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AN TON BUI (*)

On the Classical Solution of a Single Phase Stefan Type Problem for Parabolic Equations in Planar Domains with Intersecting Fixed and Free Boundaries (**)

ABSTRACT. — The purpose of this paper is to establish the existence of a local (in time) classical solution of a Stefan type single phase problem for a linear parabolic equation in planar domains with intersecting fixed and free boundaries.

Sulla soluzione classica di un problema parabolico del tipo di Stefan a una fase in domini piani con frontiere in parte fisse e in parte libere

RIASSUNTO. — Si stabilisce l'esistenza (in piccolo, relativamente al tempo) di una soluzione classica per un problema del tipo di Stefan, a una fase, riguardante equazioni lineari paraboliche in domini piani con frontiere in parte fisse e in parte libere.

0. INTRODUCTION

Let $u(x, t)$ be the temperature and $f(x, t)$ be the fixed heat source. We consider the free boundary problem:

$$(0.1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} - \nabla_x \cdot \nabla_x u = f \ \text{ in } \Omega_t, & u(x, 0) = u_0(x) \ \text{ in } \Omega = \Omega_0, \\ u(x, t) = 0 \ \text{ on } \partial\Omega_t^*, & \nabla_x u \cdot n = 0 \ \text{ on } \partial\Omega_t \backslash \partial\Omega_t^* . \end{cases}$$

with

$$(0.2)\qquad \begin{cases} \dfrac{\partial \psi}{\partial t} - \nabla_x u \cdot \nabla_x \psi = 0 & \text{for } \psi(x, t) = x_2 - \varphi(x_1, t) = 0, \\ \psi(x, 0) = \psi_0(x) . \end{cases}$$

(*) Indirizzo dell'Autore: Department of Mathematics, University of British Columbia, Vancouver, B.C. Canada V6T 1Y4.

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The domain Ω_t is given by

$$\{x : x = (x_1, x_2), -1 < x_1 < 1, 0 < x_2 < \varphi(x_1, t)\}$$

with $\partial\Omega_t^+ = \{x : -1 < x_1 < 1, x_2 = \varphi(x_1, t)\}$. Throughout the paper, the interior contact angles made by $\partial\Omega_0^+$ with $\partial\Omega_0 / \partial\Omega_0^+$ are assumed to be less than $\pi/4$.

The unknowns of the problem (0.1)-(0.2) are $\{u, \varphi\}$. Stefan-type problems for parabolic equations arise in many applications and have been studied extensively (cf. A. Fasano and M. Primicerio [4], L. Rubinstein [13]).

One phase Stefan problems have been studied by L. Cafarelli [3], A. Friedman [6], A. Friedman and D. Kinderlehrer [7] and by A. Meirmanov [11], [12]. In all of the above cited works, the initial domain is assumed to be smooth and the fixed with the free boundary have empty intersection.

The result obtained in this paper seems new and since we are dealing with classical solutions, the existence for small time only is expected as for large time the free boundary may intersect itself. A detailed outline of the paper is given in Section 1.

1. - Formulation of the problem

Let Ω be the set $\{\xi : (\xi_1, \xi_2); -1 < \xi_1 < 1, 0 < \xi_2 < \varphi_0(\xi_1)\}$ where φ_0 is a C^3-function. Denote by $\partial\Omega^+ = \{\xi : \xi_2 = \varphi_0(\xi_1), -1 < \xi_1 < 1\}$ and by $\partial\Omega^- = \partial\Omega / \partial\Omega^+$. The angles made by $\partial\Omega^\pm$ at $P^\pm = (\pm 1, \varphi_0(\pm 1))$ are denoted by $\omega(P^\pm)$.

$W^{k,2}(\Omega)$ is the usual Sobolev space and $\|\cdot\|$ is the $L^2(\Omega)$-norm. Let $\Gamma = \{P^\pm, Q^\pm\}$ where $Q^\pm = (\pm 1, 0)$ and let $\varrho(\xi)$ be the distance from a point ξ in Ω to Γ. The weighted Sobolev space $H_s^k(\Omega; \Gamma)$, $0 < s < 1$ which we shall write as $H_s^k(\Omega)$ is a Hilbert space with the norm

$$\|u\|_{H_s^k(\Omega)} = \left\{ \sum_{|\alpha| \leq k} \|\varrho^{s-k+|\alpha|} D^\alpha u\|^2 \right\}^{\frac{1}{2}}.$$

Let $(0, T)$ be a finite time-interval and let $L^2(0, T; H_s^k(\Omega))$ be the Hilbert space with the norm

$$\|u\|_{L^2(0,T; H_s^k(\Omega))} = \left\{ \int_0^T \|u(\cdot, t)\|^2_{H_s^k(\Omega)} dt \right\}^{\frac{1}{2}}.$$

Let $\tilde{\varphi}(x, t) = x_2 - \tilde{\varphi}(x_1, t)$ be a simple moving curve with $\tilde{\varphi}(x, 0) = x_2 - \varphi_0(x_1) = 0$ and intersecting the lines $x_1 = \pm 1$ at $P^\pm$. Denote by $\Omega_t = \{x : -1 < x_1 < 1; 0 < x_2 < \tilde{\varphi}(x_1, t)\}$. In this paper we consider the free boundary problem:

$$(1.1)\qquad \begin{cases} \dfrac{\partial}{\partial t}\hat{u} - \nabla_x\cdot\nabla_x\hat{u} = f(x|t=0,t) & \text{in } \Omega_t,\ 0<t<T, \\ \hat{u}=0 \quad \text{on } \partial\Omega_t^+, \qquad \nabla_x\hat{u}\cdot n = 0 & \text{on } \partial\Omega_t/\partial\Omega_t^+, \\ \hat{u}(x,0)=u_0(x) & \text{in } \Omega, \end{cases}$$

$$(1.2)\qquad \frac{\partial}{\partial t}\hat{\varphi} - \nabla_x\hat{u}\cdot\nabla_x\hat{\varphi} = 0, \qquad \hat{\varphi}(x,0)=\varphi_0(x).$$

The unknowns of the problem are $(\hat{u}, \hat{\varphi})$.

Let $\hat{v}(x,t)$ be the vector function $-\nabla_x\hat{u}$. The equation (1.2) suggests that we consider $\hat{v}$ as the velocity vector of a fictitious fluid particle. With that interpretation in mind the material derivative of $\hat{\varphi}$ is given by (1.2). Thus we are led to the introduction of Lagrangian coordinates as done in [14] by Solonnikov for fluid mechanics.

Let $v(\zeta,t)$ be the velocity vector of a fictitious particle which at $t=0$ is at the point ζ in Ω. The Eulerian and Lagrangian coordinates are related by:

$$(1.3)\qquad x = X(\zeta,t) = \zeta + \int_0^t v(\zeta,s)\,ds.$$

LEMMA 1.1: *Let v be a vector-function in $L^2(0,T;H^2_0(\Omega))$ and suppose that:*

$$(1)\qquad (T^{\frac12}+T^{\frac14})\|v\|_{L^2(0,T;H^2_0(\Omega))}, \qquad (T^{\frac12}+T^{\frac14})\left\|\frac{\partial v}{\partial t}\right\|_{L^2(0,T;H^2_0(\Omega))} < \delta < \frac14$$

with

$$T^{\frac14}\left\|\frac{\partial^2 v}{\partial t^2}\right\|_{L^2(0,T;H^2_0(\Omega))} < \delta < \frac14; \qquad 0<s<\frac16.$$

(2) $v\cdot n = 0$ *on* $\partial\Omega^- = \partial\Omega/\partial\Omega^+$.

1) $0<\varepsilon<\det U(\xi,t)<C$ *with* $U = \left(\left(\delta_{ik} + \int_0^t \frac{\partial}{\partial\xi_k} v_i(\xi,s)\,ds\right)\right)$.

2) $A(v) = (U^*)^{-1}$ *is defined and:*

(i) $\|I-A\|_{L^\infty(0,t;W^{1,\infty}(\Omega))} < M t^{\frac12}\|v\|_{L^2(0,t;W^{1,\infty}(\Omega))}$.

(ii) $\left\|\frac{\partial A}{\partial t}\right\|_{L^2(0,t;W^{1,\infty}(\Omega))} < M\|v\|_{L^2(0,t;W^{1,\infty}(\Omega))}$.

(iii) $\left\|\frac{\partial^2}{\partial t^2}A\right\|_{L^2(0,t;H^2_0(\Omega))} < M\left\{\left\|\frac{\partial v}{\partial t}\right\|_{L^2(0,t;H^2_0(\Omega))} + \|v\|_{L^2(0,t;H^2_0(\Omega))}\|v\|^2_{L^\infty(0,t;H^2_0(\Omega))}\right\}$.

M is independent of δ, v and of t; $0<t<T$.

Proof: The estimates are obtained by applying the Sobolev imbedding theorem. The computations are very tedious although simple. We shall not reproduce them.

Let $\tilde{v}(x, t) = v(\xi, t)$ with v as in Lemma 1.1 and suppose that $\tilde{v} = -\nabla_x \tilde{u}$. Then the equation (1.2) becomes:

$$\text{(1.4)} \qquad \frac{\partial}{\partial t}\psi(\xi, t) = 0\,, \qquad \psi(\xi, 0) = \psi_0(\xi)\,.$$

Hence: $\psi(\xi, t) = \psi_0(\xi) = \psi_0(X^{-1}(x, t))$.

A simple calculation as in [14] shows that (1.1) may be rewritten as:

$$\text{(1.5)} \quad \begin{cases} \dfrac{\partial}{\partial t}u(\xi, t) - v\cdot A\nabla u - A\nabla\cdot A\nabla u = f(\xi, t) & \text{in } \Omega\times(0, T)\,, \\ u(\xi, t) = 0 \quad \text{on } \partial\Omega^+\times(0, T)\,, \quad A\nabla u\cdot n = 0 & \text{on } (\partial\Omega/\partial\Omega^+)\times(0, T)\,, \\ u(\xi, 0) = u_0(\xi) \quad \text{in } \Omega\,. \end{cases}$$

We shall now give a detailed outline of the paper.

Step 1: In Section 2, a mixed elliptic boundary-value problem is considered when $0 < \omega(P^k) < \pi/4$.

Step 2: Let v be as in Lemma 1.1, we use a discretisation of the time-variable to study the initial boundary problem.

$$\text{(1.6)} \quad \begin{cases} \dfrac{\partial}{\partial t}w - \nabla\cdot A\nabla w = K \quad \text{in } \Omega\times(0, T)\,, & w(\xi, 0) = w_0 \quad \text{in } \Omega\,, \\ w = 0 \quad \text{on } \partial\Omega^+\times(0, T)\,, & A\nabla w\cdot n = g \quad \text{on } (\partial\Omega/\partial\Omega^+)\times(0, T)\,. \end{cases}$$

Step 3: The method of successive approximations is used to study the problem:

$$\text{(1.7)} \quad \begin{cases} \dfrac{\partial w}{\partial t} - v\cdot A\nabla w - A\nabla\cdot A\nabla w = f & \text{in } \Omega\times(0, T)\,, \; w(\xi, 0) = w_0\,, \\ w = 0 \quad \text{on } \partial\Omega^+\times(0, T), & A\nabla w\cdot n = g \quad \text{on } (\partial\Omega/\partial\Omega^+)\times(0, T)\,. \end{cases}$$

Step 4: Let u be the unique solution of (1.7). We define the nonlinear mapping $\mathfrak{T}(v) = -A\nabla u$. It is shown that there exists a non-empty interval $(0, T_A)$ for which $A\nabla u$ verifies the hypotheses of Lemma 1.1 and has a fixed point, i.e. $\mathfrak{T}(v) = v = -A\nabla u$. Now let:

$$\tilde{u}(x, t) = u(\xi, t) = u(X^{-1}(x, t), t)\,,$$
$$\tilde{\psi}(x, t) = \psi_0(\xi) = \psi_0(X^{-1}(w, t))\,,$$

then $\{\tilde{u}, \tilde{\psi}\}$ is the solution of (1.1)-(1.2).

2. - A mixed elliptic boundary-value problem

Let $v = (v_1, v_2)$ be a vector-function as in Lemma 1.1 and consider the mixed elliptic boundary problem:

$$(2.1)\qquad -\nabla\cdot A\nabla u = f \text{ in } \Omega\,, \qquad u = 0 \text{ on } \partial\Omega^+\,, \qquad A\nabla u\cdot n = g \text{ on } \partial\Omega^-\,.$$

Lemma 2.1: *Suppose that* $0<\omega(P^\pm)<\pi/4$ *and let* $0<s<1$ *with* $1+k-s\neq\sigma$; $k=0,1,2$ *where* $\sigma=(2n+1)\pi/2\omega(P^+)$, $(2m+1)\pi/2\omega(P^-)$. *Then there exists* C *such that*:

$$\|u\|_{H_s^{k+2}(\Omega\cap B_\mu)}\leq C\{\|\nabla\cdot\nabla u\|_{H_s^k(\Omega\cap B_\mu)}+\|\nabla u\cdot n\|_{H_s^{k+\frac{1}{2}}(\partial\Omega^-\cap B_\mu)}+\|u\|_{H_s^k(\Omega\cap B_\mu)}\}$$

for all u *in* $K_s^{k+2}(\Omega\cap B_\mu)$, $u=0$ *on* $\partial\Omega^+\cap B_\mu$, $\operatorname{supp} u\subset B_\mu$ *where* B_μ *is the disc centered at* P^+ *or* P^- *with radius* μ.

Proof: Consider the mixed problem:

$$(2.2)\qquad \begin{cases} -\Delta w = F & \text{in } N_R = B_R\cap\Omega\,, \qquad \operatorname{supp} w\subset B_R\,,\\ w=0 & \text{on } \partial\Omega^+\cap B_R\,, \qquad \nabla w\cdot n = g \quad \text{on } \partial\Omega^-\cap B_R\,, \end{cases}$$

where B_R is the disc centered at P^+, radius R. It is known that there exists a conformal mapping taking N_R onto S_R with

$$S_R=\{(r,\theta): 0<r<R,\ 0<\theta<\omega(P^+)\}\,.$$

The problem (2.2) becomes:

$$\left|\begin{array}{l} -\Delta\hat{w}=\hat{F} \quad \text{in } S_R\,, \quad \operatorname{supp}\hat{w}\subset S_R\,, \quad \hat{w}=0 \quad \text{for } \theta=0\,,\\ \dfrac{\partial}{\partial\theta}\hat{w}=\hat{g} \qquad \text{for } \theta=\omega(P^+)\,. \end{array}\right.$$

Making the change of variable $\sigma=\log r$ as in [9] and we get:

$$(2.3)\qquad \left|\begin{array}{l} -\dfrac{\partial^2}{\partial\sigma^2}\hat{u}-\dfrac{\partial^2}{\partial\theta^2}\hat{u}=\hat{f}\,; \qquad -\infty<\sigma<\infty\,,\ 0<\theta<\omega(P^+)\,,\\ \hat{u}|_{\theta=0}=0\,, \qquad \left.\dfrac{\partial\hat{u}}{\partial\theta}\right|_{\theta=\omega(P^+)}=\hat{g}\,. \end{array}\right.$$

We obtain by taking the Fourier transform with respect to σ;

$$(2.4)\qquad \lambda^2\tilde{u}-\frac{\partial^2}{\partial\theta^2}\tilde{u}=\tilde{F}\,; \qquad \tilde{u}|_{\theta=0}=0\,, \qquad \frac{\partial}{\partial\theta}\tilde{u}|_{\theta=\omega(P^+)}=\tilde{g}\,.$$

The eigenvalues of (2.4) are $\lambda = i(2n+1)\pi/2\omega(P^+)$. It is now standard to show as in [9] that for $1+k-s \neq (2n+1)\pi/2\omega(P^+)$ we have the stated estimate.

LEMMA 2.2: *Let* $0<s<1$ *and let* $1+k-s \neq 2p$. *Then:*

$$\|u\|_{H_s^{k+2}(\Omega\cap B_\mu)} < C\{\|\nabla\cdot\nabla u\|_{H_s^k(\Omega\cap B_\mu)} + \|\nabla u\cdot n\|_{H_s^{k+\frac{1}{2}}(\partial\Omega^-\cap B_\mu)} + \|u\|_{H_s^k(\Omega\cap B_\mu)}\}$$

for all u *in* $H_s^{k+2}(\Omega\cap B_\mu)$, supp $u \subset B_\mu$ *and where* B_μ *is the disc centered at* $Q^\pm$, *radius* μ.

PROOF: We proceed as in Lemma 2.1 and are led to the study of the eigenvalues of the problem:

$$\lambda^2\hat{u} - \frac{\partial^2}{\partial\theta^2}\hat{u} = \hat{f}\,; \qquad \frac{\partial\hat{u}}{\partial\theta}\bigg|_{\theta=0} = \frac{\partial\hat{u}}{\partial\theta}\bigg|_{\theta=\pi/2} = \hat{g}\,. \tag{2.5}$$

The eigenvalues are $\lambda = 2ip$ and the estimate is obtained in the same way as before.

LEMMA 2.3: *Suppose that* $0<\omega(P^\pm)<\pi/4$ *and let* $0<s<1$. *Suppose that* $1+k-s \neq 2$, $(2n+1)\pi/2\omega(P^+)$, $(2m+1)\pi/2\omega(P^-)$ *for* $k=0,1,2$. *Let* $\{f,g\}$ *be in* $H_s^k(\Omega)\times H_s^{k+\frac{1}{2}}(\partial\Omega^-)$, *then there exists a unique solution* u *of:*

$$-\Delta u = f \quad \text{in } \Omega\,, \qquad u=0 \quad \text{on } \partial\Omega^+\,, \qquad \nabla u\cdot n = g \quad \text{on } \partial\Omega^-\,.$$

Moreover:

$$\|u\|_{H_s^{k+2}(\Omega)} < C\{\|f\|_{H_s^k(\Omega)} + \|g\|_{H_s^{k+\frac{1}{2}}(\partial\Omega^-)}\}$$

PROOF: The proof using Lemmas 2.1-2.2 is standard.

THEOREM 2.1: *Let* v *be as in Lemma* 1.1 *and let* $0<\omega(P^\pm)<\pi/4$. *Suppose that* $1+k-s\neq 2$, $(2n+1)\pi/2\omega(P^+)$, $(2m+1)\pi/2\omega(P^-)$ *with* $k=0,1$ *and* $0<s<1$. *Let* $\{f,g\}$ *be in* $H_s^k(\Omega)\times H_s^{k+\frac{1}{2}}(\partial\Omega^-)$ *then there exists a unique* u, *solution of* (2.1). *Moreover:*

$$\|u\|_{H_s^{k+2}(\Omega)} < C\{\|f\|_{H_s^k(\Omega)} + \|g\|_{H_s^{k+\frac{1}{2}}(\partial\Omega^-)}\}\,.$$

PROOF: We consider the boundary-value problems:

$$-\nabla\cdot\nabla u_j = f - \nabla\cdot(I-A)\nabla u_{j-1} \quad \text{in } \Omega\,, \qquad u_j = 0 \quad \text{on } \partial\Omega^+\,,$$

$$\nabla u_j\cdot n = g + (I-A)\nabla u_{j-1}\cdot n \qquad \text{on } \partial\Omega^-\,;\ j=1,2,\ldots\,.$$

With v as in Lemma 1.1 and with k restricted to either 0 or 1, we have:

$$\|\nabla\cdot(I-A)\nabla u_{j-1}\|_{H_s^k(\Omega)} < C\delta\|u_{j-1}\|_{H_s^{k+2}(\Omega)}$$

and

$$\|(I-A)\nabla u_{j-1}\cdot n\|_{H_*^{k+\frac12}(\partial\Omega^-)} \le C\|(I-A)\nabla u_{j-1}\|_{H_*^{k+1}(\Omega)} \le C_1\delta\|u_{j-1}\|_{H_*^{k+2}(\Omega)}.$$

Set $U_j = u_j - u_{j-1}$ and applying Lemma 2.3 we obtain:

$$\sum_{j=1}^{\infty}\{\|U_j(\cdot,t)\|_{H_*^{k+2}(\Omega)}\} < \infty.$$

Therefore there exists u in $H_*^{k+2}(\Omega)$ such that:

$$(1-C\delta)\|u\|_{H_*^{k+2}(\Omega)} \le M\{\|f\|_{H_*^k(\Omega)} + \|g\|_{H_*^{k+\frac12}(\partial\Omega^-)}\}.$$

Clearly u is the unique solution of (2.1).

3. - A parabolic equation in non-smooth domains

In this section we shall use a fiscretisation of the time-variable to study the problem (1.6). Set: $h = T/N$ where N is a large positive integer and let

$$g^k(\zeta) = h^{-1}\int_{kh}^{(k+1)h} g(\zeta, t)\,dt\,; \qquad 0 \le k \le N-1.$$

Consider the elliptic boundary problems:

$$(3.1)\quad \begin{cases} w_k - w_{k-1} - h\nabla\cdot A^k\nabla w_k = hF^k & \text{in } \Omega, \qquad w_k = 0 \quad \text{on } \partial\Omega^+, \\ A^k\nabla w_k\cdot n = \varphi^k & \text{on } \partial\Omega^-, \qquad w_0 = u^0, \quad 1 \le k \le N-1. \end{cases}$$

LEMMA 3.1: *Let v and $w(P^\pm)$ be as in Theorem 2.1 and let $\{F, \varphi, \partial\varphi/\partial t\}$ be in*

$$L^2(0,T;L^2(\Omega)) \times L^2(0,T;H_*^{\frac12}(\partial\Omega^-)) \times L^2(0,T;L^2(\partial\Omega^-)).$$

Suppose u^0 is in $H_*^1(\Omega)$ $W^{1,2}(\Omega)$ with $w_0 = 0$ on $\partial\Omega^+$. Then there exists, for each k, a unique solution w_k of (3.1). Moreover:

$$\|w_k\|^2_{W^{1,2}(\Omega)} + h\sum_{j=1}^{k}\|\nabla\cdot A^j\nabla w_j\|^2 \le CE\left(F, \varphi, \frac{\partial\varphi}{\partial t}, u^0\right).$$

C is independent of t, v, h, δ and k. The expression $E(F, \varphi, \partial\varphi/\partial t, u^0)$ is defined by

$$\|u^0\|^2_{W^{1,2}(\Omega)} + \|F\|^2_{L^2(0,T;L^2(\Omega))} + \|\varphi\|^2_{L^2(0,T;H_*^{\frac12}(\partial\Omega^-))} + \left\|\frac{\partial\varphi}{\partial t}\right\|^2_{L^2(0,T;L^2(\partial\Omega^-))}.$$

PROOF: The existence of a unique solution w_k of (3.1) in $H^2_0(\Omega)$ follows from Theorem 2.1.

1) Multiplying (3.1) by w_k and integrating over Ω, we obtain:

$$\|w_k\|^2 + 2b(A^k \nabla w_k, \nabla w_k) < \|w_{k-1}\|^2 + 2b(F^k, w_k) + 2b\int_{\partial\Omega^-} \varphi^k w_k .$$

Thus

$$\text{(3.2)} \qquad \|w_k\|^2 + cb\sum_{j=1}^{k} \|w_j\|^2_{H^{1,2}(\Omega)} < \|w_0\|^2 + C(\|F\|^2_{L^2(0,T;L^2(\Omega))} + \|\varphi\|)^2_{L^2(0,T;L^2(\partial\Omega^-))} .$$

2) Multiplying (3.1) by $-\nabla\cdot A^k \nabla w_k$ and integrating over Ω, we get:

$$(A^k \nabla w_k, \nabla w_k) + \frac{b}{2}\|\nabla\cdot A^k \nabla w_k\|^2 < \frac{b}{2}\|F^k\|^2 + (A^k\nabla w_k, \nabla w_{k-1}) + \int_{\partial\Omega^-} H^k(w_k - w_{k-1}).$$

An inequality in Hardy and Littlewood, gives:

$$|(A^k \nabla w_k, \nabla w_{k-1})| < (A^k \nabla w_k, \nabla w_k)^{\frac{1}{2}}(A^k \nabla w_{k-1}, \nabla_{k-1})^{\frac{1}{2}} .$$

Hence:

$$(A^k \nabla w_k, \nabla w_k) + b\|\nabla\cdot A^k \nabla w_k\|^2 < b\|F^k\|^2 + b(b^{-1}\{A^k - A^{k-1}\}\nabla w_{k-1}, \nabla w_{k-1}) +$$

$$+\int_{\partial\Omega^-} \varphi^k w_k - \varphi^{k-1} w_{k-1} + b\int_{\partial\Omega^-} b^{-1}(\varphi^{k-1} - \varphi)^k w_{k-1} .$$

We have by taking into account (3.2):

$$c\|\nabla w_k\|^2 + Cb\sum_{j=1}^{k}\{\|w_j\|^2_{H^{1,2}(\Omega)} + \|\nabla\cdot A^j \nabla w_j\|^2\} < N\Big\{\|W_0\|^2_{H^{1,2}(\Omega)} +$$

$$+ \|F\|^2_{L^2(0,T;L^2(\Omega))} + \|\varphi\|^2_{L^2(0,T;L^2(\partial\Omega^-))} + \|\varphi^0\|_{L^2(\partial\Omega^-)}\|w_0\|_{H^{1,2}(\Omega)} + \|\varphi^k\|^2_{L^2(\partial\Omega^-)} +$$

$$+ b\sum_{j=1}^{k}(\|b^{-1}(\varphi^j - \varphi^{j-1})^k\|^2_{L^2(\partial\Omega^-)} + \|b^{-1}(A^j - A^{j-1})\|_{L^\infty(\Omega)}\|\nabla w_{j-1}\|^2)\Big\} .$$

Hence:

$$\text{(3.3)} \qquad \|\nabla w_k\|^2 + Cb\sum_{j=1}^{k}\{\|w_j\|^2_{H^{1,2}(\Omega)} + \|\nabla\cdot A^j \nabla w_j\|^2\} <$$

$$< M_1\Big\{\|w_0\|^2_{H^{1,2}(\Omega)} + \|F\|^2_{L^2(0,T;L^2(\Omega))} + \|\varphi\|_{L^2(0,T;L^2(\partial\Omega^-))} + \left\|\frac{\partial\varphi}{\partial t}\right\|^2_{L^2(0,T;L^2(\partial\Omega^-))} +$$

$$+ b\sum_{j=1}^{k}\|b^{-1}(A_j - A^{j-1})\|_{L^\infty(\Omega)}\|\nabla w_{j-1}\|^2\Big\} .$$

3) Clearly (3.3) is the discrete analog of the differential inequality:

$$g(t) < ME\left(F, \varphi, \frac{\partial \varphi}{\partial t}, w_0\right) + M\int_0^t g(s)\left\|\frac{\partial A}{\partial t}(\cdot, s)\right\|_{L^\infty(\Omega)} ds\,.$$

Applying the Gronwall lemma and taking into account Lemma 1.1 we obtain the stated estimate.

Lemma 3.2: *Suppose all the hypotheses of Lemma* 3.1 *are satisfied. Then*:

$$\sum_{i=1}^{k} h\|(w_i - w_{i-1})/h\|^2 < ME\left(F, \varphi, \frac{\partial \varphi}{\partial t}, w_0\right).$$

Proof: It is an immediate consequence of the estinate of Lemma 3.1.

Theorem 3.1: *Suppose all the hypotheses of Lemma* 3.1 *are satisfied, then there exists a unique solution* w *of*

$$\left|\begin{array}{ll} \dfrac{\partial}{\partial t} w - \nabla\cdot A\nabla w = F \quad \text{in } \Omega\times(0, T), & w(\xi, 0) = w_0 \quad \text{in } \Omega, \\ w = 0 \quad \text{on } \partial\Omega^+\times(0, T), & A\nabla w\cdot n = \varphi \quad \text{on } \partial\Omega^-\times(0, T). \end{array}\right.$$

Moreover:

$$\|w\|^2_{L^2(0,T;H^1_0(\Omega))} + \left\|\frac{\partial}{\partial t} w\right\|^2_{L^2(0,T;L^2(\Omega))} < ME\left(F, \varphi, \frac{\partial \varphi}{\partial t}, w_0\right).$$

M is independent of δ, v.

Proof: In view of Lemmas 3.1-3.2 and of Theorem 2.1 we have:

$$h\sum_{j=1}^{k} \|w_j\|^2_{H^2_0(\Omega)} < CE\left(F, \varphi, \frac{\partial \varphi}{\partial t}, w_0\right).$$

By a standard procedure we obtain w from w_k and the stated estimate follows from those of Lemmas 3.1-3.2.

We shall now proceed to get further regularity properties. Consider the problem:

$$(3.4)\quad \left|\begin{array}{l} \dfrac{\partial z}{\partial t} - \nabla\cdot A\nabla z = \dfrac{\partial F}{\partial t} + \nabla\cdot\dfrac{\partial A}{\partial t}\nabla w \qquad \text{in } \Omega\times(0, T), \\ z = 0 \quad \text{on } \partial\Omega^+\times(0, T), \quad A\nabla z\cdot n = \dfrac{\partial \varphi}{\partial t} - \dfrac{\partial A}{\partial t}\nabla w\cdot n \quad \text{on } \partial\Omega^-\times(0, T), \\ z(\xi, 0) = F(\xi, 0) + \nabla w^0 \qquad \text{in } \Omega. \end{array}\right.$$

Phyllophoraceae

Chondrus Heredia Grev.

Specimen 37; Dufour 124 = *Phyllophora Heredia* Clem.; Ardiss. e Straff. 472 = *P. Heredia* J. Ag.; Ardiss. I p. 183 = [*Phyllophora heredia* (Clemente) J. Agardh]

Gigartina Griffithsiae Lamour.

Specimen 29; Dufour 121 = *Gymnogongrus Griffithsiae* Mont.; Ardiss. e Straff. 468 = *G. Griffithsiae* Mart.; Ardiss. I p. 176 = [*Gymnogongrus griffithsiae* (Turner) Martius]

Halymenia nervosa Duby

Capraia 149; *Specimen* 36 = *Phyllophora nervosa* Grevill.; [*Phyllophora nervosa* (De Candolle) Greville]

Phyllophora nervosa Grev.

Specimen 36; Dufour 123; Ardiss. e Straff. 471; [*Phyllophora nervosa* (De Candolle) Greville]

Gigartinaceae

Gigartina acicularis Lamour.

Specimen 27; Dufour 118; Ardiss. e Straff. 463; Ardiss. I p. 167 = [*Gigartina acicularis* (Wulfen) Lamouroux]

Gigartina Teedii Lamour.

Capraia 135; *Specimen* 26; Dufour 119; Ardiss. e Straff. 464; Ardiss. I p. 168 = [*Gigartina teedii* (Roth) Lamouroux]

RHODYMENIALES

Rhodymeniaceae

Chondrus repens Grev.

Specimen 39; (De Toni IV p. 493) = [*Fauchea repens* (C. Agardh) Montagne]

Chrysymenia pinnulata Ag.

Prospetto 73; Dufour 110; (Ardiss. I p. 209) = [*Chrysymenia ventricosa* (Lamouroux) J. Agardh]

Halymenia nicaeensis Lamour.

Capraia 148; *Specimen* 55 = *Rhodymenia mediterranea* De Not.; Dufour 132 = *Rhodymenia Palmetta* Grev.; De Toni IV p. 514 = *R. Palmetta* (Esp.) Grev.; [?] (3; 38)

Lomentaria uvaria Duby

Specimen 41; Dufour 112 = *Chrysymenia uvaria* Wulf.; (Ardiss. I p. 210) = *C. uvaria* (L.) J. Ag.; [*Botryocladia botryoides* (Wulfen) J. Feldmann] (19)

* *Rhodymenia mediterranea* De Not.

Specimen 55; Dufour 132 e Ardiss. e Straff. 483 = *R. Palmetta* Grev.; [?] (3; 38)

Rhodymenia Palmetta Grev.

Specimen 56; Dufour 132; Ardiss. e Straff. 483; [?] (3; 38)

Since $\zeta = \partial w/\partial t$, we obtain by Gronwall's lemma:

$$\left\|\frac{\partial w}{\partial t}\right\|_{L^\infty(0,T;L^2(\Omega))} < M_1 K\left(F, \frac{\partial F}{\partial t}, \varphi, \frac{\partial \varphi}{\partial t}, \frac{\partial^2 \varphi}{\partial t^2}, w^0\right).$$

Hence (3.5) gives:

$$\left\|\frac{\partial w}{\partial t}\right\|_{L^2(0,T;W^{1,r}(\Omega))} < M_1 K\left(F, \frac{\partial F}{\partial t}, \varphi, \frac{\partial^2 \varphi}{\partial t^2}, w^0\right).$$

2) Since $\partial w/\partial t$ is in $L^2(0, T; W^{1,2}(\Omega))$ it also belogs to $L^2(0, T; H_s^1(\Omega))$ for $2/r < s < \frac{1}{2}$ where r is any positive large integer. Reapplying Theorem 2.1 and we have:

$$\|w\|_{L^2(0,T;H_0^2(\Omega))} < M_2 K\left(F, \frac{\partial F}{\partial t}, \varphi, \frac{\partial \varphi}{\partial t}, \frac{\partial^2 \varphi}{\partial t^2}, w^0\right).$$

3) We now show that $\partial w/\partial t$ is in $L^2(0, T; H_0^2(\Omega))$. In order to apply Theorem 3.1 to the problem (3.4) we shall now estimate $(\partial/\partial t)\{(\partial A/\partial t)\nabla w\cdot n\}$. We have:

$$\left\|\frac{\partial}{\partial t}\left\{\frac{\partial A}{\partial t}\nabla w\cdot n\right\}\right\|_{L^2(0,T;L^2(\partial\Omega^-))} < C\left\{\|\nabla v\|_{L^\infty(0,T;L^2(\partial\Omega^-))}\left\|\frac{\partial}{\partial t}(\nabla w)\right\|_{L^2(0,T;L^2(\partial\Omega^-))} + \right.$$
$$\left. + \left\|\frac{\partial}{\partial t}(\varrho^s\nabla v)\right\|_{L^2(0,T;L^2(\partial\Omega^-))}\|\varrho^{-s}\|_{L^\infty(0,T;L^2(\partial\Omega^-))}\|\nabla w\|_{L^\infty(0,T;L^2(\partial\Omega^-))}\right\}.$$

Since $0 < s < \frac{1}{2}$, we get:

$$\left\|\frac{\partial}{\partial t}\frac{\partial A}{\partial t}\nabla w\cdot n\right\|_{L^2(0,T;L^2(\partial\Omega^-))} < C\|v\|_{L^\infty(0,T;W^{1,r}(\Omega))}\left\|\frac{\partial w}{\partial t}\right\|_{L^2(0,T;W^{1,r}(\Omega))} +$$
$$+ \|w\|_{L^\infty(0,T;W^{1,r}(\Omega))}\left\|\frac{\partial}{\partial t}v\right\|_{L^2(0,T;H_0^2(\Omega))}.$$

Noting that $T^{\frac{1}{2}}\|v\|_{L^\infty(0,T;W^{1,r}(\Omega))}$, $T^{\frac{1}{2}}\|\partial v/\partial t\|_{L^2(0,T;H_0^2(\Omega))} < \delta$ and taking into account the result of the first part, we obtain:

$$\left\|\frac{\partial}{\partial t}\frac{\partial A}{\partial t}\nabla w\cdot n\right\|_{L^2(0,T;L^2(\partial\Omega^-))} < C\delta T^{-\frac{1}{2}}\left\|\frac{\partial w}{\partial t}\right\|_{L^2(0,T;H_0^2(\Omega))} +$$
$$+ K\left(F, \frac{\partial F}{\partial t}, \varphi, \frac{\partial \varphi}{\partial t}, \frac{\partial^2 \varphi}{\partial t^2}; w^0\right).$$

Applying Theorem 3.1 to (3.4) with $\zeta = \partial w/\partial t$ and we have:

$$(1 - C\delta T^{-\frac{1}{2}})\left\|\frac{\partial w}{\partial t}\right\|_{L^2(0,T;H_0^2(\Omega))} + \left\|\frac{\partial^2 w}{\partial t^2}\right\|_{L^2(0,T;L^2(\Omega))} < C_1 K\left(F, \frac{\partial F}{\partial t}, \varphi, \frac{\partial \varphi}{\partial t}, \frac{\partial^2 \varphi}{\partial t^2}; w^0\right).$$

The theorem is proved by taking δ with $0 < \delta < T^{\frac{1}{2}}/2C$.

* *Callithamnion cabellae* De Not.
Prospetto 27; Dufour 79, Ardiss. e Straff. 409 e Ardiss. I p. 68 = *C. subtilissimum* De Not.; De Toni IV p. 1347 = *Seirospora interrupta* (Sm.) Schmitz var. ? *subtilissima* (De Not.) De Toni; Preda p. 131 = *S. interrupta* (Sm.) Schmitz f. *subtilissima* (De Not.) De Toni; [*Seirospora interrupta* (Smith) Schmitz]

* *Callithamnion calcaratum* De Not.
Prospetto 31; Dufour 84 = *C. Borreri* Harvey forma; Ardiss. e Straff. 407 = *C. Borreri* Harv.; Ardiss. I p. 60 = *C. Borreri* (Smith) Harvey; De Toni IV p. 1304 e Preda p. 143 = [*Pleonosporium borreri* (Smith) Nägeli]

Callithamnion cruciatum Ag.
Specimen 113; Dufour 77 = *C. cruciatum* J. Ag.; Ardiss. e Straff. 418 e Ardiss. I p. 77: *C. cruciatum* Ag.; [*Antithamnion cruciatum* (C. Agardh) Nägeli]

* *Callithamnion flagelliferum* De Not.
Prospetto 30; Dufour 76; Ardiss. e Straff. 517 = *Spermothamnion flagelliferum* Ardiss. et Straff.; Ardiss. I p. 300, De Toni IV p. 1261 e Preda p. 164 = *S. Turneri* (Mert.) Aresch. c. *flagelliferum* Ardiss.; [*Spermothamnion repens* (Dillwyn) Rosenvinge ● var. *flagelliferum* (De Notaris) G. Feldmann-Mazoyer] (22)

Callithamnion granulatum Ag.
Capraia 189; *Specimen* 110; Dufour 85 = *C. granulatum* Ducl.; Ardiss. e Straff. 415: *C. granulatum* Ag.; Ardiss. I p. 73 = [*Callithamnion granulatum* (Ducluzeau) C. Agardh]

Callithamnion Giraudyi Solier [mscr.]
Prospetto 29; Dufour 75 = *C. variabile* Ag.; Ardiss. e Straff. 516 = *Spermothamnion Turneri* Aresch.; Ardiss. I p. 300 e De Toni IV p. 1260 = *S. Turneri* (Mert.) Aresch. b. *variabile* (Ag.) Ardiss.; [*Spermothamnion repens* (Dillwyn) Rosenvinge var. *variabile* (C. Agardh) G. Feldmann-Mazoyer] (22)

Callithamnion miniatum Mont.
Specimen 111; Dufour 84 e Ardiss. e Straff. 407 = *C. Borreri* Harv.; Ardiss. I p. 60 = *C. Borreri* (Sm.) Harv.; De Toni IV p. 1304 = [*Pleonosporium borreri* (Smith) Nägeli]

Callithamnion plumula Ag.
Specimen 112; Dufour 78; Ardiss. e Straff. 419; Ardiss. I p. 78 = *C. plumula* (Ellis) Ag.; (De Toni IV p. 1400) = *Antithamnion Plumula* (Ellis) Thuret; [*Pterothamnion plumula* (Ellis) Nägeli] (35; 14)

Callithamnion scopulorum Ag.
Mar. Lig. (2) 16; (Dufour 83); [*Aglaothamnion scopulorum* (C. Agardh) G. Feldmann-Mazoyer] (22)

Callithamnion seminudum Ag.
Capraia 190; (Dufour 84) = *C. Borreri* Harvey; (De Toni IV p. 1304) = [*Pleonosporium borreri* (Smith) Nägeli]

* *Callithamnion subtilissimum* De Not.
Prospetto 26; Dufour 79; Ardiss. e Straff. 409; Ardiss. I p. 67; De Toni IV

THEOREM 4.1: *Suppose all the hypotheses of Theorem 3.2 are satisfied. Then there exists a unique u, solution of* (4.1). *Moreover:*

$$\|u\|^2_{L^2(0,T;H^2_*(\Omega))}+\left\|\frac{\partial u}{\partial t}\right\|_{L^2(0,T;H^2_*(\Omega))}+\left\|\frac{\partial^2 u}{\partial t^2}\right\|_{L^2(0,T;L^2(\Omega))}<MK\left(f,\frac{\partial f}{\partial t},\varphi,\frac{\partial\varphi}{\partial t},\frac{\partial^2\varphi}{\partial t^2},u^0\right).$$

M is independent of v, δ and $0<\delta<T^{\frac{1}{2}}/2C$.

PROOF: Let u^k be the unique solution of (4.2) given by Theorem 3.2. Set: $U_k=u^k-u^{k-1}$ and it follows from Theorem 3.2 and Lemma 4.1 that:

$$\sum_{k=1}^{\infty}\left\{\|U_k\|_{L^2(0,T;H^2_*(\Omega))}+\left\|\frac{\partial}{\partial t}U_k\right\|_{L^2(0,T;H^2_*(\Omega))}+\left\|\frac{\partial^2}{\partial t^2}U_k\right\|_{L^2(0,T;H^2_*(\Omega))}\right\}<\infty.$$

Now a standard argument gives the existence of a solution u of (4.2) and the estimate follows from that of Theorem 3.2. This is clear from the estimates that the solution is unique.

We shall proceed to get further regularity properties for u. Consider the initial boundary problem:

$$(4.3)\quad\left|\begin{array}{ll}\dfrac{\partial z}{\partial t}-A\nabla\cdot A\nabla z-v\cdot A\nabla z=F \quad\text{in } \Omega\times(0,T), & z=0\\ & \text{on } \partial\Omega^+\times(0,T),\\ A\nabla z\cdot n=\psi=-\dfrac{\partial A}{\partial t}\nabla u\cdot n \qquad \text{on } \partial\Omega^-\times(0,T) & \\ z(\xi,0)=f(\xi,0)+\Delta u^0 \qquad \text{in } \Omega. & \end{array}\right.$$

with

$$F=\frac{\partial f}{\partial t}+\frac{\partial A}{\partial t}\nabla\cdot A\nabla u+A\nabla\frac{\partial A}{\partial t}\nabla u+\frac{\partial v}{\partial t}\cdot A\nabla u+v\cdot\frac{\partial A}{\partial t}\nabla u.$$

LEMMA 4.2: *Let F and ψ be as in* (4.3). *Then:*

1) $$\|F\|_{L^2(0,T;H^2_*(\Omega))}+\left\|\frac{\partial F}{\partial t}\right\|_{L^2(0,T;L^2(\Omega))}<C\delta\left(\|u\|_{L^2(0,T;H^2_*(\Omega))}+\right.$$
$$\left.+\left\|\frac{\partial u}{\partial t}\right\|_{L^2(0,T;H^2_*(\Omega))}+C\left\|\frac{\partial^2 f}{\partial t^2}\right\|_{L^2(0,T;L^2(\Omega))}+C\left\|\frac{\partial f}{\partial t}\right\|_{L^2(0,T;H^2_*(\Omega))}\right).$$

2) $$\|\psi\|_{L^2(0,T;H^{\frac{1}{2}}(\partial\Omega^-))}+\left\|\frac{\partial}{\partial t}\psi\right\|_{L^2(0,T;L^2(\partial\Omega^-))}<$$
$$<C\delta\left(\|u\|_{L^\infty(0,T;H^2_*(\Omega))}+\left\|\frac{\partial u}{\partial t}\right\|_{L^2(0,T;L^2(\Omega))}\right).$$

C is independent of v and of δ.

• *Griffithsia pumila* De Not.

Prospetto 33; Dufour 92; Ardiss. e Straff. 425; Ardiss. I p. 88, De Toni IV p. 1275 e Preda p. 157 = *G. irregularis* Ag.; [*Griffithsia flosculosa* (Ellis) Batters var. *irregularis* (C. Agardh) G. Feldmann-Mazoyer] (22)

Griffithsia secundiflora Ag. Iun.

Specimen 105; Dufour 179 e Ardiss. e Straff. 515 = *Bornetia secundiflora* Thuret; [*Bornetia secundiflora* (J. Agardh) Thuret]

Griffithsia sphaerica Ag.

Capraia 184; Ardiss. I p. 87 = *Griffithsia setacea* (Huds.) Ag. var.*sphaerica*; De Toni IV p. 1274 e Preda p. 156 = *Griffthsia setacea* (Ellis) Ag.; [*Griffithsia flosculosa* (Ellis) Batters var. *sphaerica* (Schousboe ex C. Agardh) G. Feldmann-Mazoyer] (22)

Spyridia clavulata J. Ag.

Prospetto 37; Dufour 103 e Ardiss. e Straff. 443 = *Centroceras clavulatum* Mont.; [*Centroceras clavulatum* (C. Agardh) Montagne]

Wrangelia penicillata Ag.

Capraia 188; Specimen 102; Dufour 177, Ardiss. e Straff. 511 e Ardiss. I p. 312 = *W. penicillata* J. Ag.; [*Wrangelia penicillata* C. Agardh]

Delesseriaceae

Aglaophyllum ocellatum Mont.

Specimen 58; Dufour 174 = *Nitophyllum punctatum* Harv.; Ardiss. e Straff. 499 = *N. punctatum* v. *ocellatum* J. Ag.; Ardiss. I p. 253 = *N. punctatum* (Stack.) Harv. var. *ocellatum* J. Ag.; Preda p. 274 = *N. punctatum* (Stackh.) Grev. α *ocellatum* J. Ag.; [*Nitophyllum punctatum* (Stackhouse) Greville]

Aglaophyllum laceratum uncinatum

Specimen 59; Dufour 173 e Ardiss. e Straff. 500 = *Nytophyllum uncinatum* J. Ag.; Ardiss. I p. 255 = *N. uncinatum* (Montg.) J. Ag.; (De Toni IV p. 650) = *N. uncinatum* (Turn.) J. Ag.; [*Acrosorium uncinatum* (Turner) Kylin] (13)

Delesseria hypoglossum Lamour.

Capraia 145; *Specimen* 60; Dufour 175; Ardiss. e Straff. 502; (De Toni IV p. 694) = *Hypoglossum woodwardii* Kütz.; [*Hypoglossum hypoglossoides* (Stackhouse) Collins et Harvey] (46)

Halymenia lacerata Duby

Capraia 150; (Dufour 173) = *Nitophyllum uncinatum* J. Ag.; (De Toni IV p. 663) = *N. laceratum* (Gmel.) Grev.: «*Specimina e mari Mediterraneo provenientia potius ad Nit. uncinatum pertinere videntur*»; [*Cryptopleura ramosa* (Hudson) Kylin ?] ? (7)

Dasyaceae

Dasya arbuscula Ag.

Specimen 101; Dufour 226 = *D. arbuscula* J. Ag.; Ardiss. e Straff. 584; *D. arbuscula* J. Ag. e *pro parte* 579: *D. Wurdemanni* Bail.; [*Dasya hutchinsiae*

$L^2(0, T; H_0^2(\Omega))$. *Then*:

$$\|A(v)\nabla u(\cdot, t)\|_{H_0^2(\Omega)} < C\left\{\|f(\cdot, t)\|_{H_0^2(\Omega)} + \left\|\frac{\partial}{\partial t}u(\cdot, t)\right\|_{H_0^2(\Omega)} + \right.$$
$$\left. + \|u(\cdot, t)\|_{H_0^2(\Omega)}\|v(\cdot, t)\|_{H_0^2(\Omega)}\right\}.$$

C is independent of v, δ and of t.

Proof: We return to the Eulerian coordinates via the transformation

$$x = X_v(\xi, t) = \xi + \int_0^t v(\xi, s)\, ds .$$

Set: $\hat{u}(x, t) = u(\xi, t) = u(X_v^{-1}(x, t), t)$. Then $\hat{u}$ is a solution of the initial boundary-value problem:

$$(5.1) \qquad \left|\begin{array}{ll} \dfrac{\partial}{\partial t}\hat{u} - \nabla_x\cdot\nabla_x\hat{u} = \hat{f} & \text{in } \Omega_t = X_v(\Omega),\ \hat{u}(x, 0) = u_0(x), \\ \hat{u} = 0 \quad \text{on } \partial\Omega_t^+, & \nabla_x\hat{u}\cdot n = 0 \quad \text{on } \partial\Omega_t^- . \end{array}\right.$$

1) We have:

$$(5.2) \qquad \frac{\partial\hat{u}}{\partial t} = \frac{\partial u}{\partial t} - v\cdot A\nabla u, \qquad \nabla_x\left(\frac{\partial\hat{u}}{\partial t}\right) = A\nabla_\xi\left(\frac{\partial u}{\partial t}\right) - A\nabla_\xi v\cdot A\nabla_\xi u .$$

From (5.2) and Lemma 1.1, we get:

$$\left\|\frac{\partial}{\partial t}\hat{u}(\cdot, t)\right\|_{H_0^2(\Omega_t)} < C\left\|\frac{\partial u}{\partial t}\right\|_{H_0^2(\Omega)} + \|v(\cdot, t)\|_{H_0^2(\Omega)}\|u(\cdot, t)\|_{H_0^2(\Omega)} .$$

Let v_ε be the spatial regularization of v and let T_ε be such that $(T_\varepsilon^{\frac{1}{2}} + T_\varepsilon^{\frac{1}{4}})\cdot\|v\|_{L^2(0, T; H_0^2(\Omega))} < \delta$. With v_ε instead of v, the curve $\partial\Omega_t^+$ is four times differentiable. From (5.1) and from Theorem 2.1, we obtain:

$$\|\hat{u}_\varepsilon(\cdot, t)\|_{H_0^2(\Omega_t)} < C\left\{\left\|\frac{\partial}{\partial t}\hat{u}_\varepsilon(\cdot, t)\right\|_{H_0^2(\Omega_t)} + \|\hat{f}(\cdot, t)\|_{H_0^2(\Omega_t)}\right\} <$$
$$< C\left\{\|f(\cdot, t)\|_{H_0^2(\Omega)} + \left\|\frac{\partial}{\partial t}u_\varepsilon(\cdot, t)\right\|_{H_0^2(\Omega)} + \|v_\varepsilon(\cdot, t)\|_{H_0^2(\Omega)}\|u_\varepsilon(\cdot, t)\|_{H_0^2(\Omega)}\right\}.$$

C is independent of ε and of $0 < t < T_\varepsilon$.

2) Set

$$v_\varepsilon^1(\eta, t) = v_\varepsilon(\zeta, t + T_\varepsilon), \qquad \eta = \zeta + \int_0^{T_\varepsilon} v_\varepsilon(\zeta, s)\, ds .$$

An easy calculation shows that there exists $T_\varepsilon^* > 0$ with

$$\{(T_\varepsilon^*)^{\frac{1}{2}} + (T_\varepsilon^*)^{\frac{1}{2}}\} \|v_\varepsilon^1\|_{L^\infty(0,T;H_0^2(\Omega))} < \delta .$$

As above, we have

$$\|\tilde{u}_\varepsilon(\cdot, t + T_\varepsilon)\|_{H_0^2(\Omega_{\varepsilon,v_\varepsilon})} < C\left\{\|f(\cdot, t + T_\varepsilon)\|_{H_0^2(\Omega)} + \left\|\frac{\partial}{\partial t} u_\varepsilon(\cdot, t + T_\varepsilon)\right\|_{H_0^2(\Omega)} + \right.$$
$$\left. + \|v_\varepsilon(\cdot, t + T_\varepsilon)\|_{H_0^2(\Omega)} \|u_\varepsilon(\cdot, t + T_\varepsilon)\|_{H_0^2(\Omega)}\right\}.$$

for $0 < t < T_\varepsilon^*$. Combining with the first step we get

$$\|\tilde{u}_\varepsilon(\cdot, t)\|_{H_0^2(\Omega_\varepsilon)} < C\left\{\|f(\cdot, t))\|_{H_0^2(\Omega)} + \left\|\frac{\partial}{\partial t} u_\varepsilon(\cdot, t)\right\|_{H_0^2(\Omega)} + \|v_\varepsilon(\cdot, t)\|_{H_0^2(\Omega)} \|u_\varepsilon(\cdot, t)\|_{H_0^2(\Omega)}\right\}.$$

for $0 < t < T_\varepsilon + T_\varepsilon^*$. After a finite number of steps we obtain the above estimate for $0 < t < T$.

Since $\nabla\tilde{u} = A\nabla u$, we have:

$$\|A\nabla_\xi u_\varepsilon)\cdot, t)\|_{H_0^2(\Omega)} < C_1\left\{\left\|\frac{\partial}{\partial t} u_\varepsilon\right\|_{H_0^2(\Omega)} + \|v_\varepsilon(\cdot, t)\|_{H_0^2(\Omega)} \|u_\varepsilon(\cdot, t)\|_{H_0^2(\Omega)} + \|f(\cdot, t)\|_{H_0^2(\Omega)}\right\}.$$

Let $\varepsilon \to 0$ and the estimate of the Lemma follows from that of Theorem 4.1.

REMARK: The above inequality does not imply that u is in $H_0^4(\Omega)$.

LEMMA 5.2: *Suppose all the hypotheses of Theorem 4.2 are satisfied. Then:*

$$\left\|\frac{\partial}{\partial t}\{A\nabla u\}(\cdot, t)\right\|_{H_0^2(\Omega)} < C\left\{\left\|\frac{\partial u}{\partial t}\right\|_{H_0^2(\Omega)} + \|u(\cdot, t)\|_{H_0^2(\Omega)} \|v(\cdot, t)\|_{W^{2,4}(\Omega)}\right\}.$$

C is independent of v, δ and of t.

PROOF: We have

$$\frac{\partial}{\partial t}(A\nabla u) = A\frac{\partial}{\partial t}\nabla u + \frac{\partial A}{\partial t}\nabla u .$$

Applying Lemma 1.1 and we get the state estimate.

LEMMA 5.3: *Suppose all the hypotheses of Theorem 4.2 are satisfied. Then:*

$$\left\|\frac{\partial^2}{\partial t^2}(A\nabla u)\right\|_{H_0^2(\Omega)} < C\left\|\frac{\partial^2 u}{\partial t^2}\right\|_{H_0^2(\Omega)} + \left\|\frac{\partial u}{\partial t}\right\|_{H_0^2(\Omega)} \|v(\cdot, t)\|_{H_0^2(\Omega)} +$$
$$+ \|\xi\|_{H_0^2(\Omega)}\left\|\frac{\partial v}{\partial t}\right\|_{H_0^2(\Omega)} + \|v\|^2_{H_0^2(\Omega)}.$$

C is independent of v, δ, t.

PROOF: We have:

$$\frac{\partial^2}{\partial t^2}(A\nabla u) = A\frac{\partial^2}{\partial t^2}\nabla u + 2\frac{\partial A}{\partial t}\frac{\partial}{\partial t}\nabla u + \frac{\partial^2 A}{\partial t^2}\nabla u .$$

Noting that $\|\varrho^i \nabla u\|_{W^{1,4}(\Omega)} \leq C\|u\|_{H_0^2(\Omega)}$, we obtain the stated result by applying Lemma 1.1 and the imbedding theorem.

LEMMA 5.4: *Suppose all the hypotheses of Theorem 4.2 are satisfied and that f is in $L^\infty(0, T; H_0^2(\Omega))$. Then there exists a non-empty interval $(0, T_*)$, independent of v, such that:*

(i) $(T_*^{\frac{1}{2}} + T_*^{\frac{1}{4}})\|w\|_{L^2(0,T_*;H_0^2(\Omega))}$, $(T_*^{\frac{1}{2}} + T_*^{\frac{1}{4}})\left\|\frac{\partial w}{\partial t}\right\|_{L^2(0,T_*;H_0^2(\Omega))} < \delta$,

(ii) $T_*^{\frac{1}{2}}\left\|\frac{\partial^2}{\partial t^2}w\right\|_{L^2(0,T_*;H_0^2(\Omega))} < \delta < \frac{1}{4}$; $w = A\nabla u$.

PROOF: Let $N(f, u_0)$ be the expression:

$$\|u_0\|_{H_0^2(\Omega)} + \|f\|_{L^\infty(0,T;H_0^2(\Omega))} + \left\|\frac{\partial f}{\partial t}\right\|_{L^2(0,T;H_0^2(\Omega))} + \left\|\frac{\partial^2 f}{\partial t^2}\right\|_{L^2(0,T;L^2(\Omega))} .$$

From the estimates of Theorem 4.1-4.2 and from Lemma 5.1 we obtain with $w = A\nabla u$.

$$(5.3)\qquad \|w\|_{L^2(0,t;H_0^2(\Omega))} < \begin{cases} CN(f, u_0)(1 + t^{\frac{1}{2}}\|v\|_{L^\infty(0,t;H(\Omega))}) , \\ C_1 N(f, u_0)(1 + \delta) . \end{cases}$$

From Lemma 5.2, we have:

$$\left\|\frac{\partial w}{\partial t}\right\|_{L^2(0,t;H_0^2(\Omega))} < CN(f, u_0)\{1 + \|v\|_{L^2(0,t;W^{1,4}(\Omega))}\} .$$

Thus,

$$(5.4)\qquad (t^{\frac{1}{2}} + t^{\frac{1}{4}})\left\|\frac{\partial w}{\partial t}\right\|_{L^2(0,t;H_0^2(\Omega))} < CN(f, u_0)\{(t^{\frac{1}{2}} + t^{\frac{1}{4}}) + \delta\} .$$

Also from Lemma 5.3 and Theorems 4.1-4.2 we get:

$$(5.5)\qquad T^{\frac{1}{2}}\left\|\frac{\partial^2}{\partial t^2}w\right\|_{L^2(0,t;H_0^2(\Omega))} < CN(f, u_0)t^{\frac{1}{2}} + $$

$$+ CN(f, u_0)(\delta + \delta^2) < CN(f, u_0)\{t^{\frac{1}{2}} + 2\delta\} .$$

It follows from (5.3)-(5.5) that for $N(f, u_0)$ sufficiently small there exists

$T_* > 0$ with $T_*^{\frac{1}{2}}$ being the positive root of:

$$CN(f, u_0)y^2 + yCN(f, u_0) + \delta\{2CN(f, u_0) - 1\} < 0 . \tag{5.6}$$

Then:

$$CN(f, u_0)\{T_*^{\frac{1}{2}} + T_*^{\frac{1}{2}} + 2\delta\} < \delta .$$

It is also clear that for $N(f, u_0)$ small, $T_*^{-\frac{1}{2}}\delta < 1/2C$ and T_* is independent of v.

The main result of the paper is the following theorem.

Theorem 5.1: *Let* $0 < \omega(P^{\pm}) < \pi/4$ *and let*

$$\|u_0\|_{H_s^4(\Omega)} + \|f\|_{L^\infty(0,T;H_s^2(\Omega))} + \left\|\frac{\partial f}{\partial t}\right\|_{L^2(0,T;H_s^2(\Omega))} + \left\|\frac{\partial^2 f}{\partial t^2}\right\|_{L^2(0,T;L^2(\Omega))}$$

be small with s as in Theorem 2.1. Suppose that $u_0 = 0 = f(\cdot, 0) + \Delta u_0 = 0$ on $\partial\Omega^+$, $\nabla u_0 \cdot n = 0$ on $\partial\Omega^-$. Then there exists a non empty interval $(0, T_*)$, a scalar function u in $L^\infty(0, T_*; H_s^4(\Omega))$ and a vector-function v in $L^2(0, T_*; H_0^3(\Omega))$ such that:

(i) $v = -A(v)\nabla_2 u$, $v\cdot n = 0$ *on* $\partial\Omega^- \times (0, T_*)$,

(ii) $$\begin{cases} \dfrac{\partial u}{\partial t} - v\cdot A\nabla u - A\nabla\cdot A\nabla u = f & \text{in } \Omega\times(0, T_*) , \\ u = 0 \quad \text{on } \partial\Omega^+\times(0, T_*) , & A\nabla u\cdot n = 0 \quad \text{on } \partial\Omega^-\times(0, T_*) , \\ u(\xi, 0) = u_0 \quad \text{in } \Omega . \end{cases}$$

Moreover $\{\partial u/\partial t, \partial^2 u/\partial t^2\}$ is in $L^2(0, T_*; H_s^3(\Omega)) \times (0, T_*; H_s^1(\Omega))$ with $\{\partial v/\partial t, \partial^2 v/\partial t^2\}$ in $L^2(0, T_*; H_s^2(\Omega)(\times L^2(0, T_*; H_s^n(\Omega))$.

Furthermore: $\{\bar{u}(x, t), \bar{\varphi}(x, t)\}$ is a solution of (1.1)-(1.2) with

$$\bar{u}(x, t) = u(\xi, t) = u(X_v^{-1}(x, t), t) , \qquad \bar{\varphi} = \varphi_0(X_v^{-1}(x, t))$$

where $X_v(\xi, t) = \xi + \int_0^t v(\xi, s)\, ds$.

Proof: 1) Let

$$\mathfrak{B} = v: \Big\{(v_1, v_2),\ v\cdot n = 0 \text{ on } \partial\Omega^-,\ (T_*^{\frac{1}{2}} + T_*^{\frac{1}{2}})\|v\|_{L^2(0,T;H_s^3(\Omega))} < \\ < \delta(T_*^{\frac{1}{2}} + T_*^{\frac{1}{2}})\left\|\frac{\partial v}{\partial t}\right\|_{L^2(0,T^*;H_s^2(\Omega))} < \delta,\ T_*^{\frac{1}{2}}\left\|\frac{\partial^2 v}{\partial t^2}\right\|_{L^2(0,T_*;H_s^1(\Omega))} < \delta\Big\}.$$

T^* as in Lemma 5.4.

It is clear that $\mathfrak{B}$ is a closed convext subset of $L^2(0, T_*; L^2(\Omega))$. For a given vector v in $\mathfrak{B}$ we have a unique solution u of (4.1) and moreover $-A(v)\nabla u$ is in $\mathfrak{B}$. Let $\mathfrak{T}$ be the mapping of $\mathfrak{B}$ into $L^2(0, T_*; L^2(\Omega))$ defined by: $\mathfrak{T}(v) = -A(v)\nabla u$ where u is the unique solution of (4.1). It follows from Lemma 5.4 that $\mathfrak{T}$ maps $\mathfrak{B}$ into $\mathfrak{B}$.

2) We show that $\mathfrak{T}$ is compact. Suppose that $\{v_n\}$ is in $\mathfrak{B}$. From Aubin's theorem we get a subsequence, denoted again by $\{v_n\}$, such that:

$$v_n \to v \text{ in } L^2(0, T_*; H_0^1(\Omega)) \qquad \text{and weakly in } L^2(0, T_*; H_0^2(\Omega)),$$

$$\frac{\partial v_n}{\partial t} \to \frac{\partial v}{\partial t} \qquad \text{weakly in } L^2(0, T_*; H_s^2(\Omega)),$$

$$\frac{\partial^2}{\partial t^2} v_n \to \frac{\partial^2}{\partial t^2} v \qquad \text{weakly in } L^2(0, T_*; H_s^0(\Omega)).$$

From the estimates of Theorems 4.1-4.2 and from Aubin's theorem we have:

$$\left\{u_n, \frac{\partial}{\partial t} u_n\right\} \to \left\{u, \frac{\partial u}{\partial t}\right\} \qquad \text{in } L^2(0, T_*; H_0^2(\Omega)) \times L^2(0, T_*; H_0^1(\Omega))$$

and weakly in $L^2(0, T_*; H_s^3(\Omega)(\times L^2(0, T_*; H_s^2(\Omega))$ with

$$\frac{\partial^2}{\partial t^2} u_n \to \frac{\partial^2}{\partial t^2} u \qquad \text{weakly in } L^2(0, T_*; H_s^1(\Omega)).$$

It is easy to check that:

(i) $w_n = A(v_n)\nabla u_n \to w = A(v)\nabla u$ in $L^2(0, T_*; L^2(\Omega))$,

(ii) $\frac{\partial u}{\partial t} - v \cdot A(u)\nabla u - A(v)\nabla \cdot A(v)\nabla u = f$ in $\Omega \times (0, T_*)$,

with $A(v)\nabla u \cdot n = 0$ on $\partial\Omega^- \times (0, T_*)$, $u = 0$ on $\partial\Omega^+ \times (0, T_*)$ and $u(\xi, 0) = u_0$. Thus, $\mathfrak{T}(v_n) \to \mathfrak{T}(v_1$ in $L^2(0, T_*; L^2(\Omega))$.

3) A proof as above shows that $\mathfrak{T}$ is continuous. By the Schauder fixed poin theorem there exists v in such that $\mathfrak{T}(v) = v = -A(v)\nabla u$.

With $\tilde{u}(x, t) = u(\xi, t) = u(X_\tau^{-1}(x, t), t)$ and $\tilde{\varphi}(x, t) = \varphi_0(\xi) = \varphi_0(X_\tau^{-1}(x, t))$, it is easy to see as mentioned in Section 1 that $\{\tilde{u}, \tilde{\varphi}\}$ is a solution of (1.1)-(1.2).

The theorem is proved.

Remarks: 1) With the data $\{f, u_0\}$ in $L^2(0, T; H_0^2(\Omega)(\times H_0^2(\Omega)$ the intersection points $P^{\pm}$ of the free and fixed boundaries are fixed. Indeed u and thus v are in the weighted Sobolev space $L^2(0, T; H_s^2(\Omega))$ and $L^2(0, T; H_0^2(\Omega))$ respectively with $0 < s < 1$. To consider the case of moving intersection points, we take the data in $L^2(0, T; H_*^s(\Omega)) \times H_*^s(\Omega)$ for large positive s and try to have u, f in $H_*^s(\Omega)$. The estimates are then mush more involved.

2) The method presented in the paper is applicable to the two-phase Stefan-type problem when there is no latent heat.

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