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On the Approximation of Semicontinuous Scorza-Dragonians by the Multifunctions of Carathéodory Type (**)

ABSTRACT. — Two theorems on the monotone approximation to a multifunction, measurable in first variable and semicontinuous in second variable are given.

Sulle funzioni semicontinue di Scorza-Dragoni e le loro approssimazioni mediante multifunzioni di Carathéodory

RIASSUNTO. — In questa Nota sono dimostrati due teoremi su approssimazioni monotone di una multifunzione di due variabili misurabili rispetto alla prima e semicontinue rispetto alla seconda.

INTRODUCTION

The well known Baire's theorem on the monotone approximation to a semicontinuous function by continuous functions asserts that a real valued function f of one variable only is lower (resp. upper) semicontinuous if and only if there exists a nondecreasing (resp. nonincreasing) sequence of continuous functions which pointwise converges to f. There exist equivalents for multifunctions of this theorem (see for example Aseev [2] and de Blasi [7]). On the other hand, if f is a real valued function of two variables, measurable in first and lower (resp. upper) semicontinuous in second variable, then it turns out that f will be a limit of a nondecreasing (resp. nonincreasing) sequence of Carathéodory type functions if and only if f has the so called Scorza-Dragoni's type property. This has been proved by Zygmunt [15]. The aim of the present paper is to give a set-valued analog of the above fact for compact convex valued multifunctions.

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2. - Preliminaries

We assume the reader is familiar with such notions concerning multifunctions as closedness, topologically or metrically lower and upper semicontinuity, Σ-measurability and weakly Σ-measurability with respect to some σ-field Σ. In case of need the necessary information can be found in Berge [3], de Blasi-Myjak [8] and Himmelberg [9]. Since for a compact valued multifunction topologically semicontinuity coincides with metrical one and weakly Σ-measurability coincides with Σ-measurability, often we shall simply say «lower» or «upper semicontinuous» and «measurable». Furthermore, the semicontinuity and measurability of real valued functions are understood in the usual sense.

Throughout the paper T denotes a metric compact Hausdorff space with the Borel σ-finite regular and complete measure μ defined on a σ-field $\mathcal{A}$ of subsets of T and X denotes a separable complete metric space. By $\mathcal{B}(X)$ we denote the σ-field of Borel subsets of X and by $\mathcal{A}\times\mathcal{B}(X)$—the product σ-field on $T\times X$. $\mathbf{R}^q$ ($\mathbf{R}=\mathbf{R}^1$) is a q-dimensional Euclidean space with the scalar product $a, b \in \mathbf{R}^q$ denoted by $a \circ b$ and with a norm denoted by $\|a\| = (a \circ a)^{\frac{1}{2}}$. $K(a, r)$ denotes the open ball centered at $a \in \mathbf{R}^q$ and with radius r. If $A \subset \mathbf{R}^q$ is a subset then $\bar{A}$ (resp. $\overline{\text{co}}\, A$) denotes the closure (resp. the closed convex hull) of A. The function $s(\cdot\,; A)\colon \mathbf{R}^q \to [0, +\infty]$ defined by $s(p; A) = \sup\{p \circ y\colon y \in A\}$, $p \in \mathbf{R}^q$, is said to be the support function of the set A. Finally we denote by $\text{Cl}\, \mathbf{R}^q$ (resp. $\text{Conv}\, \mathbf{R}^q$) the family of all nonempty closed (resp. compact convex) subsets of $\mathbf{R}^q$ and we assume that $\text{Conv}\, \mathbf{R}^q$ is endowed with Hausdorff metric d.

Now we are going to define some classes of functions and multifunctions which are of importance in what follows. So, we say that a function $f\colon T\times X \to \mathbf{R}$ is lower Carathéodory's type function (resp. upper Carathéodory's type function, Carathéodory's type function) if

(i) $f(\cdot, x)$ is $\mathcal{A}$-measurable for each $x \in X$,

(ii) $f(t, \cdot)$ is lower semicontinuous (resp. upper semicontinuous, continuous) for each $t \in T$.

At this point let us notice that the concept of Carathéodory type function is also well posed for every function with values in topological space.

Next, we say that a function $f\colon T\times X \to \mathbf{R}$ belongs to the class SD_* (resp. SD^*, SD) if f is lower Carathéodory's type function (resp. upper Carathéodory type function, Carathéodory's type function) and for every $\varepsilon > 0$ there exists a closed subset T_ε of T, with $\mu(T \setminus T_\varepsilon) < \varepsilon$, such that the restriction $f|_{T_\varepsilon \times X}$ is lower semicontinuous (resp. upper semicontinuous, continuous) in both variables jointly. For a multifunction $F\colon T\times X \to \text{Conv}\, \mathbf{R}^q$ we introduce the same classification.

3. - Some auxiliary lemmas

Lemma 1: A function $f\colon T\times X\to \mathbf{R}$ belongs to SD^* if and only if there exists a nonincreasing sequence $\{f_n\}$ of Carathéodory type functions $f_n\colon T\times X\to \mathbf{R}$ which pointwise converges to f.

Proof: See Zygmunt [15, Theorem 3].

Lemma 2: Let $A\in \operatorname{Conv}\mathbf{R}^q$ and let $\{p_1, p_2, \ldots\}$ be a dense set in the unit sphere of $\mathbf{R}^q$. Then

$$A=\bigcap_{n=1}^{\infty}\{y\colon p_n\circ y\leq s(p_n; A)\}\,, \qquad \overline{K(A,\varepsilon)}=\bigcap_{n=1}^{\infty}\{y\colon p_n\circ y\leq s(p_n; A)+\varepsilon\}\,.$$

Proof: Easily follows from the best known properties of the support function $s(\cdot\,; A)$ (see for example Blagodatskih ... [4]) which is continuous on $\mathbf{R}^q$ (see Artstein [1, Lemma 3.1]).

Lemma 3: Let $p\in \mathbf{R}^q$ and let a multifunction $F\colon T\times X\to \operatorname{Conv}\mathbf{R}^q$ belongs to SD^*. Then the function $f_p\colon T+X\to \mathbf{R}$ defined by $f_p(t,x)=$ $=s(p; F(t,x))$ belongs to SD^*.

Proof: We observe that, for every $r\in \mathbf{R}$,

$$\{(t,x)\in T\times X\colon f_p(t,x)>r\}=\{(t,x)\in T\times X\colon F(t,x)\cap\{y\in \mathbf{R}^q\colon p\circ y>r\}\neq\emptyset\}\,.$$

Now it is not difficult to deduce that $f_p\in SD^*$.

Lemma 4: If multifunctions $F_i\colon T\times X\to \operatorname{Cl}\mathbf{R}^q$, $i=1,2,\ldots,n$, $n\in \mathbf{N}$, are closed and a multifunction $G\colon T\times X\to \operatorname{Conv}\mathbf{R}^q$ is upper semicontinuous, then the multifunction $G\cap\bigcap_{i=1}^{n}F_i\colon T\times X\to \operatorname{Conv}\mathbf{R}^q$ is upper semicontinuous.

Proof: By virtue of Berge [3, Chapt. VI, § 1, Theorems 5 and 6] the multifunction $G\cap\bigcap_{i=1}^{n}F_i$ is topologically and hence metrically upper semicontinuous. Thus it is simply upper semicontinuous.

Lemma 5: If a multifunction $F\colon T\times X\to \operatorname{Conv}\mathbf{R}^q$ belongs to SD^*, then there exists a Carathéodory type function $r\colon T\times X\to [0,\infty)$ such that

$$F(t,x)\subset K(\theta, r(t,x)+1) \qquad \text{for each } (t,x)\in T\times X\,,$$

where θ denotes the origin of $\mathbf{R}^q$.

PROOF: Let us put $\varrho(t,x)=\sup\{\|y\|: y\in F(t,x)\}$, $(t,x)\in T\times X$. Thus defined function $\varrho\colon T\times X\to[0,\infty)$ belongs to SD^*. To see this, notice that for each $a\in\mathbf{R}$ we have

$$\{(t,x)\in T\times X: \varrho(t,x)>a\}=\{(t,x)\in T\times X: F(t,x)\cap K'(\theta,a)\neq\emptyset\}$$

where

$$K'(\theta,a)=\begin{cases}\mathbf{R}^q\setminus K(\theta,a) & \text{if } a>0\,,\\ \mathbf{R}^q & \text{if } a\leq 0\,.\end{cases}$$

Thus, in view of Lemma 1, there is a Carathéodory's type function $r\colon T\times X\to\mathbf{R}$ which satisfies, for each $(t,x)\in T\times X$, the inequality $\varrho(t,x)<<r(t,x)$. Then, obviously, $F(t,x)\subset K(\theta, r(t,x)+1)$.

4. - MAIN THEOREMS

THEOREM 1: Let a multifunction $F\colon T\times X\to\operatorname{Conv}\mathbf{R}^q$ be given. Then the following two statements are equivalent:

(a) $F\in SD_*$,

(b) there exists a sequence $\{F_n\}$ of Carathéodory type multifunctions $F_n\colon T\times X\to\operatorname{Conv}\mathbf{R}^q$ satisfying, for each $(t,x)\in T\times X$, the conditions:

(b_1) $F_n(t,x)\subset F(t,x)$ for $n=1,2,\ldots,$

(b_2) $F_n(t,x)\subset F_{n+1}(t,x)$ for $n=1,2,\ldots,$

(b_3) $F(t,x)=\lim\limits_{n\to\infty}F_n(t,x)=\overline{\bigcup\limits_{n=1}^{\infty}F_n(t,x)}$.

(The limit

$$A=\lim_{n\to\infty}A_n\,,$$

where $A, A_n\in\operatorname{Conv}\mathbf{R}^q$, $n\in\mathbf{N}$, means $\lim\limits_{n\to\infty}d(A_n,A)=0$).

PROOF: (b) $\Rightarrow$ (a). By Himmelberg [9, Theorem 2.3] $F(\cdot,x)$ is weakly measurable for each $x\in X$ and by Hukuhara [10, Propositions 1.2 and 7.2] $F(t,\cdot)$ is lower semicontinuous as the limit of a nondecreasing sequence $\{F_n(t,\cdot)\}$ of continuous multifunctions. Now let's fix $\varepsilon>0$. Since every Carathéodory's type compact convex valued multifunction has the Scorza-Dragoni property (see Brunovsky [5, Theorem 2.5]) we can obtain a sequence $\{T_n\}$ of closed sets such that, for $n=1,2,\ldots,$ $T_n\subset T_{n-1}$ where $T_0=T_1$, $\mu(T_{n-1}\setminus T_n)<(\frac{1}{2})^n\varepsilon$ and the restriction $F_n|_{T_n\times X}$ is continuous in both variables jointly. Then the set $T_\varepsilon = \bigcap_{n=1}^{\infty} T_n$ is closed, $T_\varepsilon \subset T$, $\mu(T \setminus T_\varepsilon) < \varepsilon$ (see Zygmunt [15]) and each multifunction F_n is continuous in both variables jointly on $T_\varepsilon \times X$. Hence the multifunction $F = \overline{\bigcup_{n=1}^{\infty} F_n}$ is lower semicontinuous on $T_\varepsilon \times X$. Thus $F \in SD_*$.

$(a) \Rightarrow (b)$. Since any multifunction belonging to SD_* is weakly $\mathcal{A} \times \mathcal{B}(X)$-measurable (see Zygmunt [16, Theorem 3]), by a Rybiński's result [14, Theorem 3] (see also Kim, ... [12, Lemma 5.2]) there is an infinite sequence $\{f_n\}$ of Carathéodory type selections $f_n\colon T \times X \to \mathbf{R}^q$ of F satisfying, for each $(t, x) \in T \times X$, the equality

$$F(t, x) = \overline{\bigcup_{n=1}^{\infty} \{f_n(t, x)\}}.$$

For every $n \in \mathbf{N}$, let $F_n\colon T \times X \to \operatorname{Conv} \mathbf{R}^q$ be the multifunction defined by

$$F_n(t, x) = \overline{\operatorname{co}}\,\{f_1(t, x), f_2(t, x), \ldots, f_n(t, x)\}.$$

Clearly, for each $t \in T$ $F_n(t, \cdot)$ is continuous and, for each $x \in X$, by Himmelberg [9, Theorem 9.1] $F_n(\cdot, x)$ is weakly measurable. Thus F_n is a Carathéodory's type multifunction. Obviously such a defined sequence $\{F_n\}$ satisfies the conditions (b_1) and (b_2) while the condition (b_3) follows from Hukuhara [10, Proposition 1.2]. This complete the proof of Theorem.

Theorem 2: Let a multifunction $F\colon T \times X \to \operatorname{Conv} \mathbf{R}^q$ be given. Then the following two statements are equivalent:

(a) $F \in SD^*$,

(b) there exists a sequence $\{F_n\}$ of Carathéodory's type multifunctions $F_n\colon T \times X \to \operatorname{Conv} \mathbf{R}^q$ satisfying, for each $(t, x) \in T \times X$, the conditions:

(b_1) $F(t, x) \subset F_n(t, x)$ for $n = 1, 2, \ldots$,

(b_2) $F_{n+1}(t, x) \subset F_n(t, x)$ for $n = 1, 2, \ldots$,

(b_3) $F(t, x) = \lim_{n\to\infty} F_n(t, x) = \bigcap_{n=1}^{\infty} F_n(t, x)$.

Proof: $(b) \Rightarrow (a)$. Similarly to the proof of part $(b) \Rightarrow (a)$ of the previous Theorem 1, employing Himmelberg's result [9, Theorem 3.5 (iii)] and Hukuhara's result [10, Proposition 1.2 and 7.1] we show that $F \in SD^*$.

$(a) \Rightarrow (b)$. Let $\{p_1, p_2, \ldots\}$ be a dense subset of a unit sphere in $\mathbf{R}^q$. Let $f_i\colon T \times X \to \mathbf{R}$, $i = 1, 2, \ldots$, be a function defined by the formula $f_i(t, x) = s(p_i; F(t, x))$, $(t, x) \in T \times X$. By Lemma 3 every f_i belongs to SD^* and, hence, by Lemma 1, there exist sequences $\{f_{i,j}\}$ of Carathéodory type functions $f_{i,j}\colon T\times X\to \mathrm{R}$ such that

$$f_i(t,x)<\ldots<f_{i,j+1}(t,x)<f_{i,j}(t,x)<\ldots<f_{i,1}(t,x)$$

and

$$\lim_{j\to\infty} f_{i,j}(t,x)=f_i(t,x) \qquad \text{for } i=1,2,\ldots,\ (t,x)\in T\times X.$$

Put

$$H_i(t,x)=\{y\in \mathrm{R}^q\colon p_i\circ y<f_i(t,x)\},$$
$$H_{i,j}(t,x)=\{y\in \mathrm{R}^q\colon p_i\circ y<f_{i,j}(t,x)+1/j\}, \qquad i,j=1,2,\ldots,\ (t,x)\in T\times X.$$

It is easy to verify that such defined multifunctions $H_i\colon T\times X\to \mathrm{Cl}\,\mathrm{R}^q$ and $H_{i,j}\colon T\times X\to \mathrm{Cl}\,\mathrm{R}^q$ are of Carathéodory type and have the following properties:

$$F(t,x)\subset (K(Ft,x),1/j)\subset H_{i,j}(t,x), \qquad i,j=1,2,\ldots,\ (t,x)\in T\times X,$$
$$H_{i,j+1}(t,x)\subset H_{i,j}(t,x), \qquad i,j=1,2,\ldots,\ (t,x)\in T\times X,$$
$$F(t,x)=\bigcap_{i=1}^{\infty} H_i(t,x)=\bigcap_{j=1}^{\infty}\Big(\bigcap_{i=1}^{j} H_{i,j}(t,x)\Big), \qquad (t,x)\in T\times X.$$

Let, further, $r\colon T\times X\to[0,\infty)$ be a function defined as in Lemma 5. Then the multifunction $G\colon T\times X\to \mathrm{Conv}\,\mathrm{R}^q$ given by formula

$$G(t,x)=K(\theta, r(t,x)+1)$$

is obviously of Carathéodory type. Now define, for each $n\in \mathrm{N}$, the multifunction $F_n\colon T\times X\to \mathrm{Conv}\,\mathrm{R}^q$ as follows

$$F_n(t,x)=G(t,x)\cap\bigcap_{i=1}^{n} H_{i,n}(t,x), \qquad (t,x)\in T\times X.$$

We claim that $\{F_n\}$ is the required sequence of Carathéodory type multifunctions. Indeed, first of all, by standard argument we easily obtain that, for each $(t,x)\in T\times X$,

(i) $F(t,x)\subset F_n(t,x)$ for $n=1,2,\ldots,$

(ii) $F_{n+1}(t,x)\subset F_n(t,x)$ for $n=1,2,\ldots,$

(iii) $\bigcap_{n=1}^{\infty} F_n(t,x)=\bigcap_{i=1}^{\infty} H_i(t,x)=F(t,x)$.

Further we conclude that, in view of Himmelberg [9, Theorem 4.1], $F_n(\cdot, x)$ is weakly $\mathcal{A}$-measurable for each $x \in X$, $n \in \mathbf{N}$, and, by Lemma 4, $F_n(t, \cdot)$ is upper semicontinuous for each $t \in T$, $n \in \mathbf{N}$. Next, since $F_n(t, x)$ has a nonempty interior (namely, it is $F(t,x) \subset \operatorname{Int} H_{i,n}(t, x)$, $i = 1, 2, \ldots, n$) it follows (see Lechicki, ... [13, Theorem B]) that $F_n(t, \cdot)$ is lower semicontinuous for each $t \in T$, $n \in \mathbf{N}$. Thus $F_n(t, \cdot)$ is continuous. Finally we see that $F_n\colon T \times X \to \operatorname{Conv} \mathbf{R}^p$ is a Carathéodory's type multifunction for $n = 1, 2, \ldots$ and $(t, x) \in T \times X$. This completes the proof of Theorem 2.

Remark: A result closely related to the above theorem, part $(a) \Rightarrow (b)$, was first given by Jarnik and Kurzweil [11, Theorem 2.5]. Namely, they proved that if a multifunction $F\colon T \times X \to \operatorname{Conv} \mathbf{R}^p$ belongs to SD^*, then there exists a sequence $\{F_n\}$ of Carathéodory type multifunctions and a measurable set $Z \subset T$ so that $\mu(Z) = 0$,

$$F_{n+1}(t, x) \subset F_n(t, x), \qquad n = 1, 2, \ldots,$$

$$F(t, x) = \bigcap_{n=1}^{\infty} F_n(t, x) \qquad \text{for } (t, x) \in (T \setminus Z) \times X,$$

$$F_n(t, x) = \{\theta\} \qquad \text{for } (t, x) \in Z \times X, \; n = 1, 2, \ldots .$$

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