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On the Existence, Uniqueness and Asymptotic Properties for Solutions of Flows of Asymmetric Fluids (**)

SUMMARY. — We consider an initial boundary-value problem for the system of equations describing nonstationary flows of asymmetric fluids. In [9] we established the existence of a weak solution of the problem in the interval $(0, T)$, where T is an arbitrary positive real. In the present paper we prove the existence of a unique global in time solution, provided the viscosity ν is greater than some $\nu^* > 0$ and the data in «small» in comparison with ν. We show also that when no external forces act then the solution decays to zero when $t \to \infty$.

Esistenza, unicità e proprietà asintotiche per le soluzioni di problemi relativi a correnti di fluidi asimmetrici

RIASSUNTO. — Si considera un problema di valori iniziali e di frontiera per il sistema di equazioni che descrivono il flusso non stazionario di fluidi asimmetrici. Nel lavoro [9] è stata dimostrata l'esistenza di una soluzione debole nell'intervallo $(0, T)$, ove T è un numero positivo arbitrario. Nel presente lavoro si dimostra l'esistenza di una unica soluzione globale, rispetto al tempo, purchè la viscosità ν sia maggiore di un conveniente $\nu^* > 0$ e il dato sia «piccolo» rispetto a ν. Si dimostra inoltre che in assenza di forze esterne la soluzione tende a zero per $t \to 0$.

0. - INTRODUCTION

In this paper we consider the system of equations describing the motion of viscous, incompressible and isotropic fluids with asymmetric stress tensor:

$$u_t - (\nu + \nu_r)\Delta u + (u \cdot \nabla)u + \nabla p = 2\nu_r \operatorname{rot} \omega + f, \tag{0.1}$$

$$\operatorname{div} u = 0, \tag{0.2}$$

$$\omega_t - (c_a + c_d)\Delta\omega - (c_0 + c_d - c_a)\nabla \operatorname{div} \omega + (u \cdot \nabla)\omega + 4\nu_r\omega = 2\nu_r \operatorname{rot} u + g. \tag{0.3}$$

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Equations (0.1)-(0.3) are conservation laws: conservation of linear momentum, mass and angular momentum, respectively, of the fluid. For the derivation of these equations and their discussion see [4], [6], [10], [11].

We add to the system (0.1)-(0.3) the following initial and boundary data

$$u|_{t=0} = u_0, \qquad u|_{S\times(0,T)} = 0, \tag{0.4}$$

$$\omega|_{t=0} = \omega_0, \qquad \omega|_{S\times(0,T)} = 0. \tag{0.5}$$

The functions $u = (u_1, u_2, u_3)$, $\omega = (\omega_1, \omega_2, \omega_3)$ and p denote, respectively, the velocity vector, the angular velocity vector of rotation of particles and the pressure of the fluid. The functions $f = (f_1, f_2, f_3)$ and $g = (g_1, g_2, g_3)$ denote external sources of linear and angular momentum, respectively. They depend explicitly upon external fields. Positive constants ν, ν_r, c_0, c_a, c_d characterize isotropic properties of the fluid; ν is the usual kinematic Newtonian viscosity, ν_r, c_0, c_a, c_d are new viscosities connected with the asymmetry of the stress tensor and, in consequence, with the appearance of the field of internal rotation ω in the considered model of the fluid; $c_0 + c_d > c_a$.

By ∇, Δ, rot and div we denote the usual gradient, Laplacian, rotation and divergence operators, so that Δu, $(u\cdot\nabla)u$, $(u\cdot\nabla)\omega$, ∇p and rot ω are vectors with components Δu_i, $u_j(\partial/\partial x_j)u_i$, $u_j(\partial/\partial x_j)\omega_i$, $(\partial/\partial x_i)p$ and $\varepsilon_{mij}(\partial/\partial x_j)\omega_m$, $(i = 1, 2, 3)$ respectively (repeated indices are summed, ε_{mij} is the alternating tensor of Levi-Civita), $\operatorname{div} u = (\partial/\partial x_j)u_j$, $u_t = (\partial/\partial t)u$.

Remark that if we put $\nu_r = c_0 = c_a = c_d = 0$ and $g = 0$ in (0.1) and (0.3) then $\omega = 0$ and the system (0.1)-(0.3) reduces to the Navier-Stokes system of classical hydrodynamics. Also, if the kinematic rotational viscosity $\nu_r = 0$, problems (0.1), (0.2), (0.4) and (0.3), (0.5) become independent of each other.

We consider the initial boundary-value problem (0.1)-(0.5) in the space-time region $Q_T = D\times(0, T)$, where D is a bounded domain in R^3, locally situated on one side of its boundary S, S is a smooth manifold of dimension 2, and $0 < T < \infty$.

Before stating the results we introduce some function spaces that we use throughout the paper.

1) L^p = usual $L^p(D)$ space $(1 < p < \infty)$ with usual norm $|\cdot|_p$,

2) H_0^1 = closure of $C_0^\infty(D; R^3)$ in the norm $\|u\|_1 = |\nabla u|_2$,

3) $\mathcal{V} = \{u \in C_0^\infty(D; R^3) : \operatorname{div} u = 0\}$,

4) H = closure of $\mathcal{V}$ in L^2,

5) V = closure of $\mathcal{V}$ in H_0^1,

6) H^2 = Sobolev space $H^2(D)$ with norm

$$\|u\|_2 = \Big(|u|_2^2 + |\nabla u|_2^2 + \sum_{i,j=1}^{3} |D_{ij}u|_2^2\Big)^{\frac{1}{2}},$$

7) H^2_0 = closure of $C^\infty_0(D; R^3)$ in H^2,

8) $W = H^2 \cap V$, with norm $\|\cdot\|_2$,

9) H^{-m} dual space to H^m_0 $(m = 1, 2)$,

10) $L^p(0, T; X)$ = Banach space of strongly measurable functions in $(0, T)$ with values in the Banach space X, for which

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p \, dt\right)^{1/p} < \infty ,$$

with the usual modification if $p = \infty$,

11) $C(0, T; X)$ = space of continuous functions in $[0, T]$ with values in X,

12) $L^p_{loc}(0, \infty; X)$ = space of functions defined in $(0, \infty)$ with values in X, whose restrictions to $(0, T)$ belong to $L^p(0, T; X)$, for every $T > 0$.

For basic properties of the above function spaces see [7], [8], [12].

By $(\cdot, \cdot)$ we denote the scalar product in L^2; $b(u, v, \omega) = ((u \cdot \nabla) v, \omega)$.

In this paper we are interested in solving

Problem 0.1: Given $u_0 \in V$, $\omega_0 \in L^2$ and $f, g \in L^\infty(0, \infty; L^2)$, find (u, ω) such that

(i) $u \in L^\infty(0, \infty; V) \cap L^2_{loc}(0, \infty; H^2)$ with $u_t \in L^2_{loc}(0, \infty; H)$,
$\omega \in L^\infty(0, \infty; L^2) \cap L^2_{loc}(0, \infty; H^1_0)$ with $\omega_t \in L^2_{loc}(0, \infty; H^{-1})$,

(ii) $u(0) = u_0$, $\omega(0) = \omega_0$,

(iii) for every $T > 0$ the following identities hold

$$\int_0^T (u_t - (\nu + \nu_r) \Delta u, \varphi) \, dt + \int_0^T b(u, u, \varphi) \, dt = \int_0^T (2\nu_r \operatorname{rot} \omega + f, \varphi) \, dt \tag{0.6}$$

for all $\varphi \in L^2(0, T; H)$.

$$\int_0^T \langle \omega_t, \psi \rangle \, dt + (c_a + c_d) \int_0^T (\nabla \omega, \nabla \psi) \, dt + (c_0 + c_d - c_a) \int_0^T (\operatorname{div} \omega, \operatorname{div} \psi) \, dt + \int_0^T b(u, \omega, \psi) \, dt + 4\nu_r \int_0^T (\omega, \psi) \, dt = \int_0^T (2\nu_r \operatorname{rot} u + g, \psi) \, dt \tag{0.7}$$

for all $\psi \in L^2(0, T; H^1_0)$.

In (0.7) $\langle \cdot, \cdot \rangle$ is the pairing between H^{-1} and H^1_0.

In view of (i) the demands in (ii) have sense, as from a general interpolation theorem (see [8], Chap. I, Theorem 3.1) it follows that $u \in C(0, T; V)$ and $\omega \in C(0, T; L^2)$ for every $T > 0$.

For convenience, in the weak formulation (0.6) of equation (0.1) we eliminated the pressure p. We refer the reader to [12], where the problem of existence (and regularity) of a distribuzion p such that (0.1) holds is considered. See also [9].

Our aim is to prove

THEOREM 0.1: There exist positive constants c_1, c_2, d such that if

$$\frac{\nu+\nu_r}{4c_1} - \nu_r > 0\,, \tag{0.8}$$

$$\frac{30\nu_r^2}{\nu+\nu_r} < \frac{c_a+c_d}{2}\,, \tag{0.9}$$

$$\frac{c_2}{(\nu+\nu_r)^3}\left(\|u_0\|_1^2 + |\omega_0|_2^2\right)^2 < \frac{1}{2}\min\left(\frac{\nu+\nu_r}{4c_1} - \nu_r,\ \frac{c_a+c_d}{2}\right) = k\,, \tag{0.10}$$

and

$$\max\left(\frac{8}{\nu+\nu_r}, \frac{d^2}{c_a+c_d}\right)\left(\|f\|^2_{L^\infty(0,\infty;L^2)} + \|g\|^2_{L^\infty(0,\infty;L^2)}\right) < (\nu+\nu_r)\,c_2^{-\frac{1}{2}}k^{\frac{3}{2}} \tag{0.11}$$

then Problem 0.1 has a unique solution (u, ω).

Moreover, if

$$f, g \in L^2(0, \infty; L^2) \quad \text{then} \quad u \in L^2(0, \infty; H^2) \quad \text{and} \quad \omega \in L^2(0, \infty; H_0^1)\,.$$

In particular, if $f = g = 0$ then the solution decays exponentially to zero when $t \to \infty$; more precisely, for some $M > 0$ and all $t > 0$

$$\|u(t)\|_1^2 + |\omega(t)|_2^2 < \left(\|u_0\|_1^2 + |\omega_0|_2^2\right)\exp(-Mt)\,. \tag{0.12}$$

The plan of the remaining sections of the paper is as follows. In Section 1 we study the linearized problem (0.6) in u. In Section 2 we consider a family of auxiliary problems in ω, which correspond to addition of the term $-\varepsilon\Delta^2\omega$ ($\varepsilon > 0$) to the left-hand side of (0.3). Estimates of u and ω obtained in Sections 1 and 2 allow us to construct, in Section 3, local in time approximate solutions $(u_\varepsilon, \omega_\varepsilon)$, $\varepsilon > 0$ of the problem (0.6)-(0.7), with $T = T(\varepsilon)$. In Section 4 we establisg global estimates of $(u_\varepsilon, \omega_\varepsilon)$, independent of ε, from which we conclude that for all $\varepsilon > 0$ solutions $(u_\varepsilon, \omega_\varepsilon)$ are global in time. Section 5 presents the proof of the existence of a solution (u, ω) of Problem 0.1, by passing to zero with ε. In Section 5 we prove the uniqueness of solutions of Problem 0.1.

For convenience, several universal numeric constants we denote by the letter C without bothering to distinguish them with subscripts. The same concerns constants which depend on ν but stay bounded when ν increases.

1. - The linearized problem (0.6) in u

In this section we consider

Problem 1.1: Given $u_0 \in V$ and $F \in L^2(0, T; L^2)$ find u such that

(i) $u \in C(0, T; V) \cap L^2(0, T; W)$ with $u_t \in L^2(0, T; H)$,

(ii) $u(0) = u_0$,

(iii) the following identity holds for all $\varphi \in L^2(0, T; H)$:

$$\int_0^T (u_t - (\nu + \nu_r)\Delta u, \varphi)\, dt = \int_0^T (F, \varphi)\, dt\,. \tag{1.1}$$

To deal with the above problem it is convenient to introduce an operator. Denote by P the orthogonal projection of L^2 onto H and define the operator $A = -P\Delta$ on W. It is well known [12] that A is an isomorphism between W and H, so that the norms $\|u\|_2$ and $|Au|_2$ are equivalent in W.

Lemma 1.1: There exists a unique solution of Problem 1.1. Moreover, the following inequalities hold

$$|u_t|_2^2 + (\nu + \nu_r)\frac{d}{dt}\|u\|_1^2 + \left(\frac{\nu + \nu_r}{2}\right)^2 |Au|_2^2 < \frac{5}{2}|F|_2^2\,, \tag{1.2}$$

$$\|u_t\|^2_{L^2(0,T;H)} + (\nu + \nu_r)\|u\|^2_{C(0,T;V)} + \left(\frac{\nu + \nu_r}{2}\right)^2 \|Au\|^2_{L^2(0,T;H)} < (\nu + \nu_r)\|u_0\|_1^2 + \frac{5}{2}\|F\|^2_{L^2(0,T;L^2)}\,. \tag{1.3}$$

Proof: A priori estimates. Assume that u is a solution of Problem 1.1. From (1.1) it follows that for all $v \in H$

$$(u_t - (\nu + \nu_r)\Delta u, v) = (F, v) \tag{1.4}$$

in the distribution sense on $(0, T)$. Set $v = u_t + \varepsilon Au$, $\varepsilon > 0$ in (1.4) (cf. [3]). Since

$$(u_t, Au) = \frac{1}{2}\frac{d}{dt}\|u\|_1^2\,.$$

we obtain

$$(1.5)\qquad |u_t|_2^2+\frac{\nu+\nu_r}{2}\frac{d}{dt}\|u\|_1^2+(\nu+\nu_r)\varepsilon|Au|_2^2\le$$
$$\le|F|_2|u_t|_2+\varepsilon|F|_2|Au|_2+\varepsilon|u_t|_2|Au|_2 .$$

Now, we estimate the terms on the right-hand side as follows:

$$|F|_2|u_t|_2\le\tfrac12|u_t|_2^2+|F|_2^2,$$
$$\varepsilon|F|_2|Au|_2\le\varepsilon\tfrac14(\nu+\nu_r)|Au|_2^2+\varepsilon(\nu+\nu_r)^{-1}|F|_2^2,$$
$$\varepsilon|u_t|_2|Au|_2\le\varepsilon\tfrac14(\nu+\nu_r)|Au|_2^2+\varepsilon(\nu+\nu_r)^{-1}|u_t|_2^2.$$

Using these inequalities in (1.5) and taking $\varepsilon=\frac14(\nu+\nu_r)$ we obtain (1.2). Inequality (1.3) follows after integrating both sides of inequality (1.4) in t over $(0, T)$. Having these estimates it is standard to prove the existence of a unique solution of Problem 1.1, see [12], Chapter III, § 3, for instance. We omit the details.

In the sequel we will substitute for F in (1.1) expressions of the form (see equation (0.1))

$$(1.6)\qquad F=F(u,\omega)=2\nu_r\operatorname{rot}\omega+f-(u\cdot\nabla)u .$$

LEMMA 1.2: Let

$$u\in C(0,T;V)\cap L^2(0,T;H^2),$$
$$\omega\in(0,T;L^2)\cap L^2(0,T;H_0^1),\qquad f\in L^2(0,T;L^2)$$

and F be as in (1.6). The following inequalities take place

$$(1.7)\qquad |F|_2^2\le12\nu_r^2\|\omega\|_1^2+3|f|_2^2+C\|u\|_1^3\|u\|_2,$$

$$(1.8)\qquad \|F\|^2_{L^2(0,T;L^2)}\le C\|\omega\|_{C(0,T;L^2)}\cdot\|\omega\|_{L^2(0,T;H_0^1)}\cdot T^{1/2}+$$
$$+3\|f\|^2_{L^2(0,T;L^2)}+C\|u\|^3_{C(0,T;V)}\cdot\|u\|_{L^2(0,T;H^2)}\cdot T^{1/2},$$

$$(1.9)\qquad \|F\|^2_{L^2(0,T;L^2)}\le12\nu_r^2\|\omega\|^2_{L^2(0,T;H_0^1)}+$$
$$+3\|f\|^2_{L^2(0,T;L^2)}+\frac12\left(\frac{\nu+\nu_r}{2}\right)\|Au\|^2_{L^2(0,T;H)}+C\|u\|^4_{C(0,T;V)}\cdot\|u\|^2_{L^2(0,T;L^2)}.$$

PROOF: We concentrate only on nontrivial points of the proof. As $H^1(D)\hookrightarrow L^6(D)$ we have

$$|(u\cdot\nabla)u|_2^2\le|u|_6^2|\nabla u|_3|\nabla u|_6\le C\|u\|_1^3\|u\|_2$$

whence (1.7) follows. To get the first term on the right-hand side of (1.8) observe that $\|\omega\|_1^2 < C\|\omega\|_2\,|\omega|_2$ [5, Theorem 10.1, p. 27]. Hence, by Hölder's inequality

$$\int_0^T \|\omega\|_1^2\,dt < C\int_0^T \|\omega\|_2\,|\omega|_2\,dt < C\|\omega\|_{C(0,T;L^2)}\|\omega\|_{L^2(0,T;H_0^2)}\cdot T^{\frac12}.$$

Similarly, using Hölder's inequality we obtain the last term on the right-hand side of (1.8). To estimate the nonlinear term $(u\cdot\nabla)u$ of F as in (1.9), notice that $|\nabla u|_3 < C|\Delta u|_2^{\frac34}\cdot|u|_2^{\frac14}$ [5, Theorem 10.1, p. 27], thus

$$|(u\cdot\nabla)u|_2^2 < |u|_6^2\,|\nabla u|_3^2 < C|u|_6^2\,|\Delta u|_2^{\frac32}\cdot|u|_2^{\frac12} < C\|u\|_1^2\,|\Delta u|_2^{\frac32}\,|u|_2^{\frac12} <$$
$$< \frac12\left(\frac{\nu+\nu_r}{2}\right)^2|\Delta u|_2^2 + C\|u\|_1^8\cdot|u|_2^2,$$

and (1.9) follows after integration in t.

2. - Auxiliary problem in ω

In this section we study the following

Problem 2.1: Given $\omega_0 \in L^2$, $g \in L^2(0,T;L^2)$ and $u \in C(0,T;V)$ find ω such that

(i) $\omega \in L^2(0,T;H_0^2) \cap C(0,T;L^2)$ with $\omega_t \in L^2(0,T;H^{-2})$,

(ii) $\omega(0) = \omega_0$,

(iii) the following equality holds for all $\psi \in L^2(0,T;H_0^2)$:

$$\int_0^T \langle \omega_t, \psi\rangle\,dt + \varepsilon\int_0^T (\Delta\omega, \Delta\psi)\,dt + (c_a + c_d)\int_0^T (\nabla\omega, \nabla\psi)\,dt +$$
$$+ (c_0 + c_d - c_a)\int_0^T (\operatorname{div}\omega, \operatorname{div}\psi)\,dt + \int_0^T b(u,\omega,\psi)\,dt +$$
$$+ 4\nu_r\int_0^T (\omega,\psi)\,dt = \int_0^T (2\nu_r \operatorname{rot} u + g, \psi)\,dt. \tag{2.1}$$

In (2.1) $\langle\cdot,\cdot\rangle$ is the pairing between H^{-2} and H_0^2:

Our aim is to prove

Lemma 2.1: There exists a unique solution of Problem 2.1. Moreover, the following inequalities hold

$$\frac{d}{dt}|\omega|_2^2+\varepsilon|\Delta\omega|_2^2+(c_0+c_d)\|\omega\|_1^2\leq \nu_r\|u\|_1^2+\frac{d^2}{c_a+c_d}|g|_2^2 \tag{2.2}$$

$(d=$ diameter of $D)$,

$$\|\omega\|^2_{C(0,T;L^2)}+\varepsilon\|\omega\|^2_{L^2(0,T;H_0^2)}+(c_a+c_d)\|\omega\|^2_{L^2(0,T;H_0^1)}\leq 2|\omega_0|_2^2+ \\ +2\nu_r\|u\|^2_{L^2(0,T;V)}+\frac{2d^2}{c_a+c_d}\|g\|^2_{L^2(0,T;L^2)} \tag{2.3}$$

and

$$\|\omega_t\|_{L^2(0,T;H^{-2})}\leq\varepsilon\|\omega\|_{L^2(0,T;H_0^2)}+ \\ +F_1(\|u\|_{L^2(0,T;H)},\|u\|_{C(0,T;V)},\|\omega\|_{L^2(0,T;H_0^1)},\|g\|_{L^2(0,T;L^2)}), \tag{2.4}$$

where F_1 is a continuous, increasing function of its arguments, with $F_1(0)=0$.

Proof: A priori estimates. Assume that ω is a solution of Problem 2.1. From (2.1) it follows that for all $v\in H_0^2$ we have

$$\langle\omega_t,v\rangle+\varepsilon(\Delta\omega,\Delta v)+(c_a+c_d)(\nabla\omega,\nabla v)+ \\ +(c_0+c_d-c_a)(\operatorname{div}\omega,\operatorname{div}v)+b(u,\omega,v)+4\nu_r(\omega,v)=(2\nu_r\operatorname{rot}u+g,v)$$

in the sense of distributions on $(0,T)$. Set $v=\omega(t)$, observe thet $b(u,\omega,\omega)=0$ and [12, Chap. III, Lemma 1.2]

$$\langle\omega_t,\omega\rangle=\frac{1}{2}\frac{d}{dt}|\omega|_2^2,$$

and then use Poincaré's inequality $|\omega|_2\leq d\|\omega\|_1$ to obtain (2.2).

Inequality (2.3) follows easily from (2.2) after integrating it in t over $(0,T)$. We obtain (2.4) directly from (2.1) after a few simple calculations.

To prove the existence of a (unique) solution of Problem 2.1 we can use a general theory of linear parabolic equations (see for example, [8], Chap. III) or we can proceed directly using the Faedo-Galerkin method based on estimates (2.3) and (2.4). We omit the details.

3. - Existence of local solutions of the approximate problems

In this section we prove the existence of a local in time solution $(u_\varepsilon,\omega_\varepsilon)$, $\varepsilon>0$ arbitrary, of

PROBLEM 3.1: Given $u_0 \in V$, $\omega_0 \in L^2$ and $f, g \in L^2(0, T; L^2)$ find $(u, \omega) = (u_\varepsilon, \omega_\varepsilon)$ such that

(i) $u \in C(0, T^*; V) \cap L^2(0, T; H^2)$, $\omega \in (0, T^*; L^2) \cap L^2(0, T^*; H_0^1)$, for some T^*, $0 < T^* < T$,

(ii) $u(0) = u_0$, $\omega(0) = \omega_0$,

(iii) identities (1.1) and (2.1) hold, with T^* in place of T, with $F = F(u, \omega)$ as in (1.6), and with $\varphi \in L^2(0, T^*; H)$, $\psi \in L^2(0, T^*; H_0^1)$.

We shall prove

THEOREM 3.1: For every $\varepsilon > 0$ there exists T^*, $0 < T^* < T$, and $(u, \omega) = (u_\varepsilon, \omega_\varepsilon)$—a solution of Problem 3.1.

PROOF. We use Schauder's principle. Let $\bar{u} \in C(0, T; V) \cap L^2(0, T; H^2)$, $\bar{\omega} \in C(0, T; L^2) \cap (0, T; H_0^1)$ and define a map Φ by $\Phi(\bar{u}, \bar{\omega}) = (u, \omega)$, where u and ω satisfy: $u(0) = u_0$, $\omega(0) = \omega_0$ together with

$$(3.1) \qquad \int_0^T (u_t - (\nu + \nu_r)\Delta u, \varphi)\, dt = \int_0^T (F(\bar{u}, \bar{\omega}), \varphi)\, dt$$

for all $\varphi \in L^2(0, T; H)$, and

$$(3.2) \qquad \int_0^T \langle \omega_t, \psi \rangle\, dt + \varepsilon \int_0^T (\Delta\omega, \Delta\psi)\, dt + (c_a + c_d) \int_0^T (\nabla\omega, \nabla\psi)\, dt +$$

$$+ (c_0 + c_d - c_a) \int_0^T (\operatorname{div} \omega, \operatorname{div} \psi)\, dt + \int_0^T b(\bar{u}, \omega, \psi)\, dt + 4\nu_r \int_0^T (\omega, \psi)\, dt =$$

$$= \int_0^T (2\nu_r \operatorname{rot} \bar{u} + g, \psi)\, dt \qquad \text{for all } \psi \in L^2(0, T; H_0^2).$$

By Lemmas 1.1 and 2.1 to each pair $(\bar{u}, \bar{\omega})$ there corresponds exactly one pair (u, ω), $u \in C(0, T; V) \cap L^2(0, T; H^2)$, $\omega \in C(0, T; L^2) \cap L^2(0, T; H_0^2)$ such that (3.1) and (3.2) hold. To prove Theorem 3.1 it suffices to show that the map Φ has a fixed point.

From (3.1), Lemma 1.1 and (1.8) it follows that

$$(3.3) \qquad \|u_t\|^2_{L^2(0,T;L^2)} + (\nu + \nu_r)\|u\|^2_{C(0,T;V)} + \left(\frac{\nu + \nu_r}{2}\right)^2 \|\Delta u\|^2_{L^2(0,T;H)} \leq$$

$$\leq (\nu + \nu_r)\|u_0\|_1^2 + C\|\bar{\omega}\|_{C(0,T;L^2)} \cdot \|\bar{\omega}\|_{L^2(0,T;H_0^1)} \cdot T^{1/2} +$$

$$+ 3\|f\|^2_{L^2(0,T;L^2)} + C\|\bar{u}\|^3_{C(0,T;V)} \cdot \|\bar{u}\|_{L^2(0,T;H^2)} \cdot T^{1/2}.$$

From (3.2) and Lemma 2.1 we have

$$(3.4)\qquad \|\omega\|^2_{C(0,T;L^2)}+\varepsilon\|\omega\|^2_{L^2(0,T;H^2_0)}+(c_a+c_d)\|\omega\|^2_{L^2(0,T;H^1_0)}<2|\omega_0|^2_2+ + 2\nu_r\|\bar{u}\|^2_{L^2(0,T;V)}+\frac{2d^2}{c_a+c_d}\|g\|^2_{L^2(0,T;L^2)}$$

with

$$(3.5)\qquad \|\omega_t\|_{L^2(0,T;H^{-2})}<\varepsilon\|\omega\|_{L^2(0,T;H^2_0)}+ + F_1(\|\bar{u}\|_{L^2(0,T;H)}, \|\bar{u}\|_{C(0,T;V)}, \|\omega\|_{L^2(0,T;H^1_0)}, \|g\|_{L^2(0,T;L^2)})\,.$$

Let K be the set

$$K=\Big\{(u,\omega):\ \|u\|^2_{C(0,T;V)}+\|u\|^2_{L^2(0,T;H^2)}+\|u_t\|^2_{L^2(0,T;H)}<2(\nu+\nu_r)\|u_0\|^2_1=m_1\,,$$
$$u(0)=u_0\,,$$
$$\|\omega\|^2_{C(0,T;L^2)}+\|\omega\|^2_{L^2(0,T;H^1_0)}+\|\omega\|^2_{L^2(0,T;H^2)}<4|\omega_0|^2_2+2\nu_r m_1=m_2\,,$$
$$\omega(0)=\omega_0\,,$$
$$\|\omega_t\|_{L^2(0,T;H^{-2})}<\varepsilon m_2+F_1(m_1, m_1, m_2, \|g\|^2_{L^2(0,T;L^2)})\Big\}\,.$$

Taking into account inequalities (3.3)-(3.5) it is easy to see that for $T=T^*$ small enough we have $\Phi(K)\subset K$. Since K is a convex and compact subset of $L^2(0,T;L^2)\times L^2(0,T;L^2)$, then to show the existence of a fixed point of Φ in K it suffices to prove that Φ is continuous in the L^2 topology. The result follows easily by compactness arguments. We omit the details.

4. - Existence of global solutions of the approximate problems

In this section we establish global in time estimates of solutions $(u_\varepsilon, \omega_\varepsilon)$, $\varepsilon>0$ of the approximate problems (Problems 3.1), provided certain relations between viscosities and the data u_0, ω_0, f and g hold. These estimates are independent of ε and they imply that for all $\varepsilon>0$ the solutions $(u_\varepsilon, \omega_\varepsilon)$ are global in time.

Lemma 4.1: Let $\varepsilon>0$ and let $(u,\omega)=(u_\varepsilon,\omega_\varepsilon)$ be a solution of Problem 3.1, as in Theorem 3.1. Then the following inequalities hold

$$(4.1)\qquad \int_0^{T^*}\|u_t(t)\|^2_2\,dt+\int_0^{T^*}\|u(t)\|^2_3\,dt< < F_2(\|u_0\|_1, |\omega_0|_2, \|f\|_{L^2(0,T^*;L^2)}, \|g\|_{L^2(0,T^*;L^2)}, \|u\|_{C(0,T^*;V)})\,,$$

$$(4.2)\qquad \int_0^{T^*}\|\omega(t)\|^2_1\,dt<F_3(\|u_0\|_1, |\omega_0|_2, \|f\|_{L^2(0,T^*;L^2)}, \|g\|_{L^2(0,T^*;L^2)})$$

where F_2 and F_3 are continuous, increasing functions of their arguments, indepenent of ε.

PROOF: Proceeding just as in [9] we obtain the following, independent of ε, estimate of (u, ω):

$$\|u\|^2_{L^\infty(0,T^*;H)} + \nu\|u\|^2_{L^2(0,T^*;V)} + \|\omega\|^2_{L^\infty(0,T^*;L^2)} + (c_a + c_d)\|\omega\|^2_{L^2(0,T^*;H_0^1)} < \tag{4.3}$$

$$< 2\left\{\frac{d^2}{\nu}\|f\|^2_{L^2(0,T^*;L^2)} + \frac{d^2}{c_a + c_d}\|g\|^2_{L^2(0,T^*;L^2)} + |u_0|^2_2 + |\omega_0|^2_2\right\}$$

(d = diameter of D). To obtain (4.1) combine (4.3) and (1.3) with $T = T^*$ and F estimated as in (1.9).

Our aim now is to establish inequality (4.9) below. From (1.7) we find

$$|F|^2_2 < 12\nu_r^2\|\omega\|^2_1 + 3|f|^2_2 + \frac{(\nu+\nu_r)^2}{8}|Au|^2_2 + \frac{C}{(\nu+\nu_r)^3}\|u\|^6_1. \tag{4.4}$$

Inequalities (4.4) and (1.2) give

$$\frac{d}{dt}\|u\|^2_1 + \frac{\nu+\nu_r}{4}|Au|^2_2 < \frac{30\nu_r^2}{\nu+\nu_r}\|\omega\|^2_1 + \frac{8}{\nu+\nu_r}|f|^2_2 + \frac{C}{(\nu+\nu_r)^3}\|u\|^6_1. \tag{4.5}$$

Assume that

$$\frac{30\nu_r^2}{\nu+\nu_r} < \frac{c_a + c_d}{2}. \tag{4.6}$$

Then, from (4.5) and (2.2) we conclude

$$\frac{d}{dt}(\|u\|^2_1 + |\omega|^2_2) + \frac{\nu+\nu_r}{4}|Au|^2_2 + \frac{c_a + c_d}{2}\|\omega\|^2_1 < \tag{4.7}$$

$$< \nu_r\|u\|^2_1 + \frac{C}{(\nu+\nu_r)^3}\|u\|^6_1 + \frac{8}{\nu+\nu_r}|f|^2_2 + \frac{d^2}{c_a + c_d}|g|^2_2.$$

Assume that

$$\frac{\nu+\nu_r}{4c_1} - \nu_r > 0 \tag{4.8}$$

where c_1 is such that $\|u\|^2_1 < c_1|Au|^2_2$ for all $u \in W$. From (4.7) and (4.8) we find

$$\frac{d}{dt}(\|u\|^2_1 + |\omega|^2_2) < \tag{4.9}$$

$$< -\left\{\min\left(\frac{\nu+\nu_r}{4c_1} - \nu_r, \frac{c_a + c_d}{2}\right) - \frac{c_2}{(\nu+\nu_r)^3}(\|u\|^2_1 + |\omega|^2_2)^2\right\}\cdot$$

$$\cdot(\|u\|^2_1 + |\omega|^2_2) + \max\left\{\frac{8}{\nu+\nu_r}, \frac{d^2}{c_a + c_d}\right\}(|f|^2_2 + |g|)^2_2.$$

Having inequality (4.9) we are in a position to obtain the main result of this section.

Theorem 4.1: Let $u_0 \in V$, $\omega_0 \in L^2$, $f, g \in L^\infty(0, \infty; L^2)$ and let inequalities (0.8)-(0.11) hold, with c_1, c_2, d as above. Then for every $\varepsilon > 0$ there exists a pair of functions $(u, \omega) = (u_\varepsilon, \omega_\varepsilon)$ such that

(i) $u \in L^\infty(0, \infty; V) \cap L^2_{loc}(0, \infty; H^2)$ with $u_t \in L^2_{loc}(0, \infty; H)$,
$\omega \in L^\infty(0, \infty; L^2) \cap L^2_{loc}(0, \infty; H^2_0)$ with $\omega_t \in L^2_{loc}(0, \infty; H^{-2})$,

(ii) $u(0) = u_0$, $\omega(0) = \omega_0$,

(iii) for every $T > 0$ the following identities hold

$$(4.10) \qquad \int_0^T (u_t - (\nu + \nu_r)\,\Delta u, \varphi)\,dt + \int_0^T b(u, u, \varphi)\,dt = \\ = \int_0^T (2\nu_r \operatorname{rot} \omega + f, g)\,dt\,, \qquad \text{for all } \varphi \in L^2(0, T; H),$$

$$(4.11) \qquad \int_0^T \langle \omega_t, \psi \rangle\,dt + \varepsilon \int_0^T (\Delta\omega, \Delta\psi)\,dt + (c_a + c_d) \int_0^T (\nabla\omega, \nabla\psi)\,dt + \\ + \int_0^T (c_0 + c_d - c_a)(\operatorname{div} \omega, \operatorname{div} \psi)\,dt + \int_0^T b(u, \omega, \psi)\,dt + \\ + 4\nu_r \int_0^T (\omega, \psi)\,dt = \int_0^T (2\nu_r \operatorname{rot} u + g, \psi)\,dt\,, \quad \text{for all } \psi \in L^2(0, T; H^2_0).$$

Proof: Theorem 3.1 assures the existence of local solutions $(u_\varepsilon, \omega_\varepsilon)$ of the problem, such that $u_\varepsilon \in C(0, T^*; V)$, $\omega_\varepsilon \in C(0, T^*; L^2)$ for some $0 < T^* < \infty$. Denote by $T^*(\varepsilon)$ the upper limit of those T^* for which $(u_\varepsilon, \omega_\varepsilon)$ exists in $(0, T^*)$. We shall show that $T^*(\varepsilon) = \infty$.

Assume, on the contrary, that $T^*(\varepsilon) < \infty$ and define $s(t) = \|u_\varepsilon(t)\|_1^2 + \|\omega_\varepsilon(t)\|_2^2$. The function $t \mapsto s(t)$ is continuous on $(0, T^*(\varepsilon))$ and, by definition of $T^*(\varepsilon)$ and Theorem 3.1, we have

$$(4.12) \qquad \lim_{t \to T^*(\varepsilon)} s(t) = \infty\,.$$

From (0.10) it follows that $s(0) < M = (\nu + \nu_r) c_2^{-\frac{1}{2}} k^{\frac{1}{2}}$. We shall contradict (4.12) showing that $s(t) < M$ for all $t > 0$. Assume that for some $t > 0$ we have $s(t) = M$. Then from (4.9) and (0.11) we find that $s'(t) < 0$. Thus $s(t) < M$ for all $t > 0$.

From (4.1) we find $u_t \in L^2_{loc}(0, \infty; H)$ and from (3.5) with $\hat{u} = u$ we obtain $\omega_t \in L^2_{loc}(0, \infty; H^{-2})$.

Corollary 4.1: Let the assumptions of Theorem 4.1 hold. Assume additionally that $f, g \in L^2(0, \infty; L^2)$. Then, for every $\varepsilon > 0$, $u_\varepsilon \in L^2(0, \infty; H^2)$, $\omega_\varepsilon \in L^2(0, \infty; H^1_0)$, with

$$(4.13) \qquad \|u_\varepsilon\|_{L^2(0,\infty;H^2)} + \|\omega_\varepsilon\|_{L^2(0,\infty;H^1_0)} < M_1 ,$$

where M_1 is independent of ε.

Proof: It is a consequence of Lemma 4.1 and the property:

$$(4.14) \qquad s(t) < M \text{ for all } t > 0; \ (M \text{ is independent of } \varepsilon) .$$

5. - Existence of solutions of Problem 0.1

In this section we prove all the assertions of Theorem 0.1 except for the uniqueness, which we treat in Section 6.

Let the assumptions of Theorem 0.1 hold.

Lemma 5.1: There exists a solution (u, ω) of Problem 0.1.

Proof: We obtain a solution (u, ω) of Problem 0.1 as a limit (in appropriate topologies) of a subsequence of the sequence $(u_\varepsilon, \omega_\varepsilon)$, $\varepsilon \to 0$ of approximate solutions, constructed in Section 4.

Fix T, an arbitrary positive real. From (4.14), (4.1) and (3.4) with $\hat{u} = u = u_\varepsilon$, $\omega = \omega_\varepsilon$ it follows that

$$(5.1) \qquad \lim_{\varepsilon \to 0} \varepsilon \|\omega_\varepsilon\|_{L^2(0,T;H^1_0)} = 0 .$$

From (5.1) and (3.5) with $\hat{u} = u = u_\varepsilon$, $\omega = \omega_\varepsilon$ we find that

$$(5.2) \qquad \|(\omega_\varepsilon)_t\|_{L^2(0,T;H^{-1})} < M_2 ; \qquad M_2 \text{ is independent of } \varepsilon .$$

From (4.14), (4.1) and (4.2), (5.2) we conclude that there exists a subsequence (u_μ, ω_μ) $(\mu \to 0)$ such that for some (u, ω)

$$(5.3) \qquad u_\mu \to u \qquad \text{weakly in } L^2(0, T; H^2) ,$$

$$(5.4) \qquad u_\mu \to u \qquad \text{weak-star in } L^\infty(0, T; V) ,$$

$$(5.5) \qquad u_\mu \to u \qquad \text{strongly in } L^2(0, T; V) ,$$

and

$$(5.6) \qquad \omega_\mu \to \omega \qquad \text{weakly in } L^2(0, T; H^1_0) ,$$

$$\omega_\mu \to \omega \quad \text{weak-star in } L^\infty(0, T; L^2), \tag{5.7}$$

$$\omega_\mu \to \omega \quad \text{strongly in } L^2(0, T; L^2), \tag{5.8}$$

$$\mu \Delta^2 \omega_\mu \to 0 \quad \text{in } L^2(0, T; H^{-2}). \tag{5.9}$$

To obtain (5.5) and (5.8) we have used Aubin's theorem [1], [7], [12].

Putting $(u, \omega) = (u_\mu, \omega_\mu)$ in (4.10) and (4.11) and using (5.3)-(5.9) we easily obtain (0.6) and (0.7) for the limit functions, for all $\varphi \in L^2(0, T; H)$, $\psi \in L^2(0, T; H_0^2)$. From (0.7) we conclude that $\omega_t \in L^2(0, T; H^{-1})$ so that (0.7) holds for all $\psi \in L^2(0, T; H_0^1)$. We must prove that $u(0) = u_0$ and $\omega(0) = \omega_0$. To prove that $u(0) = u_0$ we proceed as follows (cf. [2]). The sequence (u_μ) is bounded in $C(0, T; V)$ and in $H^1(0, T; L^2) \hookrightarrow C^{0,1}(0, T; L^2)$, As V is compactly imbedded in L^2, Arzela-Ascoli's theorem gives $u_\mu \to u$ in $C(0, T; L^2)$. Since $u_\mu(0) = u_0$, we obtain in particular $u(0) = u_0$. In the same way we prove that $\omega(0) = \omega_0$.

Thus, we have proved the existence of a solution (u, ω) on the interval $[0, T]$, $(T > 0$ arbitrarily chosen). To prove the existence of a solution on $[0, \infty)$ we proceed in the following way. Let $T_1, T_2, \ldots$ be an increasing and unbounded sequence of reals, with $T_1 > 0$. There exists a subsequence $(u_\mu^{(1)}, \omega_\mu^{(1)})$, $(\mu \to 0)$ of the sequence $(u_\varepsilon, \omega_\varepsilon) - (u_\varepsilon, \omega_\varepsilon)$ as in Theorem 4.1 —converging as in (5.3)-(5.9) on $[0, T_1]$. From the sequence $(u_\mu^{(1)}, \omega_\mu^{(1)})$ we can choose a subseguence $(u_{(2)}^\mu, \omega_\mu^{(2)})$ converging on $[0, T_2]$, and so on. We obtain a solution (u, ω) of Problem 0.1 selecting a diagonal subsequence.

Lemma 5.2: If, in addition, $f, g \in L^2(0, \infty; L^2)$ then $u \in L^2(0, \infty; H^2)$ and $\omega \in L^2(0, \infty; H_0^1)$.

Proof: The result follows directly from (4.13).

Lemma 5.3: If $f = g = 0$ then solutions of Problem 0.1 decay exponentially to zero as $t \to \infty$, namely (0.12) holds.

Proof: Let (u, ω) be a solution of Problem 0.1. The same argument as in Section 4 shows that (u, ω) satisfies inequality (4.9). If $f = g = 0$ then (4.9) and (4.14) give $z'(t) < -kz(t)$, where k is as in (0.10). Thus (0.12) holds with $M = (\nu + \nu_r) c_2^{-\frac{1}{2}} k^{\frac{1}{2}}$.

6. - The uniqueness

In this section we prove that if the assumptions of Theorem 0.1 hold then Problem 0.1 is uniquely solvable.

Suppose that (u_1, ω_1) and (u_2, ω_2) are two solutions of Problem 0.1. From the equations for the difference $(u, \omega) = (u_1 - u_2, \omega_1 - \omega_2)$ we obtain easily

$$(6.1) \qquad (\nu + \nu_r) \frac{d}{dt} \|u\|_1^2 + \left(\frac{\nu + \nu_r}{2}\right)^2 |Au|_2^2 < \frac{5}{2} |F|_2^2 ,$$

$$(6.2) \qquad \frac{1}{2} \frac{d}{dt} |\omega|_2^2 + (c_a + c_d) \|\omega\|_1^2 < |(G, \omega)| ,$$

where

$$(6.3) \qquad F = 2\nu_r \operatorname{rot} \omega - (u \cdot \nabla) u_2 - (u_1 \cdot \nabla) u ,$$

$$(6.4) \qquad G = 2\nu_r \operatorname{rot} u - (u \cdot \nabla) \omega_2 - (u_1 \cdot \nabla) \omega .$$

We have

$$(6.5) \qquad |F|_2^2 < 8\nu_r^2 \|\omega\|_1^2 + C(|\nabla u_2|_4^2 + \|u_1\|_2^2) \|u\|_1^2$$

and

$$(6.6) \qquad |(G, \omega)| < 2\nu_r (\operatorname{rot} u, \omega)| + |((u \cdot \nabla) \omega_2, \omega)| <$$
$$< \nu_r |\omega|_2^2 + \|u\|_1^2 + \frac{1}{4} \left(\frac{\nu + \nu_r}{2}\right) |Au|_2^2 + C \|\omega_2\|_1^2 \cdot |\omega|_2^2 .$$

From (6.1), (6.2) with (6.5), (6.6) and (0.9) we obtain

$$\frac{d}{dt} (\|u\|_1^2 + |\omega|_2^2) < S(t) (\|u\|_1^2 + |\omega|)_2^2$$

where the function $t \mapsto S(t)$ is (locally) integrable in $[0, \infty)$. Since $\|u(0)\|_1^2 + |\omega(0)|_2^2 = 0$, then $u = \omega = 0$ for all $t > 0$. This proves the uniqueness.

REFERENCES

[1] J. P. Aubin, *Un théorème de compacité*, C. R. Acad. Sc. Paris, 256 (1963), 5042-5044.

[2] H. Beirão da Veiga, *Sulla approssimazione delle soluzioni deboli globali della equazione di Navier-Stokes*, prepeint U.T.M. n. 155 (1984), Dip. di Matematica, Univ. di Trento.

[3] H. Beirão da Veiga, *Diffusion on viscous fluids. Existence and asymptotic properties of solutions*, Ann. Sc. Norm. Sup. Pisa, Cl. Sci., IV, 10, n. 2 (1983), 341-355.

[4] D. W. Condiff - J. S. Dahler, *Fluid mechanical aspects of antisymmetric stress*, Phys. Fluids, 11, n. 9 (1964), 842-854.

[5] A. Friedman, *Partial Differential Equations*, Holt, Rinechart and Winston Inc. (1969).

[6] H. Grad, *Statistical mechanics, thermodynamics and fluid dynamics of systems with arbitrary number of integrals*, Comm. Pure Appl. Nath., 5, n. 4 (1952), 455-494.

[7] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Paris (1969).

[8] J. L. Lions - E. Magenes, *Problèmes aux limites non homogeneous et applications*, Paris (1968).

[9] G. Lukaszewicz, *On nonstationary flows of asymmetric fluids*, Rend. Accad. Naz. Sci. XL, Mem. Mat., n. 106, vol. XII (1988), 83-97.

[10] L. G. Petrosyan, *On a model of magnetohydrodynamics of asymmetric fluids*, Appl. Mech., 12, n. 11 (1976), 103-109 (in Russian).

[11] L. G. Petrosyan, *Some Problems of Mechanics of Fluids with Antisymmetric Stress Tensor*, Erevan (1984) (in Russian).

[12] R. Temam, *Navier-Stokes Equations. Theory and Numerical Analysis*, North-Holland (1979).

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