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WOJCIECH ZYGMUNT (*)

A Note concerning the Scorza-Dragoni's Type Property of the Compact Multivalued Multifunction (**)

ABSTRACT. — The equivalence between V. V. Filippov's condition, the Scorza-Dragoni's type property and the product measurability for the compact valued multifunction $F(t, x)$ is proved.

Una Nota sulla cosiddetta proprietà di Scorza-Dragoni per multifunzioni a valori compatti

RIASSUNTO. — In questa Nota è dimostrata l'equivalenza tra la condizioni di V. V. Filippov, la proprietà di Scorza-Dragoni e la prodotto-misurabilità di una multifunzione $F(t, x)$ a valori compatti.

In the present note we show that the condition for the compact valued multifunction introduced and studied recently by V. V. Filippov [F1, F2] is equivalent to the product measurability of this multifunction. Thereby we give the answer to the question put by ourselves in [Z] if the compact valued and product measurable multifunction has the Scorza-Dragoni's type property.

We start with recalling the notions which play the main role in our considerations. So, given two topological spaces $\mathbb{X}$, $\mathbb{Y}$, a multifunction Φ from $\mathbb{X}$ into $\mathbb{Y}$ is said to be upper continuous if, for every open set $A \subset \mathbb{Y}$, the set $\Phi^+(A) = \{x \in \mathbb{X}: \Phi(x) \subset A\}$ is open in $\mathbb{X}$. If, moreover, $(\mathbb{X}, \Sigma)$ is a measurable space, the multifunction Φ is said to be Σ-measurable (resp. weakly Σ-measurable) if, for every closed (resp. open) set $A \subset \mathbb{Y}$, the set $\Phi^-(A) = \{x \in \mathbb{X}: \Phi(x) \cap A \neq \emptyset\}$ is Σ-measurable, i.e. $\Phi^-(A) \in \Sigma$. It is known (see for instance [H, Theorem 3.1]) that in the case of the compact valued multifunction the notions of measurability and weak measurability are equivalent.

(*) Indirizzo dell'Autore: Instytut Matematyki, Uniwersytet Marii Curie-Sklodowskiej Plac Marii Curie-Sklodowskiej 1, 20-031 Lublin (Polonia).

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Henceforth we accept the following notations:

— T is a Hausdorff compact metric space,
— μ is a Borel σ-finite regular and complete measure defined on a σ-field $\mathcal{A}$ of subsets of T,
— X is a separable complete metric space,
— $\mathcal{B}(X)$ is a σ-field of Borel subsets of X,
— $\mathcal{A}\times\mathcal{B}(X)$ is the product σ-field on $T\times X$,
— Y is a separable metric space,
— Comp Y is a family of all nonempty compact subsets of Y.

Let us consider a multifunction $F\colon T\times X\to\operatorname{Comp} Y$. We say that F belongs to the class SD^* (i.e. it has the Scorza-Dragoni's type property in upper continuous case) if for every $\varepsilon>0$ there exists a closed subset T_ε of T, with $\mu(T\setminus T_\varepsilon)<\varepsilon$, such that F is upper continuous on $T_\varepsilon\times X$ in both variables jointly. We shall also say that the multifunction F is C^* type if

(i) $F(\cdot, x)$ is $\mathcal{A}$-measurable for each fixed $x\in X$, and

(ii) $F(t,\cdot)$ is upper continuous for each fixed $t\in T$.

The below result has been proved by V. V. Filippov in [F1].

Theorem 1: Let a multifunction $F\colon T\times X\to\operatorname{Comp} Y$ be C^* type. Then $F\in SD^*$ if and only if F satisfies the following condition:

(**) for every open set $U\subset X$ and for every open set $V\subset Y$ the set $M^*(U,V)=\{t\in T\colon F(t,U)\subset V\}$ is $\mathcal{A}$-measurable, i.e. $M^*(U,V)\in\mathcal{A}$.

We shall show that V. V. Filippov's condition (**) is equivalent to the product measurability of F. Namely, we have

Theorem 2: Let a multifunction $F\colon T\times X\to\operatorname{Comp} Y$ be a C^* type. Then F is $\mathcal{A}\times\mathcal{B}(X)$-measurable if and only if F satisfies V. V. Filippov's condition (**).

Proof: Assume that F is $\mathcal{A}\times\mathcal{B}(X)$-measurable and take open sets $U\subset X$ and $V\subset Y$. Put $W=Y\setminus V$. In view of the assumptions, the set $F^-(W)\cap T\times U$ belongs to the σ-field $\mathcal{A}\times\mathcal{B}(X)$. Then using the projection theorem [CV, Theorem III.23] we have

$$\{t\in T\colon F(t,U)\cap W\neq\emptyset\}=$$
$$=\Big\{t\in T\colon \bigvee_{x\in U} F(t,x)\cap W\neq\emptyset\Big\}=\operatorname{pr}_T\left(F^-(W)\cap T\times U\right)\in\mathcal{A},$$

where pr_T denotes the projection $T\times X$ onto T. Hence

$$M^*(U,V)=T\setminus\{t\in T\colon F(t,U)\cap W\neq\emptyset\}\in\mathcal{A}.$$

Conversely, assume now that, for every open set $U \subset X$ and for every open set $V \subset Y$, the set $M^*(U, V)$ belongs to $\mathcal{A}$. For each closed set $M \subset Y$ define a relation $\Phi_M : T \to\!\!\!\!\to X$ as follows: $\Phi_M(t) = \{x \in X : F(t, x) \cap W \neq \emptyset\}$. Since the multifunction F is upper continuous, the set $\Phi_M(t)$ is closed in X for each $t \in T$. We claim that thus defined closed valued relation is weakly $\mathcal{A}$-measurable. Indeed, for every open set $U \subset X$ we have

$$\Phi_M^-(U) = \{t \in T : \Phi_M(t) \cap U \neq \emptyset\} = \left\{t \in T : \bigvee_{x \in U} F(t, x) \cap W \neq \emptyset\right\} =$$

$$= \{t \in T : F(t, U) \cap W \neq \emptyset\} = T \setminus M^*(U, V) \in \mathcal{A},$$

where $V = Y \setminus W$. Hence, by Himmelberg's theorem [H, Theorem 3.5], it follows that Φ_M has a measurable graph, i.e. $\operatorname{Gr} \Phi_M \in \mathcal{A} \times \mathcal{B}(X)$. But

$$\operatorname{Gr} \Phi_M = \{(t, x) \in T \times X : x \in \Phi_M(t)\} =$$

$$= \{(t, x) \in T \times X : F(t, x) \cap W \neq \emptyset\} = F^-(W).$$

Since the closed set W is arbitrary, which means that F is $\mathcal{A} \times \mathcal{B}(X)$-measurable. The proof of the theorem is completed.

We close this note with a corollary which combines the above Theorems 1 and 2 and Theorem 4 (ii) from [Z].

Corollary: For a C^* type compact valued multifunction $F : T \times X \to\!\!\!\!\to \operatorname{Comp} Y$ the following statements are equivalent:

(*a*) $F \in SD^*$,

(*b*) F satisfies V. V. Filippov's condition (**),

(*c*) F is weakly $\mathcal{A} \times \mathcal{B}(X)$-measurable.

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