A Remark on the Impact of Two Vibrating Strings (***) (*)

**SUMMARY.** — We correct a formula in a previous work.

**Un'osservazione sull'urto tra due corde vibranti**

**Scontro.** — Si corregge una formula di un precedente lavoro.

The recent paper [1] of L. Amerio on the impact of two vibrating strings led us to reconsider our previous work [2] on the subject. Then we found that the introductory formula (1.6) of paper [2] about the impact reaction is, in general, wrong. This error, however, does not imply any consequence on the remainder of the paper, because under the successive hypothesis (1.16) of equal propagation speed (the only case studied in [2]) the given formula reduces to the correct one. The same remarks hold for note [3], where the same results have been presented. The occurrence of this error is rather surprising, because in the paper [4] of one of the authors the correct expression for a single string was given. In this short note we establish the true result, for all cases.

The system

\[ \mu_i \frac{\partial^2 y_i}{\partial t^2} - T_i \frac{\partial^2 y_i}{\partial x^2} = p_i(x, t) + f_i, \quad (i = 1, 2), \]

\[ y_1(x, t) > y_2(x, t), \]

\[ -f_1 = f_2 \geq 0 \text{ in the sense of distributions,} \]

\[ \text{supp} f_i \subseteq \{(x, t) : y_1(x, t) = y_2(x, t)\}. \]


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(with suitable initial-boundary value conditions) was taken as a model of the motion of two strings, vibrating in the same plane, and hitting together. The domain is irrelevant to our purpose, so we can assume to work in $\mathbb{R}^2$.

Here $\mu_i$ are the linear densities of the strings, $T_i$ the tensions, $\beta_i$ the external active forces and $J = J_1 = J_2$ the (impulsive) reaction between the strings.

In order to evaluate $J$, which is obviously a non negative measure, we must consider the duality $\langle J, \theta \rangle$ with a suitable test function $\theta \in \mathcal{D}(B^2)$.

To this aim let

$$x(t) = \begin{cases} \frac{\alpha}{1 - t^2} & \text{for } |t| < 1, \\ 0 & \text{for } |t| > 1, \end{cases}$$

where $\alpha$ is such that $\int_0^1 x(t) \, dt = 1$, and $x_n(t) = n x(nt)$.  

It is well known that, for $n \to \infty$, $x_n(t) \to \delta$, the Dirac measure at the origin. Let moreover

$$\omega_n(t) = \int_{-1/n}^{1/n} \left(-x_n(t + 1/n) + x_n(t - 1/n)\right) \, dt,$$

and notice that $-1 < \omega_n(t) < \omega_{n+1}(t) < 0$, $\omega_n(t) = 0$ for $|t| > 2/n$, $\omega_n(0) = 1$ for all $n \to \infty$ and $\forall t \neq 0$. If we multiply the derivative $\omega_n'(t) = -x_n(t + 1/n) + x_n(t - 1/n)$ by a function $g(t)$ having a jump discontinuity at the origin, and integrate with respect to $t$, then we get

$$\int_0^1 g(t) \omega_n(t) \, dt \to [g].$$

Taking now for $\theta$ the product $\theta(x, t) = \psi(x) \omega_n(t - \psi(x))$, where $t = \psi(x)$ is the equation of an impact arc $A$, and $\psi(x)$ is an arbitrary test function, and evaluating $\langle J_i, \theta \rangle$ ($i = 1, 2$), we get:

$$\langle J_i, \theta \rangle = \langle \mu_i \partial_x^2 y_i \partial_t^2 - T_i \partial_x y_i \partial_t^2 - \beta_i (x, t, y_i), \theta \rangle =$$

$$= \langle \mu_i y_{i,tt} - T_i y_{i,xt} - \beta_i (x, t, y_i), \psi(x) \omega_n(t - \psi(x)) \rangle =$$

$$= -\langle \mu_i y_{i,tt}, \psi(x) \omega'_n(t - \psi(x)) \rangle + \langle T_i y_{i,xt}, \psi(x) \omega_n(t - \psi(x)) \rangle -$$

$$- \langle \beta_i (x, t, y_i), \psi(x) \omega_n(t - \psi(x)) \rangle =$$

$$= -\langle \mu_i y_{i,tt}, \psi(x) \omega'_n(t - \psi(x)) \rangle + \langle T_i y_{i,xt}, \psi(x) \omega_n(t - \psi(x)) \rangle +$$

$$+ \langle T_i y_{i,xt}, \psi(x) \omega'_n(t - \psi(x)) - \psi'(x)(t - \psi(x)) \rangle - \langle \beta_i (x, t, y_i), \psi(x) \omega_n(t - \psi(x)) \rangle +$$

$$= -\langle \mu_i y_{i,tt}, \psi'(x) T_i y_{i,xt}, \psi(x) \omega'_n(t - \psi(x)) \rangle +$$

$$+ \langle T_i y_{i,xt}, \psi'(x)(t - \psi(x)) - \beta_i (x, t, y_i), \psi(x) \omega_n(t - \psi(x)) \rangle .$$
Letting \( n \to \infty \), the second term vanishes, by Lebesgue dominated convergence theorem; hence we obtain:

\[
\lim \langle J, \theta \rangle = -\lim \langle \mu_1 y_{1t} + \varphi'(x) T_t y_{1u}, \varphi(x) \psi_4(t - \varphi(x)) \rangle = -\int_A \left[ \mu_1 y_{1t} + \varphi'(x) T_t y_{1u} \right] \varphi(x) \, dx,
\]

where the jump is taken across the impact arc \( A: t = \varphi(x) \), that is for instance \( \left[ y_{1t} \right] = y_{1t}(x, \varphi(x)^+) - y_{1t}(x, \varphi(x)^-) \). From \( J_1 = -J_2 \) and adding, it follows:

\[
\int_A \left[ \left[ \mu_1 y_{1t} + \varphi'(x) T_t y_{1u} \right] + \left[ \mu_2 y_{2t} + \varphi'(x) T_t y_{2u} \right] \right] \varphi(x) \, dx = 0,
\]

from which, by the arbitrariness of \( \varphi(x) \), we obtain the equality:

\[
\left[ \mu_1 y_{1t} + \varphi'(x) T_t y_{1u} \right] + \left[ \mu_2 y_{2t} + \varphi'(x) T_t y_{2u} \right] = 0.
\]

By differentiating the identities \( y_i(x, \varphi(x)^+) = y_i(x, \varphi(x)^-) \Rightarrow \left[ y_i \right] = 0 \) \( (i = 1, 2) \), we get at once: \( \left[ J_{12} + \varphi'(x) J_{1u} \right] = 0 \Rightarrow \left[ J_{12} \right] = -\varphi'(x) \left[ J_{1u} \right] \), so that we can eliminate the derivatives with respect to \( x \) and obtain:

\[
\left( \mu_1 - \varphi'^2(x) T_{1u} \right) \left[ y_{1u} \right] + \left( \mu_2 - \varphi'^2(x) T_{2u} \right) \left[ y_{2u} \right] = 0.
\]

If we introduce the conventional « reduced densities »:

\[
m_i = \mu_i - \varphi'^2(x) T_i = \mu_i(1 - \varphi'^2(x) \epsilon_i^2),
\]

where \( \epsilon_i = \sqrt{T_i/\mu_i} \) represent the propagation speeds along the strings, we can write the previous formula as

\[
m_1 \left[ y_{1u} \right] + m_2 \left[ y_{2u} \right] = m_1 \left( y_{1u}^+ - y_{1u}^- \right) + m_2 \left( y_{2u}^+ - y_{2u}^- \right) = 0,
\]

or equivalently:

\[
m_1 y_{1u}^+ + m_2 y_{2u}^+ = m_1 y_{1u}^- + m_2 y_{2u}^-.
\]

This equation is like (1.6) of paper [2], but with the « reduced densities » \( m_i \) instead of \( \mu_i \). This condition, together with Newton's law

\[
y_{1u}^+ - y_{1u}^- = -b(y_{1u}^- - y_{1u}^+),
\]

gives again

\[
y_{1u}^+ = a_{11} y_{1u}^- + a_{12} y_{2u}^-,
\]

\[
y_{2u}^+ = a_{21} y_{1u}^- + a_{22} y_{2u}^-,
\]
where the coefficients
\[ a_{11} = \frac{m_1 - b m_2}{m_1 + m_2}, \quad a_2 = \frac{(1 + b) m_2}{m_1 + m_2}, \quad a_{21} = \frac{(1 + b) m_1}{m_1 + m_2}, \quad a_{22} = \frac{m_2 - b m_1}{m_1 + m_2}, \]

have the same expression (1.14) in [2], except again for \( m_i \) instead of \( \mu_i \). The same equations hold for the derivatives \( y_{1t} = -\varphi'(x) y_{1t} \), so that in (1.14) the coefficients \( a_{ij} \) relating the values of \( y_{1t} \) after and before the impact always agree with the \( \alpha_{ij} \). Notice that in general \( m_i \), hence also \( a_{ij} \), depend on \( x \).

Observe moreover that the singular case \( m_1 + m_3 = 0 \) corresponds to the equality \( (\mu_1 + \mu_2) - \rho(x)(T_1 + T_3) = 0 \); in such case \( 1/\varphi'(x) = \overline{\nu} \), a characteristic speed which plays a notable role in Amerio's paper [1].

Under the hypothesis \( \dot{\epsilon}_1 = \dot{\epsilon}_2 \), \( m_i \) are proportional to \( \mu_i \), and \( a_{ij} \) do not depend on \( x \) and agree with the coefficients given in [2], so that the analysis made for this case continues to hold.

REFERENCES