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On the Stability of Stochastic Differential Equations « with Memory » (**) (***)

ABSTRACT. — The paper [25] by K. Yamada contains a stability theorem for the solutions of a sequence of stochastic differential equations. Here we generalize Yamada's result in several ways.

Sulla stabilità di certe equazioni differenziali stocastiche « con memoria »

RIASSUNTO. — L'articolo [25] di K. Yamada contiene un teorema di stabilità per le soluzioni di una successione di equazioni differenziali stocastiche. In questo articolo il risultato di Yamada viene generalizzato da più punti di vista.

INTRODUCTION

The paper [25] by K. Yamada contains a stability theorem, in the sense of probability laws' convergence, for the solutions of a sequence of stochastic differential equations. This theorem finds its motivation in several works on approximation theory of optimal stochastic control problems (cf. [3], [4], [26]): so it has immediate applications to stochastic control problems and actually, paper [25] contains also an example of such applications.

It's worth while noting that this kind of problem was discussed in more general settings by several authors (cf., for example, [9], [17], [18], Chap. 7) but the convergence conditions they assume for the coefficients and the driving terms of the approximating equations, generally are different and stronger than those in [25], when applied to the equations considered there.

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In this paper we generalize Yamada's result in several ways. We consider equations with coefficients that may depend, at every time t, on the whole past of the solution before time t (these are called equations « with memory »); on the contrary, those considered by Yamada, have coefficients depending, for every t, only on $X(t^-)$, X denoting the solution process (equations « without memory »).

Moreover, the convergence assumptions we set for the driving terms of the approximating equations, are less stringent than the correponding ones in [25] (see Remark (1.14)).

Also the regularity properties we demand for the coefficients of the limit equation, when applied to equations « without memory », are weaker then those in [25].

As to the coefficients of our equations, the convergence mode we assume here, is the natural extension to our case of the corresponding hypothesis assumed by Yamada (*) in [25] (nevertheless cf. Remark (1.15)).

0. - Preliminaries and notations

Let T be a subset of the real line. A *stochastic basis* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in T}, P)$ (resp. $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in T})$) is a probability space $(\Omega, \mathcal{F}, P)$ (resp. a measurable space $(\Omega, \mathcal{F})$) endowed with a filtration $(\mathcal{F}_t)_{t\in T}$ on $(\Omega, \mathcal{F})$.

By the terms *P-completion* and *usual P-augmentation* of a given stochastic basis, we mean the bases defined in the usual Strasbourg manner (cf. [8], (0.21) and [24], II.40).

Let $\mathbf{R}_+ = [0, +\infty[$. We denote by $\mathbf{D} = \mathbf{D}(\mathbf{R}_+, \mathbf{R})$ the space of all càdlàg (« continus à droite avec limite à gauche », in French) mappings from $\mathbf{R}_+$ into $\mathbf{R}$ (the set of real numbers).

By $\mathfrak{D}$ we denote the Borel σ-algebra of $\mathbf{D}$, when this space is endowed with the Skorokhod topology (cf. [22], [2], [13], [23]); by $(\mathfrak{D}_t)_{t\in\mathbf{R}_+}$ the « canonical » filtration of $(\mathbf{D}, \mathfrak{D})$.

If d is an integer >1, $\mathbf{D}^d$ will always denote the product space $\mathbf{D}^d = (\mathbf{D}(\mathbf{R}_+, \mathbf{R}))^d$ endowed with the product topology, the space $\mathbf{D}$ being supposed to carry the Skorokhod topology; the notation $\mathfrak{D}^d = (\mathfrak{D})^{\otimes d}$ will stand for the Borel σ-algebra of $\mathbf{D}^d$, while $(\mathfrak{D}_t^d)_{t\in\mathbf{R}_+} = ((\mathfrak{D}_t)^{\otimes d})_{t\in\mathbf{R}_+}$ will denote the « canonical » filtration of $(\mathbf{D}^d, \mathfrak{D}^d)$.

By a $(\mathfrak{D}_t)$*-predictable step function* $a(t, f)$ from $\mathbf{R}_+ \times \mathbf{D}$ into $\mathbf{R}$ we mean a function which can be written as follows, for all $(t, f) \in \mathbf{R}_+ \times \mathbf{D}$:

$$a(t, f) = \psi_0(f)\, 1_{\{0\}}(t) + \sum_{k\geq 0} \psi_k(f)\, 1_{]t_k, t_{k+1}]}(t),$$

where $(t_k)_{k\geq 0}$ is a finite and increasing family of real numbers such that $t_0 = 0$ and, for each $k \geq 0$, ψ_k: $(\mathbf{D}, \mathfrak{D}_{t_k}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ is measurable.

(*) We mention here also the paper [27] by the same author. There he studies a different model; the methods are similar to those used in [25]. Interesting applications are also given.

We end these preliminaries with a few notations.

Let X be a càdlàg $\mathbb{R}^d$-valued (resp. $\mathbb{R}$-valued) measurable process defined on the probability space $(\Omega, \mathcal{F}, P)$. Then X can be considered as a random element of $(\mathbf{D}^d, \mathfrak{D}^d)$ (resp. $(\mathbf{D}, \mathfrak{D})$) defined on $(\Omega, \mathcal{F}, P)$, by associating to each ω in Ω the path $t \mapsto X(t, \omega)$ of X at ω: we will denote this random element by $X_\cdot$.

The *law* of X is the probability measure Λ defined on $\mathfrak{D}^d$ (resp. $\mathfrak{D}$) as the image of P by the random element $X_\cdot$.

If $(\mu_n)_n$ is a sequence of probability measures on $\mathfrak{D}^d$ (or on $\mathfrak{D}$) sometimes we will write $\mu_n \underset{n}{\Rightarrow} \mu$ to denote the weak convergence, when n tends to infinity, of sequence $(\mu_n)_n$ to the probability measure μ on $\mathfrak{D}^d$ (or on $\mathfrak{D}$, respectively).

Similarly, we write $X_n \underset{n}{\Rightarrow} X$ to mean that $\Lambda_n \underset{n}{\Rightarrow} \Lambda$, $(\Lambda_n)_n$ (resp. Λ) denoting the laws (resp. the law) of the sequence of (càdlàg measurable) processes $(X_n)_n$ (resp. of the (càdlàg measurable) process X).

We write $Y_n \xrightarrow[n]{P} Y$ when the sequence $(Y_n)_n$ of random variables from $(\Omega, \mathcal{F}, P)$ into $\mathbb{R}$ (or $\mathbb{R}^d$) converges in probability to Y when n tends to infinity.

In addition to Skorokhod topology, on the space $\mathbf{D}$ we will consider the topology of uniform convergence on compact sets which will be denoted by τ_u. On the contrary the notation τ, will stand for Skorokhod topology.

For all definitions and results in the general theory of stochastic processes we don't explicitly quote, we refer to [6], [8] or [16].

1. - Basic result

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P)$ be a stochastic basis satisfying the usual hypotheses (cf. [16], p. 3) and let W be an $(\mathcal{F}_t)$-Wiener process defined on it.

Suppose also that the space $(\Omega, \mathcal{F}, P)$ be endowed with a sequence of filtrations $(\mathcal{F}_t^n)_{t\in\mathbb{R}_+}$, $n \geq 1$, such that, for every n, the stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t^n)_{t\in\mathbb{R}_+}, P)$ verifies the usual hypotheses.

We are given two sequences $(A^n)_n$, $(M^n)_n$ of càdlàg, real-valued and $(\mathcal{F}_t^n)$-adapted processes, where, for every $n \geq 1$, M^n is a square integrable $(\mathcal{F}_t^n)$-martingale (i.e., for every t, the random variable $M^n(t)$ is square integrable with respect to P and moreover $\sup_{t\in\mathbb{R}^+} E_P[|M^n(t)|^2] < +\infty$) and A^n is an increasing process.

We assume that $A^n(0) = M^n(0) = 0$ for all n.

We shall denote by $\mu_n(\omega, dt, dx)$ the random measure of jumps of martingale M^n

$$\mu_n(\omega, dt, dx) = \sum_{s>0} 1_{\{\Delta M^n(s,\omega)\neq 0\}}(s)\,\varepsilon_{(s,\,\Delta M^n(s,\omega))}(dt, dx)$$

where $\varepsilon_{(s,x)}$ is the Dirac measure concentrated at point (s, x) and $\Delta M^n(s, \omega) = M^n(s, \omega) - M^n(s^-, \omega)$.

We will write $\nu_n(\omega, dt, dx)$ for the compensator i.e. the dual $(\mathcal{F}_t^n)$-predictable projection of measure μ_n.

We are also given two sequences $(r_n)_n$, $(\sigma_n)_n$, $n \geq 1$, of mappings from $\mathbf{R}_+ \times \mathbf{D}$ into $\mathbf{R}$.

About sequence $(\sigma_n)_n$ we set the following assumption:

(1.1) *For each $n \geq 1$, there exists a sequence $(a_i^n)_i$ of $(\mathfrak{D}_t)$-predictable step functions from $\mathbf{R}_+ \times \mathbf{D}$ into $\mathbf{R}$, such that, for all (t, f)*

$$\sigma_n(t, f) = \lim_i a_i^n(t, f) .$$

From this assumption, it follows that, for each $n \geq 1$, the mapping σ_n is $(\mathfrak{D}_t)$-predictable on the basis $(\mathbf{D}, \mathfrak{D}, (\mathfrak{D}_t)_{t \in \mathbf{R}_+})$ (i.e., in the terminology of [21], it is a $(\mathfrak{D}_t)$-predictable path functional).

We also assume that, for each $n \geq 1$, the function r_n is $(\mathfrak{D}_t)$-predictable.

As a consequence of the $(\mathfrak{D}_t)$-predictability property of sequences $(\sigma_n)_n$, $(r_n)_n$, we have the following property:

(1.2) *Let Y be a real-valued, càdlàg and $(\mathcal{A}_t)$-adapted process defined on a generic stochastic basis $(\breve{\Omega}, \mathcal{A}, (\mathcal{A}_t)_{t \in \mathbf{R}_+})$.*

Consider the processes a_n, b_n defined as follows, for every $(t, \breve{\omega})$ ($\breve{\omega}$ denoting the generic element of $\breve{\Omega}$)

$$a_n(t, \breve{\omega}) := r_n(t, Y_\cdot(\breve{\omega})) , \qquad b_n(t, \breve{\omega}) := \sigma_n(t, Y_\cdot(\breve{\omega})) .$$

Then, for every $n \geq 1$, processes a_n, b_n are $(\mathcal{A}_t)$-predictable and, for each element $(t, \breve{\omega})$ of $\mathbf{R}_+ \times \breve{\Omega}$, $a_n(t, \breve{\omega})$ and $b_n(t, \breve{\omega})$ depend only on the values $Y(s, \breve{\omega})$ with $s < t$.

The above property follows easily from Prop. 6.4, p. 69 in [18].

Now let us set the assumptions of our stability theorem.

(1.3) *Let $(X^n)_n$ be a given sequence of càdlàg and $(\mathcal{F}_t^n)$-adapted processes such that, for every $n \geq 1$ and all $t \in \mathbf{R}_+$, we have*

$$\text{(1.3.1)} \qquad X^n(t) = X^n(0) + \int_{]0,t]} r_n(s, X_\cdot^n(\omega))\, dA^n(s) + \int_{]0,t]} \sigma_n(s, X_\cdot^n(\omega))\, dM^n(s) .$$

Moreover we assume that

$$\text{(1.3.2)} \qquad \sup_n \boldsymbol{E}_P[|X^n(0)|^2] < +\infty .$$

(1.4) *Let $\langle M^n \rangle$ denote the Meyer process of M^n. Then, for any $t \geq 0$, the sequence of random variables $(\langle M^n \rangle(t))_n$ converges to the constant t in the norm of $L^1(\Omega, \mathcal{F}, P)$ when n tends to infinity.*

(1.5) *For any* $t>0$ *and* $\varepsilon>0$, *the sequence of random variables*

$$\left(\int_{]0,t]}\int_{|x|>\varepsilon} x^2\nu_n(ds,dx)\right)_n$$

converges in probability to 0, *as* n *tends to infinity.*

(1.6) *Functions* r_n, σ_n *are bounded uniformly in* n. *There exists two* $(\mathfrak{D}_t)$-*predictable mappings* r, σ *from* $\mathbf{R}_+\times\mathbf{D}$ *into* $\mathbf{R}$, *to which sequences* $(r_n)_n$, $(\sigma_n)_n$ *converge in the following sense, when* n *tends to infinity*:

For each path f *in* $\mathbf{D}$, *there exists a Lebesgue nullset* N_f *such that one has*

$$\lim_n r_n(t_n,f_n)=r(t,f)\,, \qquad \lim_n \sigma_n(t_n,f_n)=\sigma(t,f)$$

whenever $t\in\mathbf{R}_+-N_f$ *and* $(t_n)_n$, $(f_n)_n$ *are arbitrary sequences of real numbers, of paths in* $\mathbf{D}$, *respectively, verifying the conditions*

$$t_n \xrightarrow[n]{} t \text{ in the usual topology of } \mathbf{R}_+\,,$$

$$f_n \xrightarrow[n]{} f \text{ in the } \tau_u \text{ topology on } \mathbf{D}\,.$$

Moreover r, σ *are both bounded.*

(1.7) *As* n *tends to infinity, the sequence of random variables* $(A^n(t))_n$ *converges in probability to the constant* t, *for every* t *in* $\mathbf{R}_+$.

Moreover, for every $n\geq 1$, *process* A^n *is* $(\mathcal{F}_t^n)$-*predictable.*

(1.8) *Let* λ *be a given probability measure on* $\mathbf{R}$. *We suppose that, for the Itô stochastic differential equation*

$$X(t)=X(0)+\int_0^t r(s,X_\cdot(\omega))\,ds+\int_0^t \sigma(s,X_\cdot(\omega))\,dw(s)$$

existence and uniqueness holds in the sense of probability law and in connection with the initial distribution λ.

(1.9) Notation: *For every* n, *we denote by* Λ_n *the law on* $(\mathbf{D},\mathfrak{D})$ *of process* X^n *defined in* (1.3) *and by* λ_n *the law of the real random variable* $X^n(0)$.

The (unique) law of any solution of the equation in (1.8) *with* λ *as initial distribution, is denoted by* Λ.

The theorem we want to prove is the following:

(1.10) Theorem: *Let* $(\lambda_n)_n$ *be the sequence of probability measures on* $\mathbf{R}$ *introduced above and suppose that, as* n *tends to infinity,* λ_n *weakly converges to* λ, λ *being the probability measure defined in* (1.8).

Then, under conditions (1.1), (1.3), (1.4), (1.5), (1.6), (1.7) and (1.8), the sequence of the above defined laws $(\Lambda_n)_n$ weakly converges to the law Λ, when n tends to infinity.

We end this section with a few comments.

(1.11) Remark: Consider the deterministic process H that has, for all ω, the path H, such that $H(t) = t$ for all t. Because of the continuity of this process, from Lemma 1 in [15], it follows that the convergence assumption (1.7) is equivalent to the following one:

As n tends to infinity, sequence $(A^n)_n$ converges to the process H in the sense of the « compact convergence in probability » (cf. [12], 28.3), *i.e., for any t*

$$\sup_{0 \le s \le t} |A^n(s) - s| \xrightarrow[n]{P} 0 .$$

By the same reason, assumptions (1.4), (1.5) are equivalent to the following ones, respectively:

As n tends to infinity, sequence $(\langle M^n \rangle)_n$ converges to the process H in the sense of the « compact convergence in probability » and, for every t, sequence $(\langle M^n \rangle (t))_n$ is a P-uniformly integrable family of random variables.

For any $t > 0$ and any $\varepsilon > 0$, the sequence of processes

$$\Big(\int_{]0,t]} \int_{|x| > \varepsilon} x^2 \nu_n(ds, dx) \Big)_n$$

converges to 0 (that is the process with all paths identically zero) in the sense of the « compact convergence in probability », as n tends to infinity.

(1.12) Remark: Assumption (1.1) is verified, for example, when for each n and each f the mapping $t \mapsto \sigma_n(t, f)$ is left-continuous.

In [25] the equations considered are « without memory »: in other words the coefficients r_n, σ_n depend on f only through $f(t^-)$, i.e. for every n and every $(t, f) \in \mathbf{R}_+ \times \mathbf{D}$, $r_n(t, f) = r_n(t, f(t^-))$ and $\sigma_n(t, f) = \sigma_n(t, f(t^-))$, so r_n, σ_n are defined on $\mathbf{R}_+ \times \mathbf{R}$.

In this case assumption (1.1) is also verified when

$$\sigma_n(t_h, x_k) \xrightarrow[h,k]{} \sigma_n(t, x)$$

for all $(t, x) \in \mathbf{R}_+ \times \mathbf{R}$ and all sequences $(t_h)_h$, $(x_k)_k$ such that

$$t = \lim_{\substack{h \\ t_h \ge t}} t_h , \qquad x = \lim_k x_k .$$

Such hypothesis, which is not explicitly assumed in [25], is often verified in applications (cf. [25], § 3).

(1.13) Remark: Conditions (1.4) and (1.5) guarantee the weak convergence, as n tends to infinity, of the laws of M_n to the Wiener measure. These were first established by Rebolledo (see [20], Th. 3 or [19], Th. 5, p. 51; cf. also [14], Corollary 2).

(1.14) Remark: Besides conditions (1.4) and (1.5), the stability theorem in [25] requires one of the following conditions to hold:

$$\int_{]0,t]} \int_{\mathbb{R}-\{0\}} x^2 \nu_n(ds, dx) \xrightarrow[n]{P} t \qquad \text{for any } t > 0\,;$$

$$\int_{]0,t]} \int_{\mathbb{R}-\{0\}} x^2 \nu_n(ds, dx) \xrightarrow[n]{P} 0 \qquad \text{for any } t > 0\,.$$

The former is equivalent to the following condition

$$\langle M^n \rangle(t) \xrightarrow[n]{P} t$$

when all the martingales M^n are purely discontinuous, (while the latter is automatically verified when all the M^n are continuous). This is due to the fact that, for all t,

$$\langle M^n \rangle(t) = \langle {}^c M^n \rangle(t) + \int_{]0,t]} \int_{\mathbb{R}-\{0\}} x^2 \nu_n(ds, dx)$$

${}^c M^n$ denoting the continuous part of martingale M^n ([10], Th. 2).

Yamada's result also need the continuity of function σ appearing as a coefficient in the martingale-term integral of the limit equation.

All these assumptions are removed here.

(1.15) Remark: Hypothesis (1.6) corresponds to both assumptions (A4), (A5) of [25], but, when applied to equations without memory, our assumption is slightly weaker than the corresponding one in [25] (cf. [25], p. 260).

The convergence condition (1.6) could seem a little strange at first. However concrete stochastic control problems justify it: see [25], § 3, [26], [3]. (Remark 3 in [25] gives a sufficient condition for (1.6) to hold in the case of equations without memory).

2. - Auxiliary results

The results on stochastic integration we list in this paragraph, are independent, so to say, from the application we make here to our particular problem and are in itself of some interest.

(2.1) THEOREM: *Let X, Z respectively denote a càdlàg adapted process, a (càdlàg adapted) semimartingale both defined on a stochastic basis satisfying the usual hypotheses.*

Let σ be a bounded real-valued mapping on $\mathbf{R}_+ \times \mathrm{D}$, which is everywhere the limit of a given sequence $(\sigma_n)_n$ of $(\mathcal{D}_t)$-predictable step functions.

Then the fact that a given process Y verifies relation

$$Y_t(\omega) = \int_{]0,t]} \sigma(s, X_\cdot(\omega))\, dZ_s(\omega),$$

is a property that depends only on the global law of $(X_\cdot, Y_\cdot, Z_\cdot)$ as a random element of $(\mathrm{D}^3, \mathcal{D}^3)$.

PROOF: To begin with, make the following remark: if Y, Y^n are càdlàg processes and, as n tends to infinity, (Y^n) converges to Y in the sense of the « compact convergence in probability » (cf. Remark (1.11)) then also the sequence $(Y^n_\cdot)$ converges in probability to $Y_\cdot$, where, according to our notations (see § 0), $Y_\cdot$ and $Y^n_\cdot$ denote processes Y and Y^n, respectively, considered as random elements of $(\mathrm{D}, \mathcal{D})$.

Since for every n, mapping σ_n is a step function, process Y^n defined as follows

$$Y^n_t(\omega) = \int_{]0,t]} \sigma_n(s, X_\cdot(\omega))\, dZ_s(\omega)$$

can be written in the form

$$Y^n_\cdot = \varphi_n \circ (X_\cdot, Z_\cdot), \tag{2.2}$$

where φ_n is a suitable measurable mapping from $(\mathrm{D}^2, \mathcal{D}^2)$ into $(\mathrm{D}, \mathcal{D})$, depending only on σ_n.

To verify the latter statement, clearly it suffices to consider the case when σ_n is of the following form

$$(t, \omega) \mapsto a(f)\, 1_{]s,\infty[}(t),$$

where $s \in \mathbf{R}_+$ and a is a real-valued measurable function on $(\mathrm{D}, \mathcal{D}_s)$: in that case one directly checks the property.

By virtue of the initial remark, from the « dominated convergence property » of stochastic integrals (see [16], 24.2 or [12], 29.11), it follows that sequence $(Y^n_\cdot)$ converges in probability to the stochastic integral

$$\int_{]0,\cdot]} \sigma(s, X_\cdot(\omega))\, dZ_s(\omega)$$

considered as a random element of $(\mathrm{D}, \mathcal{D})$.

Thus, a given càdlàg process Y coincides with the latter stochastic integral, if and only if $Y_.$, as a random element of $(\mathrm{D}, \mathfrak{D})$, is the limit in probability of the sequence (2.2).

Since this property depends only on the global law of the triplet $(X_.\ Y_.,\ Z_.)$, the proof is complete. □

(2.3) Theorem: *Let X, Z respectively denote a measurable process and a (raw) finite variation process (cf. [21], IV,7) that are both real-valued, càdlàg and defined on a given probability space.*

Let σ be a bounded measurable function from $(\mathrm{R}_+ \times \mathrm{D}, \mathcal{B}(\mathrm{R}_+) \otimes \mathfrak{D})$ into $(\mathrm{R}, \mathcal{B}(\mathrm{R}))$. Then the process

$$\int_{]0,t]} \sigma(s, X_.(\omega))\, dZ_s(\omega)\,,$$

considered as a random element of $(\mathrm{D}, \mathfrak{D})$, can be set in the form $\varphi \circ (X_., Z_.)$, where $\varphi\colon (\mathrm{D}^2, \mathfrak{D}^2) \to (\mathrm{D}, \mathfrak{D})$ is a suitable measurable mapping depending only on σ (and $X_.$, $Z_.$ denote processes X, Z considered as random elements of $(\mathrm{D}, \mathfrak{D})$).

Thus the fact that a given process Y verifies relation

$$Y_t(\omega) = \int_{]0,t]} \sigma(s, X_.(\omega))\, dZ_s(\omega)\,,$$

is a property that depends only on the global law of $(X_., Y_., Z_.)$ as a random element of $(\mathrm{D}^3, \mathfrak{D}^3)$.

Proof: Denote by B the subset of D formed by the paths that are of bounded variation on every interval of the form $[0, n]$, where n is a positive integer.

If we express the total variation on $[0, n]$ of a generic path by means of the path restriction to the dyadic points in $[0, n]$, we easily see that B is an element of $\mathfrak{D}$.

Then denote by φ the mapping from D^2 into D, defined as follows:

— for each element (f, g) of D^2, $\varphi(f, g)$ is the path

$$h(t) = \begin{cases} \displaystyle\int_{]0,t]} \sigma(s, f)\, g(ds) & \text{when } g \in B\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Then one has

$$\int_{]0,t]} \sigma(s, X_.(\omega))\, dZ_s(\omega) = \varphi(X_.(\omega), Z_.(\omega))\,.$$

On the other hand, mapping φ is Borel measurable: this is very easy to verify in the case that σ has the form $1_{]0,s]\times A}$, where $s \in \mathbf{R}_+$ and $A \in \mathfrak{D}$; by a monotone class argument, the statement follows in general.

The proof is now complete. □

(2.4) Proposition: *Let* $(X^1, \ldots, X^n)$ *be an* $\mathbf{R}^n$*-valued càdlàg process defined on a given probability space* $(\Omega, \mathcal{F}, P)$.

Denote by $(\mathcal{X}_t)_{t\in\mathbf{R}_+}$ *the filtration generated by the given process.*

Then the fact that X^n *is a square integrable* $(\mathcal{X}_t)$*-martingale, is a property that depends only on the global law of* $(X^1_\cdot, \ldots, X^n_\cdot)$*, as a random element of* $(\mathbf{D}^n, \mathfrak{D}^n)$.

Moreover, X^n *is a square integrable* $(\mathcal{X}_t)$*-martingale if and only if* X^n *is a square integrable martingale with respect to the usual P-augmentation of* $(\mathcal{X}_t)_{t\in\mathbf{R}_+}$.

Proof: Process X^n is an $(\mathcal{X}_t)$-martingale if and only if, X^n_t is P-integrable for every t and

$$E_P[(X^n_t - X^n_s)\cdot g] = 0$$

for any real numbers s, t with $s < t$ and any bounded, $\mathcal{X}_s$-measurable function g from Ω into $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$.

This condition is equivalent to the following one:

$$E_P[(X^n_t - X^n_s)\cdot \varphi_s \circ (X^1_\cdot, \ldots, X^n_\cdot)] = 0 ,$$

for every bounded $\mathfrak{D}^{\otimes n}_s$-measurable function φ_s from $\mathbf{D}^n$ into $\mathbf{R}$.

But this property, as well as X^n_t's integrability, clearly depends only on the global law of $(X^1_\cdot, \ldots, X^n_\cdot)$ as a random element of $(\mathbf{D}^n, \mathfrak{D}^n)$.

Square integrability property of X^n, also depends only on the law of $X^n_\cdot$.

To check the second statement of the proposition, it suffices (see [8], Chap. IX, § 2) to prove that, if X^n is a square integrable $(\mathcal{X}_t)$-martingale, then it is also a (square integrable) martingale with respect to the usual P-augmentation of $(\mathcal{X}_t)_{t\in\mathbf{R}_+}$. This can be done by a standard limit argument (cf. [8]) using the fact that the random variables $(X^n_t)_{t\in\mathbf{R}_+}$ are a uniformly integrable family. □

3. - Proof of the theorem. Preliminary lemmas

To prove Theorem (1.10), we will show that for every subsequence of the given sequence $(A_n)_n$ (1.9), there exists a further subsequence weakly converging to A.

For every n, set

$$(3.1) \qquad I^n_t = \int_{]0,t]} r_n(s, X^n_s)\, dA^n(s) , \qquad J^n_t = \int_{]0,t]} \sigma_n(s, X^n_s)\, dM^n(s)$$

and let V^n be the random element of $(\mathbf{D}^5, \mathfrak{D}^5)$ defined as follows:

$$V^n = (X^n_\cdot, I^n_\cdot, J^n_\cdot, A^n_\cdot, M^n_\cdot) . \tag{3.2}$$

We have the following

(3.3) Lemma: *The laws $(\Gamma_n)_n$ of the random elements $(V^n)_n$ constitute a tight sequence of probability measures on $(\mathbf{D}^5, \mathfrak{D}^5)$.*

Proof: It suffices to prove that sequence $(V^n)_n$ verifies the Aldous-Rebolledo conditions (see [1], [19]).

For this, one shows that these conditions are satisfied by each sequence of processes appearing in the definition of V^n. The proof makes use of the assumptions (1.3.2), (1.4), (1.7) and of Remark (1.11). It's an application of Lenglart's inequality ([11]) and it can be carried out exactly in the same way as the proof of Lemma 6 in [14], p. 677 (cf. also [25], Lemma 1). □

(3.4) Lemma: *Under conditions (1.4) and (1.5), the laws of the processes M^n converge weakly to the law of the Wiener process W, when n tends to infinity.*

Proof: See [20], Th. 3 or [19], Th. 5, p. 51. □

(3.5) Lemma: *Let θ be any cluster point of the sequence of laws $(A_n)_n$ ((1.9)). Then θ is concentrated on $C = C(\mathbf{R}_+, \mathbf{R})$, the space of continuous functions from $\mathbf{R}_+$ to $\mathbf{R}$.*

Proof: In view of Lemma (3.3) and replacing sequence $(V^n)_n$ by suitable subsequences, we may suppose that $A_n \underset{n}{\Rightarrow} \theta$ and that there exists a law Γ on $(\mathbf{D}^5, \mathfrak{D}^5)$ such that $\Gamma_n \underset{n}{\Rightarrow} \Gamma$.

Let $V := (X_\cdot, I_\cdot, J_\cdot, A_\cdot, M_\cdot)$ be a random element of $(\mathbf{D}^5, \mathfrak{D}^5)$ (defined on a suitable probability space) which has law Γ: clearly θ is $X_\cdot$'s law.

Let $T = [0, t]$ be an arbitrary interval and $\mathbf{D}[T]$ be the space of all càdlàg mappings from T into $\mathbf{R}$, endowed with the Skorokhod topology; the notation $\mathfrak{D}[T]$ stands for the Borel σ-algebra of $\mathbf{D}[T]$, hence $\mathfrak{D}[T] \otimes \mathfrak{D}[T]$ will be the Borel σ-algebra of the product space $\mathbf{D}[T] \times \mathbf{D}[T]$ endowed with the product topology.

Denote by $M^n[T]$, $J^n[T]$ the restrictions to T of M^n, J^n, respectively. Let k be a uniform bound for sequence $(\sigma_n)_n$ ((1.6)).

From the definition of J^n, we see that the random element $(J^n_\cdot[T], M^n_\cdot[T])$ of $(\mathbf{D}[T] \times \mathbf{D}[T], \mathfrak{D}[T] \otimes \mathfrak{D}[T])$ has law concentrated on the closed set F defined as follows

$$F = \{(f_1, f_2) \in \mathbf{D}[T] \times \mathbf{D}[T] : \delta(f_2) \leq k\delta(f_1)\} .$$

where δ: $D[T] \to R$ is the continuous mapping

$$\delta(f) = \sup_{t \in T} |\Delta f(t)|.$$

As a consequence of the fact that sequence $(V^n)_n$ satisfies the Aldous-Rebolledo conditions, every component of V has no fixed time of discontinuity, when considered as a process with paths in D. Thus, as n tends to infinity, the random element $(J^n_\cdot[T], M^n_\cdot[T])$ weakly converges to the random element $(J_\cdot[T], M_\cdot[T])$, $J_\cdot[T]$, $M_\cdot[T]$ denoting the restrictions $J[T]$, $M[T]$ to T of J, M, respectively, considered as random elements of $D[T]$.

Thus the law of $(J_\cdot[T], M_\cdot[T])$ is concentrated on F, i.e. almost every path $(J_\cdot, M_\cdot)$ verifies the inequality

$$\delta(J_\cdot) \leq k\delta(M_\cdot).$$

Owing to the previous Lemma, $\delta(M_\cdot) = 0$ and therefore $\delta(J_\cdot) = 0$ with probability one: thus J is almost surely continuous on T and then on all of R_+.

Exactly as in the above case, it is easy to prove that also process I is almost surely continuous.

Since $(X^n_\cdot, I^n_\cdot, J^n_\cdot) \Rightarrow (X_\cdot, I_\cdot, J_\cdot)$ and a.s., for every n

$$X^n_\cdot = X^n(0) + I^n_\cdot + J^n_\cdot,$$

we have a.s.

$$X_\cdot = X(0) + I_\cdot + J_\cdot,$$

and thus X is a.s. continuous. □

(3.6) Lemma: *Property which is considered in assumption* (1.4) *depends only on the laws of the processes* $(M^n)_n$.

When property (1.4) *holds, also property considered in* (1.5), *depends only on the laws of the processes* $(M^n)_n$.

Proof: Because of the positivity of the random variables $\langle M^n \rangle(t)$, property (1.4) is equivalent to the following one:

— For any t, sequence $E_P[\langle M^n \rangle(t)]$ converges to t, when n tends to infinity.

But, for every n

$$E_P[\langle M^n \rangle(t)] = E_P[(M^n(t))^2].$$

Since the quantity on the right depends only on M^n's law, the first statement in the Lemma is proved.

To check the second one, fix $\varepsilon > 0$ and set

$$^{\varepsilon}N^n(t) = \int_{]0,t]} \int_{|x|>\varepsilon} x^2 \nu_n(ds, dx)$$

and recall Remark (1.11). Since property (1.4) holds, sequence $(\langle M^n \rangle(t))_n$ is P-uniformly integrable. But, for every t, we have:

$$\langle M^n \rangle(t) = \langle {}^c M^n \rangle(t) + \int_{]0,t]} \int_{E} x^2 \nu_n(ds, dx), \tag{3.7}$$

where ${}^c M^n$ denotes the continuous part of martingale M^n and $E = \mathbf{R} - \{0\}$. Since, for every n, ${}^{\varepsilon}N^n(t) \leq \langle M^n \rangle(t)$, also sequence $({}^{\varepsilon}N^n(t))_n$ is P-uniformly integrable.

Bearing in mind that the random variables ${}^{\varepsilon}N^n(t)$ are positive, condition (1.5) is equivalent to the following one:

— For any $t \geqq 0$ and $\varepsilon > 0$, sequence $E_P[{}^{\varepsilon}N^n(t)]$ converges to 0, as n tends to infinity.

But, for every t

$$E_P[{}^{\varepsilon}N^n(t)] = E_P[{}^{\varepsilon}S^n(t)], \tag{3.8}$$

where

$${}^{\varepsilon}S^n(t) := \sum_{s \leqq t} |\Delta M^n(s)|^2 \cdot 1_{\{|\Delta M^n(s)| > \varepsilon\}}.$$

Let α be the mapping from $(\mathbf{D}, \mathfrak{D})$ into $(\mathbf{D}, \mathfrak{D})$, defined as follows

$$\forall f \in \mathbf{D}, \qquad \alpha_s(f) = \sum_{r \leqq s} |\Delta f(r)|^2 1_{\{|\Delta f(r)| > \varepsilon\}}(r):$$

hence ${}^{\varepsilon}S^n_{\bullet} = \alpha \circ M^n_{\bullet}$.

Since α is measurable, the law of ${}^{\varepsilon}S^n_{\bullet}$ depends only on the law of M^n. Thus conclusion follows from (3.8). □

Now let $(\Lambda_n)_n$ be an arbitrary subsequence of the given sequence of laws (1.9) and consider the corresponding subsequence $(\Gamma_n)_n$ of $V^{n'}$ laws.

In view of Lemma (3.3), there exists a further subsequence (still indexed by n) $(\Gamma_n)_n$ and a law Γ on $(\mathbf{D}^5, \mathfrak{D}^5)$ such that

$$\Gamma_n \underset{n}{\Rightarrow} \Gamma .$$

In the following we will be always concerned with this fixed subsequence $(\Gamma_n)_n$ *and we will show that* $\pi_1(\Gamma) = \Lambda$, *where* π_1 *denotes the projection from* $\mathbf{D}^5$ *to the first factor* $\mathbf{D}$.

Since, for every n, $\pi_1(\Gamma_n) = \Lambda_n$, conclusion of Theorem (1.10) will follow.

By virtue of Skorokhod representation theorem (cf. [22]), on a complete probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ there exists random elements $(\tilde{V}^n)_n$, $\tilde{V}$ of $(D^6, \mathcal{D}^6)$, such that

$$(3.9)\quad \begin{cases} \text{for every } n\ \ \tilde{V}^n(\tilde{P}) = \Gamma_n, \quad \tilde{V}(\tilde{P}) = \Gamma, \text{ and} \\ \lim_n \tilde{V}^n(\tilde{\omega}) = \tilde{V}(\tilde{\omega}) \text{ in } D^6, \quad \text{for } \tilde{P}\text{-almost all } \tilde{\omega} \in \tilde{\Omega}. \end{cases}$$

Let us set, for every $n \geq 1$

$$(3.10)\quad \begin{cases} \tilde{V}^n = (\tilde{X}^n_., \tilde{I}^n_., \tilde{J}^n_., \tilde{A}^n_., \tilde{M}^n_.) \text{ and} \\ \tilde{V} = (\tilde{X}_., \tilde{I}_., \tilde{J}_., \tilde{A}_., \tilde{M}_.). \end{cases}$$

(3.11) Definition: *We denote by*

$$(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\in R_+}, \tilde{P}) \quad (\textit{resp. } (\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}^n_t)_{t\in R_+}, \tilde{P}))$$

the usual $\tilde{P}$*-augmentation of the basis generated by* $\tilde{V}$ *(resp. by* $\tilde{V}^n$*, for every* n*).*

Denote by $\tilde{X}, \tilde{I}, \tilde{J}, \ldots$ etc. (resp. $\tilde{X}^n, \tilde{I}^n, \tilde{J}^n, \ldots$ etc.) the random elements $\tilde{X}_., \tilde{I}_., \tilde{J}_., \ldots$ etc. (resp. $\tilde{X}^n_., \tilde{I}^n_., \tilde{J}^n_., \ldots$ etc., $n \geq 1$) considered as processes on $R_+ \times \tilde{\Omega}$.

(3.12) Lemma: *For every* $n \geq 1$*, process* $\tilde{A}^n$ *is* $(\tilde{\mathcal{F}}^n_t)$*-adapted and* $\tilde{P}$*-almost surely increasing; process* $\tilde{M}^n$ *is a square integrable martingale on the basis* $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}^n_t)_{t\in R_+}, \tilde{P})$.

Proof: For a given càdlàg process, the property of having increasing paths, can be expressed by means of the process restriction to the rational numbers in R_+: thus it depends only on the law of the process. Conclusion for process $\tilde{A}^n$ follows. What stated about $\tilde{M}^n$, is a consequence of the definition of $\tilde{M}^n$ and of Proposition (2.4). $\square$

In the sequel, for each $n \geq 1$*, processes* $\tilde{X}^n, \tilde{A}^n, \tilde{M}^n, \tilde{I}^n, \tilde{J}^n$ *will be always considered as defined on the basis* $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}^n_t)_{t\in R_+}, \tilde{P})$ *and their properties will be referred to this basis (even if this is not explicitly expressed).*

From (1.3.1) and the definition of $\tilde{V}^n$, the relation

$$\tilde{X}^n(t) = \tilde{X}^n(0) + \tilde{I}^n(t) + \tilde{J}^n(t)$$

is $\tilde{P}$-a.s. satisfied for any $t \in R_+$ and any $n \geq 1$.

As a consequence of the above Lemma and of Theorems (2.1), (2.3), the last relation can be written in the following way

$$(3.13)\quad \tilde{X}^n(t) = \tilde{X}^n(0) + \int_{]0,t]} r_n(s, \tilde{X}^n_s)\, d\tilde{A}^n(s) + \int_{]0,t]} \sigma_n(s, \tilde{X}^n_s)\, d\tilde{M}^n(s).$$

On account of Lemma (3.6), properties corresponding to (1.4) and (1.5) hold for sequence $(\bar{M}^n)_n$.

Moreover, from assumption (1.7) and property (3.9), we easily see that the law of $\bar{A}$ is the same as the law of H, where H is the deterministic process defined in Remark (1.11).

Because of the properties of Skorokhod topology, we have

(3.14) *As n tends to infinity, sequence $(\bar{A}^n)_n$ P-almost surely converges to the deterministic process H* (1.11) *in the topology of uniform convergence on compact sets.*

Because of (3.10), also the following equality is P-almost surely verified: for any $t \geq 0$

$$\bar{X}(t) = \bar{X}(0) + I(t) + J(t). \tag{3.15}$$

4. - Proof of the theorem. Limit procedure

To simplify notations, from now on we will always omit the bar in the symbols.

First we want to show that process I is identified as the process $\int r(s, X_.)\,ds$.

(4.1) Lemma: *The second component I of the random element V in* (3.10), *as a process, is P-indistinguishable from process $\int r(s, X_.)\,ds$.*

Proof: Process I is P-almost surely continuous (cf. Proof of Lemma (3.5)) and, by (3.5), so is process X: thus, from (3.9) and the properties of Skorokhod topology, as $n \to +\infty$ processes X^n, I^n P-a.s. converge to X, I, respectively, in the topology of uniform convergence on compact sets.

If we set, for any t

$$B^n(t) = \inf\{s : A^n(s) > t\},$$

for any ω, $n \geq 1$ and t, we have (see [5], Chap. IV, Th. 43):

$$\int_{[0,t]} r_n(s, X^n_.(\omega))\, dA^n(s) = \int_0^{A^n(t)} r_n(B^n(s), X^n_.(\omega))\, ds. \tag{4.2}$$

Since sequence $(A^n)_n$ P-a.s. converges to the deterministic process H in the τ_u-topology ((3.14)), it's easy to see that, for P-almost every ω, the following relation holds

$$\lim_{n\to\infty} B^n(t) = t \qquad \text{for all } t \in \mathbf{R}_+. \tag{4.3}$$

Hence, because of assumption (1.6), P-a.s. we have

$$\text{(4.4)} \qquad \lim \int_0^t r_n(B^n(s), X^n_\cdot(\omega))\, ds = \int_0^t r(s, X_\cdot(\omega))\, ds \qquad \text{for each } t.$$

For a fixed (t, ω), set $E_n =]t, A^n(t, \omega)]$ if $t < A^n(t, \omega)$, $E_n =]A^n(t, \omega), t]$ otherwise.

Because of (4.2), we may write:

$$\int_{]0,t]} r_n(s, X^n_\cdot(\omega))\, dA^n(s) = \int_{]0,t]} r_n(B^n(s), X^n_\cdot(\omega))\, ds \pm \int_{E_n} r_n(B^n(s), X^n_\cdot(\omega))\, ds$$

and finally, recalling the uniform boundedness assumption on $(r_n)_n$, from the previous relation, (4.4) and (3.14), we have P-a.s.

$$\lim_{n\to\infty} \int_{]0,t]} r_n(s, X^n_\cdot(\omega))\, dA^n(s) = \int_0^t r(s, X_\cdot(\omega))\, ds \qquad \text{for each } t.$$

But process I^n is P-indistinguishable from process $\int r_n(s, X^n_\cdot)\, dA^n$ (Lemma (2.3)), so we get the conclusion from the last relation and the initial remark. □

Now we quote the following

(4.5) Definition (see [7], Chap. II, Def. 7.1, p. 89): *We say a stochastic basis* $(\hat{\Omega}, \bar{\mathcal{F}}, (\bar{\mathcal{F}}_t)_{t\in\mathbb{R}_+}, \bar{P})$ *an extension of the stochastic basis* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P)$, *if there exists a measurable mapping*

$$\pi: (\hat{\Omega}, \bar{\mathcal{F}}) \to (\Omega, \mathcal{F})$$

such that

(*a*) *for every* t, $\pi^{-1}(\mathcal{F}_t) \subset \bar{\mathcal{F}}_t$;

(*b*) $P = \pi(\bar{P})$;

(*c*) *for every random variable* $X \in L_\infty(\Omega, \mathcal{F}, P)$

$$E_{\bar{P}}[\bar{X} | \bar{\mathcal{F}}_t] = E_P[X | \mathcal{F}_t] \circ \pi, \qquad \bar{P}\text{-a.s.},$$

where we set $\bar{X} := X \circ \pi$.

Next we want to prove the following:

(4.6) Proposition: *Consider processes* X, J *in the definition* (3.10) *of* V. *Then there exists an extension* $(\hat{\Omega}, \bar{\mathcal{F}}, (\bar{\mathcal{F}}_t)_{t\in\mathbb{R}_+}, \bar{P})$ *of the basis* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P)$

defined in (3.11) (*) *and a* $(\bar{\mathcal{F}}_t)$*-Wiener process* $\bar{W}$ *such that the stochastic integral* $\int \sigma(s, \bar{X}_\cdot)\, d\bar{W}$ *is* $\bar{P}$*-indistinguishable from the process* $\bar{J}(t, \bar{\omega})$, *where we set*

$$\bar{X}_\cdot = X_\cdot \circ \pi\,, \qquad \bar{J}_\cdot = J_\cdot \circ \pi$$

with $\pi\colon (\bar{\Omega}, \bar{\mathcal{F}}) \to (\Omega, \mathcal{F})$ *denoting a mapping that verifies the appropriate properties of the previous Definition.*

To prove the above Proposition, we proceed in several steps.

(4.7) Lemma: *Let* $C_c^2(\mathrm{R})$ *denote the space of the real twice continuously differentiable functions with a compact support.*

Set $E = \mathrm{R} - \{0\}$. *Fix any* $\varphi \in C_c^2(\mathrm{R})$ *and pose, for* $n \geq 1$

$$\Phi^n(t) = \varphi(J^n(t)) - \varphi(J^n(0)) - \tfrac{1}{2}\int_0^t \varphi''(J^n(s^-))\, \sigma_n^2(s, X_\cdot^n)\, d\langle {}^cM^n\rangle(s) -$$

$$-\int_{]0,t]}\int_E \left[\varphi(J^n(s^-) + \sigma_n(s, X_\cdot^n)x) - \varphi(J^n(s^-)) - \varphi'(J^n(s^-))\, \sigma_n(s, X_\cdot^n)x\right] \nu_n(ds, dx)\,,$$

$$\Phi(t) = \varphi(J(t)) - \varphi(J(0)) - \tfrac{1}{2}\int_0^t \varphi''(J(s))\, \sigma^2(s, X_\cdot)\, ds\,,$$

where X^n, J^n, X, J, M^n *are the processes defined in* (3.10) *and* ${}^cM^n$, ν_n *denotes the continuous part of* M^n, *the compensator of the jumps measure of* M^n, *respectively.*

Then, for each n, Φ^n *is a* $(\mathcal{F}_t^n)$*-martingale.*

Moreover, for any t, *sequence* $(\Phi^n(t))_n$ *converges in probability to* $\Phi(t)$ *when* n *tends to infinity.*

Proof: For any t and $n \geq 1$ one has

$$E_P\Big[\int_{]0,t]}\int_E x^2 \nu_n(ds, dx)\Big] < +\infty\,.$$

Bearing in mind the properties of the stochastic integral with respect to random measures, from the last relation one can easily deduce the martingale property of Φ^n by a standard application of the transformation formula for semimartingales (see [16], Th. 25.7, p. 179) and Taylor's expansion (cf. [25], p. 267).

Now let us prove the second statement in the Lemma.

(*) Recall what we have agreed upon at the beginning of this paragraph.

Fix $t \in \mathbf{R}_+$ and set, for every $n \geq 1$:

$$(4.8) \qquad \Xi^n(t) := \frac{1}{2}\int_0^t \varphi''(J^n(s^-))\,\sigma_n^2(s, X_\cdot^n)\,d\langle {}^cM^n\rangle(s) +$$

$$+\int_{]0,t]}\int_E \left[\varphi(J^n(s^-) + \sigma_n(s, X_\cdot^n)x) - \varphi(J^n(s^-)) - \varphi'(J^n(s^-))\,\sigma_n(s, X_\cdot^n)x\right] \nu_n(ds, dx)\,.$$

As n tends to infinity, $J_\cdot^n$ P-a.s. converges to $J_\cdot$ in the τ_u topology: thus to verify the second statement it suffices to prove that

$$(4.9) \qquad \Xi^n(t) \xrightarrow[n]{P} \frac{1}{2}\int_0^t \varphi''(J(s))\,\sigma^2(s, X_\cdot)\,ds\,.$$

Because of Taylor's expansion, for any n and ω, we may rewrite (4.8) as follows:

$$(4.10) \qquad \Xi^n(t) = \frac{1}{2}\int_0^t \varphi''(J^n(s^-))\,\sigma_n^2(s, X_\cdot^n)\,d\langle {}^cM^n\rangle(s) +$$

$$+\frac{1}{2}\int_{]0,t]}\int_E \varphi''(J^n(s^-) + \theta_n(s,\omega)\sigma_n(s, X_\cdot^n)x)\,\sigma_n^2(s, X_\cdot^n)x^2\,\nu_n(ds, dx)$$

where $\theta_n(s,\omega)$ denotes a suitable real number such that $0 \leq \theta_n \leq 1$.

Then we have for all ω:

$$(4.11) \qquad \left|\Xi^n(t) - \frac{1}{2}\int_0^t \sigma^2(s, X_\cdot)\varphi''(J(s))\,ds\right| \leq$$

$$\leq \frac{1}{2}\left|\int_{]0,t]}\int_E \varphi''(J^n(s^-) + \theta_n(s,\omega)\sigma_n(s, X_\cdot^n)x)\,\sigma_n^2(s, X_\cdot^n)x^2\,\nu_n(ds, dx) -\right.$$

$$\left.-\int_{]0,t]}\int_E \varphi''(J^n(s^-))\,\sigma_n^2(s, X_\cdot^n)x^2\,\nu_n(ds, dx)\right| + \frac{1}{2}\left|\int_0^t \varphi''(J^n(s^-))\,\sigma_n^2(s, X_\cdot^n)\,d\langle {}^cM^n\rangle(s) +\right.$$

$$\left.+\int_{]0,t]}\int_E \varphi''(J^n(s^-))\,\sigma_n^2(s, X_\cdot^n)x^2\,\nu_n(ds, dx) - \int_0^t \sigma^2(s, X_\cdot)\varphi''(J(s))\,ds\right| =$$

$$= \zeta^n(t) + \left|\Psi^n(t) - \frac{1}{2}\int_0^t \sigma^2(s, X_\cdot)\varphi''(J(s))\,ds\right|,$$

where processes ζ^n, Ψ^n are defined in an obvious manner.

First we show that

$$(4.12) \qquad \zeta^n(t) \xrightarrow[n]{P} 0\,.$$

Consider an arbitrary subsequence, still denoted by $(\zeta^n(t))_n$. Since φ'' is uniformly continuous, for an arbitrary $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that the following holds

$$|\varphi''(J^n(s^-) + \theta_n(s, \omega)\sigma_n(s, X^n_\cdot)x) - \varphi''(J^n(s^-))| < \varepsilon$$

for every real number x satisfying $|x| \leq \delta(\varepsilon)$, all ω, s and *all* n.

From this relation we deduce for *all* n

$$\zeta^n(t) = \frac{1}{2}\left| \int_{]0,t]} \int_{R} \varphi''(J^n(s^-) + \theta_n(s, \omega)\sigma_n(s, X^n_\cdot)x)\, \sigma_n^2(s, X^n_\cdot)x^2 \nu_n(ds, dx) - \right.$$

$$\left. - \int_{]0,t]} \int_{R} \varphi''(J^n(s^-))\, \sigma_n^2(s, X^n_\cdot)x^2 \nu_n(ds, dx) \right| \leq$$

$$\leq \frac{\varepsilon}{2} K^2 \int_{]0,t]} \int_{|x| \leq \delta(\varepsilon)} x^2 \nu_n(ds, dx) + \frac{1}{2} \int_{]0,t]} \int_{|x| > \delta(\varepsilon)} 2LK^2 x^2 \nu_n(ds, dx),$$

where K, L denotes an upper bound for the values of $(|\sigma_n|_n)$, $|\varphi''|$ respectively.

Thus

$$\limsup_n \zeta^n(t) \leq \frac{\varepsilon}{2} K^2 \limsup_n \int_{]0,t]} \int_{|x| \leq \delta(\varepsilon)} x^2 \nu_n(ds, dx) + \tag{4.13}$$

$$+ LK^2 \limsup_n \int_{]0,t]} \int_{|x| > \delta(\varepsilon)} x^2 \nu_n(ds, dx).$$

Now recall assumptions (1.4), (1.5) and the final remarks in § 3: passing to a further suitable subsequence, we may suppose that P-a.s.

$$\lim_{n \to \infty} \langle M^n \rangle(t) = t \quad \text{and} \quad \lim_{n \to \infty} \int_{]0,t]} \int_{|x| > \delta(\varepsilon)} x^2 \nu_n(ds, dx) = 0. \tag{4.14}$$

Because of these relations and (3.7), P-a.s. we have

$$\lim_{n \to \infty} \left[\langle {}^c M^n \rangle(t) + \int_{]0,t]} \int_{|x| \leq \delta(\varepsilon)} x^2 \nu_n(ds, dx) \right] = t,$$

and finally, from (4.13) and (4.14), we obtain for the latter subsequence, P-a.s.

$$\limsup_n \zeta^n(t) \leq \frac{\varepsilon}{2} K^2 t.$$

Since ε was arbitrary, P-a.s. $\lim_{n \to \infty} \zeta^n(t) = 0$ and we get (4.12).

Now to show that

$$\Psi^n(t) \xrightarrow[n]{P} \frac{1}{2} \int_0^t \sigma^2(s, X_\cdot) \varphi''(J(s))\, ds, \tag{4.15}$$

remark that relation (3.7) implies

$$\Psi^n(t) = \tfrac{1}{2}\int_0^t \varphi''(J^n(s^-))\,\sigma_n^2(s, X_\cdot^n)\,d\langle{}^cM^n\rangle(s) +$$

$$+ \tfrac{1}{2}\int_{]0,t]}\int_E \varphi''(J^n(s^-))\,\sigma_n^2(s, X_\cdot^n)\,x^2\,\nu_n(ds, dx) = \tfrac{1}{2}\int_{]0,t]} \varphi''(J^n(s^-))\,\sigma_n^2(s, X_\cdot^n)\,d\langle M^n\rangle(s)\,.$$

Passing to a suitable subsequence, we may suppose that, as $n \to +\infty$, P-a.s.

$$\lim_{n\to+\infty}\Big[\sup_{0\le s\le t}|\langle M^n\rangle(s) - s|\Big] = 0\,.$$

Since processes X^n, J^n P-a.s. converge to X, J, respectively, in the τ_u topology and processes J is P-a.s. continuous, it's easy to see, as in the proof of Lemma (4.1), that for the latter subsequence $(\Psi^n(t))_n$ one has, P-a.s.

$$\lim_{n\to\infty}\Psi^n(t) = \frac{1}{2}\int_0^t \varphi''(J(s))\,\sigma^2(s, X_\cdot)\,ds\,.$$

Relation (4.15) follows. Because of (4.11) and (4.12) we then have (4.9) and the proof is complete. □

Now, because of (4.10) and the uniform boundedness assumption on the coefficients $(\sigma_n)_n$, we have, for any n and t

$$|\Phi^n(t)| \le K_1 + K_1\langle M^n\rangle(t)$$

where $\Phi^n(t)$ is the process defined in the above Lemma and K_1 is a suitable positive constant independent of n.

Because of the above inequality and assumption (1.4), for every t the sequence $(\Phi^n(t))_n$ is a P-uniformly integrable family: using the properties stated in the previous Lemma and the same kind of argument as in the proof of Proposition (2.4), one easily proves the following

(4.16) Lemma: *For every* $\varphi \in C_c^2(\mathbb{R})$, *process* Φ *defined in the preceding Lemma, is a* $(\mathcal{F}_t)$*-martingale, where* $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ *is the usual P-augmentation of the filtration generated by the random element* V (cf. (3.11) (*)).

Since P-a.s. $\lim_{n\to\infty} J^n(0) = J(0)$ and, for all n, $P(\{J^n(0) = 0\}) = 1$, we have $P(\{J(0) = 0\}) = 1$. As a consequence of the latter Lemma, process $J(t) - J(0)$ is a locally square integrable local $(\mathcal{F}_t)$-martingale on the basis

(*) Recall what we have agreed upon at the beginning of this paragraph.

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P)$ (*) and thus the same holds for process J. From the same Lemma, we can also easily see that the Meyer process of J is $\int \sigma^2(s, X_\cdot)\, ds$.

Thus, by THEOREM II-7.1′ in [7], there exists an extension $(\tilde\Omega, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\in\mathbb{R}_+}, \tilde P)$ of the basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P)$ and an $(\tilde{\mathcal{F}}_t)$-Wiener process $\tilde W$, such that the process $\tilde J(t, \tilde\omega)$ is $\tilde P$-indistinguishable from the stochastic integral

$$\int \sigma(s, \tilde X_\cdot(\tilde\omega))\, d\tilde W(s)\,,$$

where we set

$$\tilde X_\cdot = X_\cdot \circ \pi\,, \qquad \tilde J_\cdot = J_\cdot \circ \pi\,,$$

$\pi\colon (\tilde\Omega, \tilde{\mathcal{F}}) \to (\Omega, \mathcal{F})$ denoting a mapping with the appropriate properties of Definition (4.5).

Thus Proposition (4.6) is proved. □

Now, because of (3.15) and Lemma (4.1)

$$X_\cdot(\omega) = X(0, \omega) + \int_0^\cdot r(s, X_\cdot(\omega))\, ds + J_\cdot(\omega)$$

for P-almost every $\omega \in \Omega$.

Let $\tilde P$ the probability introduced in Proposition (4.6): owing to (4.5) (*b*), $P = \pi(\tilde P)$ and then from the preceding relation we obtain

$$X_\cdot(\pi\tilde\omega) = X(0, \pi\tilde\omega) + \int_0^\cdot r(s, X_\cdot(\pi\tilde\omega))\, ds + J_\cdot(\pi\tilde\omega)$$

that is

$$\tilde X_\cdot(\tilde\omega) = \tilde X(0, \tilde\omega) + \int_0^\cdot r(s, \tilde X_\cdot(\tilde\omega))\, ds + \tilde J_\cdot(\tilde\omega)$$

for $\tilde P$-almost every $\tilde\omega \in \tilde\Omega$.

Finally, from Proposition (4.6), we conclude that $\tilde P$-a.s.

$$\tilde X_\cdot(\tilde\omega) = \tilde X(0, \tilde\omega) + \int_0^\cdot r(s, \tilde X_\cdot(\tilde\omega))\, ds + \int_0^\cdot \sigma(s, \tilde X_\cdot(\tilde\omega))\, d\tilde W(s)\,.$$

In other words we have found a solution of the equation in (1.8): this solution is realized by processes $\tilde X$, $\tilde W$ defined on the stochastic basis $(\tilde\Omega, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\in\mathbb{R}_+}, \tilde P)$.

Since $X(0)$, $\tilde X(0)$ have the same distribution λ, the law of process $\tilde X$ is Λ ((1.8), (1.9)).

But processes X and $\tilde X$ have the same law and since the law of X is $\pi_1(\Gamma)$ (recall (3.10)), we obtain $\pi_1(\Gamma) = \Lambda$.

This concludes the proof of Theorem (1.10).

(*) See the footnote at the preceding page.

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