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A Remark on the Impact of Two Vibrating Strings (***) (***)

SUMMARY. — We correct a formula in a previous work.

Un'osservazione sull'urto tra due corde vibranti

SUNTO. — Si coregge una formula di un precedente lavoro.

The recent paper [1] of L. Amerio on the impact of two vibrating strings led us to reconsider our previous work [2] on the subject. Then we found that the introductory formula (1.6) of paper [2] about the impact reaction is, in general, wrong. This error, however, does not imply any consequence on the remainder of the paper, because under the successive hypothesis (1.16) of *equal* propagation speed (the only case studied in [2]) the given formula reduces to the correct one. The same remarks hold for note [3], where the same results have been presented. The occurrence of this error is rather surprising, because in the paper [4] of one of the authors the correct expression for a single string was given. In this short note we establish the true result, for all cases.

The system

$$\mu_i \partial^2 y_i/\partial t^2 - T_i \partial^2 y_i/\partial x^2 = p_i(x, t, y_i) + J_i, \qquad (i = 1, 2),$$

$$y_2(x, t) \geq y_1(x, t),$$

$$-J_1 = J_2 \geq 0 \text{ in the sense of distributions},$$

$$\operatorname{supp} J_i \subseteq \{(x, t) : y_1(x, t) = y_2(x, t)\},$$

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(with suitable initial-boundary value conditions) was taken as a model of the motion of two strings, vibrating in the same plane, and hitting together. The domain is irrelevant to our purpose, so we can assume to work in $\mathbb{R}^2$.

Here μ_i are the linear densities of the strings, T_i their tensions, p_i the external active forces and $J = -J_1 = J_2$ the (impulsive) reaction between the strings.

In order to evaluate J, which is obviously a non negative measure, we must consider the duality $\langle J, \vartheta\rangle$ with a suitable test function $\vartheta \in \mathfrak{D}(\mathbb{R}^2)$.

To this aim let

$$\alpha(t) = \begin{cases} \varrho \exp\left(-1/(1-t^2)\right) & \text{for } |t|<1\,, \\ 0 & \text{for } |t|>1\,, \end{cases}$$

where ϱ is such that $\int_{-\infty}^{+\infty}\alpha(t)\,dt = 1$, and $\alpha_n(t) = n\alpha(nt)$.

It is well known that, for $n\to\infty$, $\alpha_n(t)\to\delta$, the Dirac measure at the origin. Let moreover

$$\omega_n(t) = \int_{-\infty}^{t}\{-\alpha_n(t+1/n) + \alpha_n(t-1/n)\}\,dt\,,$$

and notice that $-1<\omega_n(t)<\omega_{n+1}(t)<0$, $\omega_n(t)=0$ for $|t|>2/n$, $\omega_n(0)=1$ $\forall n$, $\omega_n(t)\to 0$ for $n\to\infty$ and $\forall t\neq 0$. If we multiply the derivative $\omega_n'(t) = -\alpha_n(t+1/n)+\alpha_n(t-1/n)$ by a function $g(t)$ having a jump discontinuity $[g] = g(0^+)-g(0^-)$ at the origin, and integrate with respect to t, then we get for $n\to\infty$:

$$\int_{-\infty}^{+\infty} g(t)\,\omega_n'(t)\,dt \to [g]\,.$$

Taking now for ϑ the product $\vartheta(x,t) = \psi(x)\omega_n(t-\varphi(x))$, where $t=\varphi(x)$ is the equation of an impact arc Λ, and $\psi(x)$ is an arbitrary test function, and evaluating $\langle J_i, \vartheta\rangle$ $(i=1,2)$, we get:

$$\begin{aligned}
\langle J_i,\vartheta\rangle &= \langle \mu_i\partial^2 y_i/\partial t^2 - T_i\partial^2 y_i/\partial x^2 - p_i(x,t,y_i), \vartheta\rangle = \\
&= \langle \mu_i y_{itt} - T_i y_{ixx} - p_i(x,t,y_i), \psi(x)\omega_n(t-\varphi(x))\rangle = \\
&= -\langle \mu_i y_{it}, \psi(x)\,\omega_n'(t-\varphi(x))\rangle + \langle T_i y_{ix}, \partial\{\psi(x)\omega_n(t-\varphi(x))\}/\partial x\rangle - \\
&\quad -\langle p_i(x,t,y_i), \psi(x)\,\omega_n(t-\varphi(x))\rangle = \\
&= -\langle \mu_i y_{it}, \psi(x)\,\omega_n'(t-\varphi(x))\rangle + \langle T_i y_{ix}, \psi'(x)\,\omega_n(t-\varphi(x))\rangle + \\
&\quad + \langle T_i y_{ix}, \psi(x)\,\omega_n'(t-\varphi(x))(-\varphi'(x))\rangle - \langle p_i(x,t,y_i), \psi(x)\,\omega_n(t-\varphi(x))\rangle = \\
&= -\langle \mu_i y_{it} + \varphi'(x)\,T_i y_{ix}, \psi(x)\,\omega_n'(t-\varphi(x))\rangle + \\
&\qquad\qquad + \langle T_i y_{ix}\psi'(x) - p_i(x,t,y_i)\psi(x), \omega_n(t-\varphi(x))\rangle\,.
\end{aligned}$$

Letting $n \to \infty$, the second term vanishes, by Lebesgue dominated convergence theorem; hence we obtain:

$$\lim \langle J_i, \theta \rangle = -\lim \langle \mu_i y_{it} + \varphi'(x) T_i y_{ix}, \psi(x) \omega_n'(t-\varphi(x)) \rangle =$$
$$= -\int_{\Lambda} [\mu_i y_{it} + \varphi'(x) T_i y_{ix}] \psi(x)\, dx ,$$

where the jump is taken across the impact arc Λ: $t = \varphi(x)$, that is for instance $[y_{it}] = y_{it}(x, \varphi(x)^+) - y_{it}(x, \varphi(x)^-)$. From $J_1 = -J_2$ and adding, it follows:

$$\int_{\Lambda} \{[\mu_1 y_{1t} + \varphi'(x) T_1 y_{1x}] + [\mu_2 y_{2t} + \varphi'(x) T_2 y_{2x}]\} \psi(x)\, dx = 0 ,$$

from which, by the arbitrariness of $\psi(x)$, we obtain the equality:

$$[\mu_1 y_{1t} + \varphi'(x) T_1 y_{1x}] + [\mu_2 y_{2t} + \varphi'(x) T_2 y_{2x}] = 0 .$$

By differentiating the identities $y_i(x, \varphi(x)^+) = y_i(x, \varphi(x)^-) \Rightarrow [y_i] = 0$ $(i = 1, 2)$, we get at once: $[y_{ix} + \varphi'(x) y_{it}] = 0 \Rightarrow [y_{ix}] = -\varphi'(x)[y_{it}]$, so that we can eliminate the derivatives with respect to x and obtain:

$$(\mu_1 - \varphi'^2(x) T_1)[y_{1t}] + (\mu_2 - \varphi'^2(x) T_2)[y_{2t}] = 0 .$$

If we introduce the conventional «reduced densities»:

$$m_i = \mu_i - \varphi'^2(x) T_i = \mu_i \{1 - \varphi'^2(x) c_i^2\} ,$$

where $c_i = \sqrt{T_i / \mu_i}$ represent the propagation speeds along the strings, we can write the previous formula as

$$m_1 [y_{1t}] + m_2 [y_{2t}] = m_1 \{y_{1t}^+ - y_{1t}^-\} + m_2 \{y_{2t}^+ - y_{2t}^-\} = 0 ,$$

or equivalently:

$$m_1 y_{1t}^+ + m_2 y_{2t}^+ = m_1 y_{1t}^- + m_2 y_{2t}^- .$$

This equation is like (1.6) of paper [2], but with the «reduced densities» m_i instead of μ_i. This condition, together with Newton's law

$$y_{1t}^+ - y_{2t}^+ = -h(y_{1t}^- - y_{2t}^-) ,$$

gives again

$$y_{1t}^+ = a_{11} y_{1t}^- + a_{12} y_{2t}^- ,$$
$$y_{2t}^+ = a_{21} y_{1t}^- + a_{22} y_{2t}^- ,$$

where the coefficients

$$a_{11}=\frac{m_1-bm_2}{m_1+m_2}, \qquad a_{12}=\frac{(1+b)m_2}{m_1+m_2}, \qquad a_{21}=\frac{(1+b)m_1}{m_1+m_2}, \qquad a_{22}=\frac{m_2-bm_1}{m_1+m_2},$$

have the same expression (1.14) in [2], except again for m_i instead of μ_i. The same equations hold for the derivatives $y_{ix}=-\varphi'(x)y_{it}$, so that in (1.14) the coefficients $\bar{a}_{ij}$ relating the values of y_{ix} after and before the impact always agree with the a_{ij}. Notice that in general m_i, hence also a_{ij}, depend on x.

Observe moreover that the singular case $m_1+m_2=0$ corresponds to the equality $(\mu_1+\mu_2)-\varphi'^2(x)(T_1+T_2)=0$; in such case $1/\varphi'(x)=\bar{V}$, a characteristic speed which plays a notable role in Amerio's paper [1].

Under the hypothesis $c_1=c_2$, m_i are proportional to μ_i, and a_{ij} do not depend on x and agree with the coefficients given in [2], so that the analysis made for this case continues to hold.

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