a_{11} \dfrac{\partial^2 U^0}{\partial x_2^2} - 2a_{12} \dfrac{\partial^2 U^0}{\partial x_1 \partial x_2} - \\ \qquad - \left( \dfrac{\partial a_{12}}{\partial x_2} - \dfrac{\partial a_{22}}{\partial x_1} \right) \dfrac{\partial U^0}{\partial x_1} - \left( \dfrac{\partial a_{21}}{\partial x_1} - \dfrac{\partial a_{11}}{\partial x_2} \right) \dfrac{\partial U^0}{\partial x_2} = - \dfrac{\partial f_1}{\partial x_2} + \dfrac{\partial f_2}{\partial x_1} , \\ U^0(0, x_2) = \Phi_1(x_2) , \qquad U^0(a, x_2) = \Psi_1(x_2) , \qquad x_2 \in ]0, a_2[ , \\ U^0(x_1, 0) = \Phi_2(x_1) , \qquad U^0(x_1, b) = \Psi_2(x_1) , \qquad x_1 \in ]0, a_1[ , \end{array} \right.$$

provided that

$$(1.7)\qquad \frac{\partial U^0}{\partial x_2} > 0 , \qquad \frac{\partial U_0}{\partial x_1} < 0 \qquad \text{in } \Omega .$$

In (1.6) the traces are given by:

$$(1.8)\qquad \left| \begin{array}{lll} \Phi_1(x_2) = \displaystyle\int_0^{x_2} \varphi_1(t)\, dt , & \Psi_1(x_2) = \displaystyle\int_0^{x_2} \psi_1(t)\, dt , & x_2 \in ]0, a_2[ , \\ \Phi_2(x_1) = \displaystyle\int_0^{x_1} \varphi_2(t)\, dt , & \Psi_2(x_1) = \displaystyle\int_0^{x_1} \psi_2(t)\, dt , & x_1 \in ]0, a_1[ , \end{array} \right.$$

where $\varphi_1, \psi_1, \varphi_2, \psi_2$ are the normal components of the trace $\varphi$ on the sides of $\Omega$.

Analogous results can be obtained, with suitable modifications, for $t(x)$ not identically zero in $\Omega$, and hence for the case in which the changes in the travel demand also interest internal points of the network.

## 2. - Stability with respect to the cost

We assume that the travel demand $(\varphi, t)$ does not changes, whereas the cost changes from $c$ to $c^*$, with:

$$c(x, u(x)) = (Au|u) + 2(f|u), \qquad c^*(x, u(x)) = (A^*u|u) + 2(f^*|u),$$

for $u \in K^{L^2(\Omega)}$. We intend to compare the corresponding equilibrium patterns $u^0$ and $u^{0*}$; to this end we make, besides (1.4), the following hypotheses:

*a*) $a_{ij}, a^*_{ij} \in L^\infty(\Omega)$, $(a_{ij} = a_{ji},\ a^*_{ij} = a^*_{ji},\ i, j = 1, 2)$;

*b*) $f_i, f^*_i \in L^2(\Omega)$, $(i = 1, 2)$.

Moreover, we suppose that the solution $u^{0*}$ of the variational inequality (1.3), corresponding to the cost $c^*$, exists (the existence of $u^{0*}$ is guaranteed, for example, if we assume coerciveness also for $A^*$).

Then, we have the following:

Theorem 2.1: Under the above mentioned assumptions, there exists $\bar{\nu} \in \mathbb{R}_+$ such that:

$$\|u^0 - u^{0*}\|_{L^2(\Omega)} \leq \bar{\nu}\{\|(A - A^*)u^{0*}\|_{L^2(\Omega)} + \|f - f^*\|_{L^2(\Omega)}\}. \tag{2.1}$$

Proof: By (1.3), we have:

$$\int_\Omega \{(A^*u^0|u - u^0) + (f|u - u^0)\}\, dx \geq 0, \qquad \forall u \in K^{L^2(\Omega)}, \tag{2.2}$$

$$\int_\Omega \{(A^*u^{0*}|u - u^{0*}) + (f^*|u - u^{0*})\}\, dx \geq 0, \qquad \forall u \in K^{L^2(\Omega)}. \tag{2.3}$$

Adding (2.2) (with $u = u^{0*}$) to (2.3) (with $u = u^0$) we have:

$$\int_\Omega \{(Au^0 - A^*u^{0*}|u^{0*} - u^0) + (f - f^*|u^{0*} - u)\}\, dx \geq 0, \tag{2.4}$$

and then

$$\int_\Omega \{(Au^0 - Au^{0*}|u^0 - u^{0*})\, dx \leq \tag{2.5}$$
$$\leq \int_\Omega \{(Au^{0*} - A^*u^{0*}|u^{0*} - u^0) + (f - f^*|u^{0*} - u^0)\}\, dx.$$

So, by (1.4), we have:

$$\nu\int_{\Omega}\|u^0-u^{0*}\|^2\,dx<\int_{\Omega}\Big\{(Au^0-Au^{0*}|u^0-u^{0*})\,dx<$$

$$<\Big(\int_{\Omega}\|(Au^{0*}-A^*u^{0*})+(f-f^*)\|^2\,dx\Big)^{\frac12}\cdot\Big(\int_{\Omega}\|u^{0*}-u^0\|^2\,dx\Big)^{\frac12}\Big\}.$$

Hence:

$$\Big(\int_{\Omega}\|u^0-u^{0*}\|^2\,dx\Big)^{\frac12}<\frac{1}{\nu}\Big\{\Big(\int_{\Omega}\|Au^{0*}-A^*u^{0*}\|^2\,dx\Big)^{\frac12}+\Big(\int_{\Omega}\|(f-f^*)\|^2\,dx\Big)^{\frac12}\Big\}$$

i.e. the thesis, with $\vartheta=1/\nu$.

Another type of result is the following:

THEOREM 2.2: Under the above mentioned assumption, we have:

$$\int_{\Omega}\{(A^*u^{0*}-Au^{0*}|u^{0*}-u^0)+(f^*-f|u^{0*}-u^0)\}\,dx<0\,. \tag{2.6}$$

PROOF: By (2.4):

$$0<\int_{\Omega}\{(Au^0-Au^{0*}+Au^{0*}-A^*u^{0*}|u^{0*}-u^0)+(f-f^*|u^{0*}-u^0)\}\,dx=$$

$$=\int_{\Omega}\{(Au^0-Au^{0*}|u^{0*}-u^0)+(Au^{0*}-A^*u^{0*}|u^{0*}-u^0)+(f-f^*|u^{0*}-u^0)\}\,dx\,.$$

So, by means of the coerciveness of $A$, one obtains:

$$0\leq\int_{\Omega}(Au^{0*}-Au^0|u^{0*}-u^0)\,dx\leq$$

$$\leq\int_{\Omega}\{(Au^{0*}-A^*u^{0*}|u^{0*}-u^0)+(f-f^*)|u^{0*}-u^0)\}\,dx\,.$$

This theorem allows us, for example, to prove that if one improves or worsens the cost along a direction, the average flow along the same direction, suitably weighted, increases or decreases correspondingly.

Indeed, let us assume:

$$A^*=A+\begin{pmatrix}\lambda(x) & 0\\ 0 & 0\end{pmatrix},\qquad f^*=f \tag{2.7}$$

with $\lambda(x)\neq0$ for each $x\in\Omega$.

The cost $c^*$ corresponding to the matrix $A^*$ presents here an improvement or worsening along the $x_1$-direction with respect to the cost $c$ of $A$ (see Note (¹)), in accord with the sign of $\lambda(x)$, that we assume constant in $\Omega$.

In this case (2.6) becomes:

$$\int_{\Omega} \lambda u_1^{0*}(u_1^{0*} - u_1^0)\, dx \le 0 \tag{2.8}$$

and so, if $u_1^{0*} \neq 0$, a.e. in $\Omega$:

$$\frac{\int_{\Omega} \lambda u_1^{0*}(u_1^{0*} - u_1^0)\, dx}{\int_{\Omega} \lambda u_1^{0*}\, dx}\ \mathrm{sgn}\,(\lambda) \le 0\,, \tag{2.9}$$

where one has an average evaluation of the change of the equilibrium pattern along the $x_1$-direction.

## 3. - Stability with respect to the travel demand

In this Section we assume

$$\Omega = ]0, a_1[ \times ]0, a_2[$$

and we denote the new travel demand on $\partial\Omega$ by $\varphi_1^*, \psi_1^*, \varphi_2^*, \psi_2^*$. Moreover, we suppose that $t(x)$ does not change in $\Omega$ and hence, without loss of generality, we can assume $t(x)$ identically zero in $\Omega$.

Besides the hypotheses mentioned in Sect. 1, we assume:

$$\left|\begin{array}{ll} a_{ij} \in C^1(\bar\Omega) & i, j = 1, 2\,, \\ f_i \in C^1(\bar\Omega) & i = 1, 2\,, \\ \varphi_i, \psi_i, \varphi_i^*, \psi_i^* \in C^1(\partial\Omega) & i = 1, 2\,, \end{array}\right. \tag{3.1}$$

and we suppose that the equilibrium patterns $u^0$ and $u^{0*}$, corresponding, respectively, to the old and to the new travel demands, can be obtained from the potentials $U^0$ and $U^{0*}$, solutions of the respective Dirichlet problems of type (1.6)-(1.8).

Let us denote by $W^0$ the solution of the problem

$$\left|\begin{array}{ll} \Delta W^0 = 0 \qquad \text{in } \Omega\,, & \\ W^0(x_1, 0) = \int_0^{x_1} \{\varphi_2(t) - \varphi_2^*(t)\}\, dt & \\ & x_1 \in ]0, a_1[\,, \\ W^0(x_1, a_2) = \int_0^{x_1} \{\psi_2(t) - \psi_2^*(t)\}\, dt & \\ W^0(0, x_2) = \int_0^{x_2} \{\varphi_1(t) - \varphi_1^*(t)\}\, dt & \\ & x_2 \in ]0, a_2[\,, \\ W^0(a_1, x_2) = \int_0^{x_2} \{\psi_1(t) - \psi_1^*(t)\}\, dt & \end{array}\right. \tag{3.2}$$

and by $\zeta_1(x_2)$, $\eta_1(x_2)$, $\zeta_2(x_1)$, $\eta_2(x_1)$ the traces:

$$
(3.3)\qquad \left|\begin{array}{ll}
\dfrac{\partial W^0}{\partial x_1}(0, x_2) = \zeta_1(x_2) & \\
 & x_2 \in ]0, a_2[\,, \\
\dfrac{\partial W^0}{\partial x_1}(a_1, x_2) = \eta_1(x_2) & \\
\dfrac{\partial W^0}{\partial x_2}(x_1, 0) = \zeta_2(x_1) & \\
 & x_1 \in ]0, a_1[\,. \\
\dfrac{\partial W^0}{\partial x_2}(x_1, a_2) = \eta_2(x_1) &
\end{array}\right.
$$

Then, we have the following:

THEOREM 3.1: Under the above mentioned assumptions, there exists $r \in \mathbb{R}^+$, depending on the data, such that:

$$
\|u^0 - u^{0*}\|_{H^2(\Omega, \mathbb{R}^2)} < r \sum_{i=1}^{2} \max_{[0, a_i]} \{|\varphi_i - \varphi_i^*| + |\psi_i - \psi_i^*| + |\varphi_i' - \varphi_i^{*\prime}| + |\psi_i' - \psi_i^{*\prime}| + |\zeta_i| + |\eta_i| + |\zeta_i'| + |\eta_i'|\}\,.
$$

PROOF: The difference $V^0 = U^0 - U^{0*}$ is the solution of the problem:

$$
(3.4)\qquad \left|\begin{array}{ll}
L(V_0) = 0 \qquad \text{in } \Omega\,, & \\
V_0(x_1, 0) = \displaystyle\int_0^{x_1} \{\varphi_2(t) - \varphi_2^*(t)\}\, dt & \\
 & x_1 \in ]0, a_1[\,, \\
V^0(x_1, a_2) = \displaystyle\int_0^{x_1} \{\psi_2(t) - \psi_2^*(t)\}\, dt & \\
V^0(0, x_2) = \displaystyle\int_0^{x_2} \{\varphi_1(t) - \varphi_1^*(t)\}\, dt & \\
 & x_2 \in ]0, a_2[\,. \\
V^0(a_1, x_2) = \displaystyle\int_0^{x_2} \{\psi_1(t) - \psi_1^*(t)\}\, dt &
\end{array}\right.
$$

Let us splite $V^0$ as $V^0 = W^0 + V_1^0$, where $W^0$ is the solution of the problem (3.2) whereas $V_1^0$ is the solution in $H^2(\Omega)$ of the problem:

$$
\left|\begin{array}{ll}
L(V_1^0) = -L(W^0) & \text{in } \Omega\,, \\
V_1^0 = 0 & \text{on } \partial\Omega\,.
\end{array}\right.
$$

As is well known (cfr., e.g., [8]), there exists $K_1 \in \mathbb{R}^+$ such that:

$$
\|V_1^0\|_{H^2(\Omega)} < K_1 \|L(W^0)\|_{L^2(\Omega)}\,.
$$

Furthermore, from the assumption (3.1), it results:

$$
\|L(W^0)\|_{L^2(\Omega)} < K_2 \|W^0\|_{H^2(\Omega)}\,,
$$

where $K_2$ is a positive constant which depends on the coefficients of $L$. Hence, we have

$$\|V^0\|_{H^2(\Omega)} < K_3 \|W^0\|_{H^2(\Omega)} \tag{3.5}$$

with $K_3 = 1 + K_1 K_2$.

Let us observe that it results $\partial^2 W^0/\partial x_1^2 = -\partial^2 W^0/\partial x_2^2$, and that the functions $\partial W^0/\partial x_1$, $\partial W^0/\partial x_2$, $\partial^2 W^0/\partial x_1^2$, $\partial^2 W^0/\partial x_2^2$, $\partial^2 W^0/(\partial x_1 \partial x_2)$ satisfy the following Dirichlet problems:

$$\left\{\begin{array}{ll} \Delta \dfrac{\partial W^0}{\partial x_1} = 0 \qquad \text{in } \Omega, & \\ \dfrac{\partial W^0}{\partial x_1}(x_1, 0) = \varphi_2(x_1) - \varphi_2^*(x_1) & \\ & x_1 \in\, ]0, a_1[\,, \\ \dfrac{\partial W^0}{\partial x_1}(x_1, a_2) = \psi_2(x_1) - \psi_2^*(x_1) & \\ \dfrac{\partial W^0}{\partial x_1}(0, x_2) = \zeta_1(x_2) & \\ & x_2 \in\, ]0, a_2[\,, \\ \dfrac{\partial W_0}{\partial x_1}(a_1, x_2) = \eta_1(x_2) & \end{array}\right. \tag{3.6}$$

$$\left\{\begin{array}{ll} \Delta \dfrac{\partial W^0}{\partial x_2} = 0 \qquad \text{in } \Omega, & \\ \dfrac{\partial W^0}{\partial x_2}(x_1, 0) = \zeta_2(x_1) & \\ & x_1 \in [0, a_1[\,, \\ \dfrac{\partial W_0}{\partial x_2}(x_1, a_2) = \eta_2(x_1) & \\ \dfrac{\partial W_0}{\partial x_2}(0, x_2) = \varphi_1(x_2) - \varphi_1^*(x_2) & \\ & x_2 \in\, ]0, a_2[\,, \\ \dfrac{\partial W^0}{\partial x_2}(a_1, x_2) = \psi_1(x_2) - \psi_1^*(x_2) & \end{array}\right. \tag{3.7}$$

$$\left\{\begin{array}{ll} \Delta \dfrac{\partial^2 W^0}{\partial x_1^2} = 0 \qquad \text{in } \Omega, & \\ \dfrac{\partial^2 W^0}{\partial x_1^2}(x_1, 0) = \varphi_2'(x_1) - \varphi_2^{*\prime}(x_1) & \\ & x_1 \in\, ]0, a_1[\,, \\ \dfrac{\partial^2 W^0}{\partial x_1^2}(x_1, a_2) = \psi_2'(x_1) - \psi_2^{*\prime}(x_1) & \\ \dfrac{\partial^2 W^0}{\partial x_1^2}(0, x_2) = -\dfrac{\partial^2 W^0}{\partial x_2^2}(0, x_2) = \varphi_1^{*\prime}(x_2) - \varphi_1'(x_2) & \\ & x_2 \in\, ]0, a_2[\,, \\ \dfrac{\partial^2 W^0}{\partial x_1^2}(a_1, x_2) = -\dfrac{\partial^2 W^0}{\partial x_2^2}(a_1, x_2) = \psi_1^{*\prime}(x_2) - \psi_1'(x_2) & \end{array}\right. \tag{3.8}$$

$$
(3.9)\quad \left|\begin{array}{ll}
\Delta \dfrac{\partial^2 W^0}{\partial x_2^2} = 0 \qquad \text{in } \Omega\,, & \\
\dfrac{\partial^2 W^0}{\partial x_2^2}(x_1, 0) = -\dfrac{\partial^2 W^0}{\partial x_1^2}(x_1, 0) = \varphi_2^{*\prime}(x_1) - \varphi_2'(x_1) & \\
& x_1 \in\, ]0, a_1[\,, \\
\dfrac{\partial^2 W^0}{\partial x_2^2}(x_1, a_2) = -\dfrac{\partial^2 W^0}{\partial x_1^2}(x_1, a_2) = \psi_2^{*\prime}(x_1) - \psi_2'(x_1) & \\
\dfrac{\partial^2 W^0}{\partial x_2^2}(0, x_2) = \varphi_1'(x_2) - \varphi_1^{*\prime}(x_2) & \\
& x_2 \in\, ]0, a_2[\,, \\
\dfrac{\partial^2 W^0}{\partial x_2^2}(a_1, x_2) = \psi_1'(x_2) - \psi_1^{*\prime}(x_2) &
\end{array}\right.
$$

$$
(3.10)\quad \left|\begin{array}{ll}
\Delta \dfrac{\partial^2 W^0}{\partial x_1\, \partial x_2} = 0 \qquad \text{in } \Omega\,, & \\
\dfrac{\partial^2 W^0}{\partial x_1\, \partial x_2}(x_1, 0) = \zeta_2'(x_1) & \\
& x_1 \in\, ]0, a_1[\,, \\
\dfrac{\partial^2 W^0}{\partial x_1\, \partial x_2}(x_1, a_2) = \eta_2'(x_1) & \\
\dfrac{\partial^2 W^0}{\partial x_1\, \partial x_2}(0, x_2) = \zeta_1'(x_2) & \\
& x_2 \in\, ]0, a_2[\,. \\
\dfrac{\partial^2 W^0}{\partial x_1\, \partial x_2}(a_1, x_2) = \eta_1'(x_2)) &
\end{array}\right.
$$

Now, making use of the maximum principle we obtain:

$$
\begin{aligned}
(3.11)\quad \|W^0\|_{H^2(\Omega)} \leq \sqrt{a_1 a_2}\Big\{ & \max_{[0,a_1]} \big[(a_1+1)|\varphi_2 - \varphi_2^*| + (a_1+1)|\psi_2 - \psi_2^*| + \\
& + 2|\varphi_2' - \varphi_2^{*\prime}| + 2|\psi_2' - \psi_2^{*\prime}| + |\zeta_2| + |\eta_2| + |\zeta_2'| + |\eta_2'|\big] + \\
& + \max_{[0,a_2]} \big[(a_2+1)|\varphi_1 - \varphi_1^*| + (a_2+1)|\psi_1 - \psi_1^*| + \\
& + 2|\varphi_1' - \varphi_1^{*\prime}| + 2|\psi_1' - \psi_1^{*\prime}| + |\zeta_1| + |\eta_1| + |\zeta_1'| + |\eta_1'|\big]\Big\}.
\end{aligned}
$$

Finally, by virtue of (1.5), we obtain the thesis.

## 4. - An example of optimal design

We present a simple example to show how it is possible to improve the estimate of Theorem 3.1.

We use such a result to obtain a control condition between the travel costs and the change in the travel demands.

We consider the grid

$$\Omega = ]0, a[\times]0, \pi[$$

and the cost

$$c(x, u(x)) = (Au|u) \tag{4.1}$$

with $A = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$, where $\alpha_1$, $\alpha_2$, are positive constants.

The changes of the travel demands are given by:

$$\left| \begin{array}{ll} \varphi_1^*(x_2) = \varphi_1(x_2) + \lambda & \\ & x_2 \in [0, \pi[\,, \\ \psi_1^*(x_2) = \psi_1(x_2) + \lambda(1 - \cos x_2) & \\ \varphi_2^*(x_1) = \varphi_2(x_1) & \\ & x_1 \in ]0, a[\,, \\ \psi_2^*(x_1) = \psi_2(x_1) & \end{array} \right. \tag{4.2}$$

where $\varphi_1$, $\psi_1$, $\varphi_2$, $\psi_2$ are the old demands and $\lambda$ is a positive parameter.

The difference $u^{0*} - u^0$ can be derived from the potential:

$$V^0(x_1, x_2) = U^{0*}(x_1, x_2) - U^0(x_1, x_2) \tag{4.3}$$

for which an easy calculation gives

$$V^0(x_1, x_2) = \lambda \left( x_2 - \frac{\operatorname{sh} Kx_1}{\operatorname{sh} Ka} \sin x_2 \right) \tag{4.4}$$

with $K = \sqrt{\alpha_1/\alpha_2}$.

Thus we obtain:

$$\|u^{0*} - u^0\|_{L^\infty(\Omega, \mathbb{R}^2)} = \lambda(1 + K \coth Ka) \tag{4.5}$$

and hence, if we require that:

$$\|u^{0*} - u^0\|_{L^\infty(\Omega, \mathbb{R}^2)} \leq \delta \tag{4.6}$$

where $\delta$ is a prescribed control parameter, it is sufficient that it results:

$$\lambda(1 + K \coth Ka) \leq \delta\,. \tag{4.7}$$

This relation allows us to act on the parameters $\lambda$, $K$, and $a$ in order to satisfy the required control.

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