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On the Approximation of Semicontinuous Scorza-Dragonians by the Multifunctions of Carathéodory Type (**)}

ABSTRACT. — Two theorems on the monotone approximation to a multifunction, measurable in first variable and semicontinuous in second variable are given.

Sulle funzioni semicontinue di Scorza-Dragonni e le loro approssimazioni mediante multifunzioni di Carathéodory

RIASSUNTO. — In questa Nota sono dimostrati due teoremi su approssimazioni monotone di una multifunzione di due variabili misurabili rispetto alla prima e semicontinue rispetto alla seconda.

INTRODUCTION

The well known Baire's theorem on the monotone approximation to a semicontinuous function by continuous functions asserts that a real valued function \( f \) of one variable only is lower (resp. upper) semicontinous if and only if there exists a nondecreasing (resp. nonincreasing) sequence of continuous functions which pointwise converges to \( f \). There exist equivalents for multifunctions of this theorem (see for example Aseev [2] and de Blasi [7]). On the other hand, if \( f \) is a real valued function of two variables, measurable in first and lower (resp. upper) semicontinuous in second variable, then it turns out that \( f \) will be a limit of a nondecreasing (resp. nonincreasing) sequence of Carathéodory type functions if and only if \( f \) has the so called Scorza-Dragonni's type property. This has been proved by Zygmunt [15]. The aim of the present paper is to give a set-valued analog of the above fact for compact convex valued multifunctions.


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2. - Preliminaries

We assume the reader is familiar with such notions concerning multifunctions as closedness, topologically or metrically lower and upper semicontinuity, \( \Sigma \)-measurability and weakly \( \Sigma \)-measurability with respect to some \( \sigma \)-field \( \Sigma \). In case of need the necessary information can be found in Berge [3], de Blasi-Mijak [8] and Himmelberg [9]. Since for a compact valued multifunction topologically semicontinuity coincides with metrical one and weakly \( \Sigma \)-measurability coincides with \( \Sigma \)-measurability, often we shall simply say «lower» or «upper semicontinuous» and «measurable». Furthermore, the semicontinuity and measurability of real valued functions are understood in the usual sense.

Throughout the paper \( T \) denotes a metric compact Hausdorff space with the Borel \( \sigma \)-finite regular and complete measure \( \mu \) defined on a \( \sigma \)-field \( \mathcal{A} \) of subsets of \( T \) and \( X \) denotes a separable complete metric space. By \( \mathcal{B}(X) \) we denote the \( \sigma \)-field of Borel subsets of \( X \) and by \( \mathcal{A} \times \mathcal{B}(X) \)—the product \( \sigma \)-field on \( T \times X \). \( \mathbb{R}^d \) (\( \mathbb{R} = \mathbb{R}^1 \)) is a \( d \)-dimensional Euclidean space with the scalar product \( \langle a, b \rangle \in \mathbb{R}^d \) denoted by \( a \cdot b \) and with a norm denoted by \( \| a \| = (a \cdot a)^{1/2} \). \( K(a, r) \) denotes the open ball centered at \( a \in \mathbb{R}^d \) and with radius \( r \). If \( A \subseteq \mathbb{R}^d \) is a subset then \( \overline{A} \) (resp. \( \text{co} \ A \)) denotes the closure (resp. the closed convex hull) of \( A \). The function \( s(\cdot; A): \mathbb{R}^d \rightarrow [0, + \infty] \) defined by \( s(p; A) = \sup \{ p(y) : y \in A \}, \quad p \in \mathbb{R}^d \), is said to be the support function of the set \( A \). Finally we denote by \( \text{Conv} \mathbb{R}^d \) (resp. \( \text{Conv} \mathbb{R}^d \)) the family of all nonempty closed (resp. compact convex) subsets of \( \mathbb{R}^d \) and we assume that \( \text{Conv} \mathbb{R}^d \) is endowed with Hausdorff metric \( d \).

Now we are going to define some classes of functions and multifunctions which are of importance in what follows. So, we say that a function \( f: T \times X \rightarrow \mathbb{R} \) is lower Carathéodory's type function (resp. upper Carathéodory's type function, Carathéodory's type function) if

(i) \( f(\cdot, x) \) is \( \mathcal{A} \)-measurable for each \( x \in X \),

(ii) \( f(t, \cdot) \) is lower semicontinuous (resp. upper semicontinuous, continuous) for each \( t \in T \).

At this point let us notice that the concept of Carathéodory type function is also well posed for every function with values in topological space.

Next, we say that a function \( f: T \times X \rightarrow \mathbb{R} \) belongs to the class \( SD^* \) (resp. \( SD^+, SD \)) if \( f \) is lower Carathéodory's type function (resp. upper Carathéodory type function, Carathéodory's type function) and for every \( \varepsilon > 0 \) there exists a closed subset \( T_\varepsilon \) of \( T \), with \( \mu(T \setminus T_\varepsilon) < \varepsilon \), such that the restriction \( f|_{T_\varepsilon \times X} \) is lower semicontinuous (resp. upper semicontinuous, continuous) in both variables jointly. For a multifunction \( F: T \times X \rightarrow \text{Conv} \mathbb{R}^d \) we introduce the same classification.
3. SOME AUXILIARY LEMMAS

**Lemma 1:** A function \( f: T \times X \to \mathbb{R} \) belongs to \( SD^* \) if and only if there exists a nonincreasing sequence \( \{f_n\} \) of Carathéodory type functions \( f_n: T \times X \to \mathbb{R} \) which pointwise converges to \( f \).

**Proof:** See Zygmunt [15, Theorem 3].

**Lemma 2:** Let \( A \in \text{Conv} \mathbb{R}^d \) and let \( \{p_1, p_2, \ldots\} \) be a dense set in the unit sphere of \( \mathbb{R}^d \). Then

\[
A = \bigcap_{n=1}^{\infty} \{ y : p_n \cdot y < s(p_n; A) \}, \quad \overline{K(A, \varepsilon)} = \bigcap_{n=1}^{\infty} \{ y : p_n \cdot y < s(p_n; A) + \varepsilon \}.
\]

**Proof:** Easily follows from the best known properties of the support function \( s(\cdot; A) \) (see for example Blagodatskii ... [4]) which is continuous on \( \mathbb{R}^d \) (see Artstein [1, Lemma 3.1]).

**Lemma 3:** Let \( p \in \mathbb{R}^d \) and let a multifunction \( F: T \times X \to \text{Conv} \mathbb{R}^d \) belongs to \( SD^* \). Then the function \( f_p: T + X \to \mathbb{R} \) defined by \( f_p(t, x) = s(p; F(t, x)) \) belongs to \( SD^* \).

**Proof:** We observe that, for every \( r \in \mathbb{R} \),

\[
\{(t, x) \in T \times X : f_p(t, x) > r\} = \{(t, x) \in T \times X : F(t, x) \cap \{ y \in \mathbb{R}^d : p \cdot y > r\} \neq \emptyset\}.
\]

Now it is not difficult to deduce that \( f_p \in SD^* \).

**Lemma 4:** If multifunctions \( F_i: T \times X \to \text{Cl} \mathbb{R}^d, \ i = 1, 2, \ldots, n, \ n \in \mathbb{N} \), are closed and a multifunction \( G: T \times X \to \text{Conv} \mathbb{R}^d \) is upper semicontinuous, then the multifunction \( G \cap \bigcap_{i=1}^{n} F_i: T \times X \to \text{Conv} \mathbb{R}^d \) is upper semicontinuous.

**Proof:** By virtue of Berge [3, Chapt. VI, § 1, Theorems 5 and 6] the multifunction \( G \cap \bigcap_{i=1}^{n} F_i \) is topologically and hence metrically upper semicontinuous. Thus it is simply upper semicontinuous.

**Lemma 5:** If a multifunction \( F: T \times X \to \text{Conv} \mathbb{R}^d \) belongs to \( SD^* \), then there exists a Carathéodory type function \( r: T \times X \to [0, \infty) \) such that

\[
F(t, x) \subset K(\theta, r(t, x) + 1)
\]

for each \( (t, x) \in T \times X \),

where \( \theta \) denotes the origin of \( \mathbb{R}^d \).
PROOF: Let us put \( \varrho(t, x) = \sup \{ y : y \in F(t, x) \} \), \((t, x) \in T \times X\). Thus defined function \( \varrho : T \times X \rightarrow [0, \infty) \) belongs to \( SD^\ast \). To see this, notice that for each \( a \in \mathbb{R} \) we have

\[
\{(t, x) \in T \times X : \varrho(t, x) > a \} = \{(t, x) \in T \times X : F(t, x) \cap K^\ast(\theta, a) \neq \emptyset \}
\]

where

\[
K^\ast(\theta, a) = \begin{cases} 
\mathbb{R}^n \setminus K(\theta, a) & \text{if } a > 0, \\
\mathbb{R}^n & \text{if } a < 0.
\end{cases}
\]

Thus, in view of Lemma 1, there is a Carathéodory's type function \( r : T \times X \rightarrow \mathbb{R} \) which satisfies, for each \((t, x) \in T \times X\), the inequality \( \varrho(t, x) < r(t, x) \). Then, obviously, \( F(t, x) \subset K(\theta, r(t, x) + 1) \).

4. MAIN THEOREMS

THEOREM 1: Let a multifunction \( F : T \times X \rightarrow \text{Conv } \mathbb{R}^n \) be given. Then the following two statements are equivalent:

(a) \( F \in SD^\ast \),
(b) there exists a sequence \( \{ F_n \} \) of Carathéodory type multifunctions \( F_n : T \times X \rightarrow \text{Conv } \mathbb{R}^n \) satisfying, for each \((t, x) \in T \times X\), the conditions:

\[
(b_1) \quad F_n(t, x) \subset F(t, x) \quad \text{for } n = 1, 2, \ldots,
\]

\[
(b_2) \quad F_n(t, x) \subset F_{n+1}(t, x) \quad \text{for } n = 1, 2, \ldots,
\]

\[
(b_3) \quad F(t, x) = \lim_{n \to \infty} F_n(t, x) = \bigcup_{n=1}^{\infty} F_n(t, x).
\]

(The limit

\[
A = \lim_{n \to \infty} A_n,
\]

where \( A, A_n \in \text{Conv } \mathbb{R}^n, \quad n \in \mathbb{N} \), means \( \lim d(A_n, A) = 0 \).

PROOF: \((b) \Rightarrow (a)\). By Himmelberg [9, Theorem 2.3] \( F(\cdot, x) \) is weakly measurable for each \( x \in X \) and by Hukuhara [10, Propositions 1.2 and 7.2] \( F(t, \cdot) \) is lower semicontinuous as the limit of a nondecreasing sequence \( \{F_n(\cdot, \cdot)\} \) of continuous multifunctions. Now let's fix \( \varepsilon > 0 \). Since every Carathéodory's type compact convex valued multifunction has the Scorza-Dragoni property (see Brunovsky [5, Theorem 2.5]) we can obtain a sequence \( \{T_n\} \) of closed sets such that, for \( n = 1, 2, \ldots, \quad T_n \subset T_{n-1} \) where \( T_0 = T_1, \mu(T_{n-1} \setminus T_n) < (\varepsilon)^n \) and the restriction \( F_{n|T_n \times X} \) is continuous in both variables
jointly. Then the set \( T_s \subset \bigcap_{n=1}^{\infty} T_n \) is closed, \( T_s \subset T \), \( \mu(T_s \setminus T_n) < \varepsilon \) (see Zygmunt [15]) and each multifunction \( F_n \) is continuous in both variables jointly on \( T_s \times X \). Hence the multifunction \( F = \bigcup_{n=1}^{\infty} F_n \) is lower semicontinuous on \( T_s \times X \). Thus \( F \in SD_\ast \).

(a) \( \Rightarrow \) (b). Since any multifunction belonging to \( SD_\ast \) is weakly \( \mathcal{A} \times \mathcal{B}(X) \)-measurable (see Zygmunt [16, Theorem 3]), by a Rybiński's result [14, Theorem 3] (see also Kim, ... [12, Lemma 5.2]) there is an infinite sequence \( \{ f_n \} \) of Carathéodory type selections \( f_n : T \times X \rightarrow \mathbb{R}^d \) of \( F \) satisfying, for each \( (t, x) \in T \times X \), the equality

\[
F(t, x) = \bigcup_{n=1}^{\infty} \{ f_n(t, x) \}.
\]

For every \( n \in \mathbb{N} \), let \( F_n : T \times X \rightarrow \text{Conv} \mathbb{R}^d \) be the multifunction defined by

\[
F_n(t, x) = \overline{\text{co}} \{ f_1(t, x), f_2(t, x), ..., f_n(t, x) \}.
\]

Clearly, for each \( t \in T \), \( F_n(t, -) \) is continuous and, for each \( x \in X \), by Himmelberg [9, Theorem 9.1] \( F_n(\cdot, x) \) is weakly measurable. Thus \( F_n \) is a Carathéodory's type multifunction. Obviously such a defined sequence \( \{ F_n \} \) satisfies the conditions \( (b_1) \) and \( (b_2) \) while the condition \( (b_3) \) follows from Hukuhara [10, Proposition 1.2]. This complete the proof of Theorem.

**Theorem 2:** Let a multifunction \( F : T \times X \rightarrow \text{Conv} \mathbb{R}^d \) be given. Then the following two statements are equivalent:

(a) \( F \in SD_\ast \),

(b) there exists a sequence \( \{ F_n \} \) of Carathéodory's type multifunctions \( F_n : T \times X \rightarrow \text{Conv} \mathbb{R}^d \) satisfying, for each \( (t, x) \in T \times X \), the conditions:

(b1) \( F(t, x) \subset F_n(t, x) \) for \( n = 1, 2, ..., \)

(b2) \( F_{n+1}(t, x) \subset F_n(t, x) \) for \( n = 1, 2, ..., \)

(b3) \( F(t, x) = \bigcap_{n=1}^{\infty} F_n(t, x) \).

**Proof:** (b) \( \Rightarrow \) (a). Similarly to the proof of part (b) \( \Rightarrow \) (a) of the previous Theorem 1, employing Himmelberg's result [9, Theorem 3.5 (iii)] and Hukuhara's result [10, Proposition 1.2 and 7.1] we show that \( F \in SD_\ast \).

(a) \( \Rightarrow \) (b). Let \( \{ p_1, p_2, ... \} \) be a dense subset of a unit sphere in \( \mathbb{R}^d \). Let \( f_i : T \times X \rightarrow \mathbb{R} \), \( i = 1, 2, ..., \) be a function defined by the formula \( f_i(t, x) = i(p_i; F(t, x)) \), \( (t, x) \in T \times X \). By Lemma 3 every \( f_i \) belongs to \( SD_\ast \) and,
hence, by Lemma 1, there exist sequences \( \{f_{i,j}\} \) of Carathéodory type functions \( f_{i,j} : T \times X \to \mathbb{R} \) such that

\[
f_i(t, x) < \ldots < f_{i,j+1}(t, x) < f_{i,j}(t, x) < \ldots < f_{i,1}(t, x)
\]

and

\[
\lim_{j \to \infty} f_{i,j}(t, x) = f_i(t, x) \quad \text{for } i = 1, 2, \ldots, (t, x) \in T \times X.
\]

Put

\[
H_i(t, x) = \{ y \in \mathbb{R}^s : p_i \circ y < f_i(t, x) \},
\]

\[
H_{i,j}(t, x) = \{ y \in \mathbb{R}^s : p_i \circ y < f_{i,j}(t, x) + \frac{1}{j} \}, \quad i, j = 1, 2, \ldots, (t, x) \in T \times X.
\]

It is easy to verify that such defined multifunctions \( H_i : T \times X \to \text{Cl} \, \mathbb{R}^s \) and \( H_{i,j} : T \times X \to \text{Cl} \, \mathbb{R}^s \) are of Carathéodory type and have the following properties:

\[
F(t, x) \subset (K(F_i, x), 1/j) \subset H_{i,j}(t, x), \quad i, j = 1, 2, \ldots, (t, x) \in T \times X,
\]

\[
H_{i,j+1}(t, x) \subset H_{i,j}(t, x), \quad i, j = 1, 2, \ldots, (t, x) \in T \times X,
\]

\[
F(t, x) = \bigcap_{i=1}^{\infty} H_i(t, x) = \bigcap_{i=1}^{\infty} \left( \bigcap_{j=1}^{\infty} H_{i,j}(t, x) \right), \quad (t, x) \in T \times X.
\]

Let, further, \( r : T \times X \to [0, \infty) \) be a function defined as in Lemma 5. Then the multifunction \( G : T \times X \to \text{Conv} \, \mathbb{R}^s \) given by formula

\[
G(t, x) = K(\emptyset, r(t, x) + 1)
\]

is obviously of Carathéodory type. Now define, for each \( s \in \mathbb{N} \), the multifunction \( F_s : T \times X \to \text{Conv} \, \mathbb{R}^s \) as follows

\[
F_s(t, x) = G(t, x) \cap \bigcap_{i=1}^{s} H_{i,s}(t, x), \quad (t, x) \in T \times X.
\]

We claim that \( \{F_s\} \) is the required sequence of Carathéodory type multifunctions. Indeed, first of all, by standard argument we easily obtain that, for each \( (t, x) \in T \times X \),

(i) \( F(t, x) \subset F_s(t, x) \) for \( s = 1, 2, \ldots, \)

(ii) \( F_{i+1}(t, x) \subset F_i(t, x) \) for \( i = 1, 2, \ldots, \)

(iii) \( \bigcap_{s=1}^{\infty} F_s(t, x) = \bigcap_{i=1}^{\infty} H_i(t, x) = F(t, x) \).
Further we conclude that, in view of Himmelberg [9, Theorem 4.1], $F_n(t, x)$ is weakly $A$-measurable for each $x \in X, n \in \mathbb{N}$, and, by Lemma 4, $F_n(t, \cdot)$ is upper semicontinuous for each $t \in T, n \in \mathbb{N}$. Next, since $F_n(t, x)$ has a nonempty interior (namely, it is $F(t, x) \cap \text{Int } H_{\alpha}(t, x)$, $i = 1, 2, \ldots, n$) it follows (see Lechicki, ... [13, Theorem B]) that $F_n(t, \cdot)$ is lower semicontinuous for each $t \in T, n \in \mathbb{N}$. Thus $F_n(t, \cdot)$ is continuous. Finally we see that $F_n : T \times X \rightarrow \text{Conv } \mathbb{R}^n$ is a Carathéodory's type multifunction for $n = 1, 2, \ldots$ and $(t, x) \in T \times X$. This completes the proof of Theorem 2.

**Remark:** A result closely related to the above theorem, part (a) $\Rightarrow$ (b), was first given by Jarník and Kurzweil [11, Theorem 2.5]. Namely, they proved that if a multifunction $F : T \times X \rightarrow \text{Conv } \mathbb{R}^n$ belongs to $SD^*$, then there exists a sequence $\{F_n\}$ of Carathéodory type multifunctions and a measurable set $Z \subset T$ so that $\mu(Z) = 0$,

$$F_n(t, x) \subset F_n(t, x), \quad n = 1, 2, \ldots,$$

$$F(t, x) = \bigcap_{n=1}^{\infty} F_n(t, x) \text{ for } (t, x) \in (T \setminus Z) \times X,$$

$$F_n(t, x) = \emptyset \text{ for } (t, x) \in Z \times X, \quad n = 1, 2, \ldots.$$  

**REFERENCES**


