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Periodic Solutions of the Oberbeck-Boussinesq Model Relative to the Flow of a Viscous Fluid Subject to Heating (**) (***)

SUMMARY. — The periodic solutions of the Oberbeck-Boussinesq model relative to the flow of a viscous fluid subject to heating are considered. The model is formulated mathematically by a system consisting of an inequality of Navier-Seokes type and a « heat » equation. An existence and uniqueness result of a periodic solution in an appropriate functional class is proved, provided the data are sufficiently small.

Soluzioni periodiche del modello di Oberbeck-Boussinesq relativo al moto di un fluido viscoso soggetto a riscaldamento

RIASSUNTO. — Si studiano le soluzioni periodiche del modello di Oberbeck-Boussinesq che descrive il moto di un fluido viscoso soggetto a riscaldamento. Questo modello è costituito da un sistema formato da una disequazione di tipo Navier-Stokes e da un'equazione del calore. Si dimostra che, qualora i dati siano sufficientemente piccoli, tale sistema ammette una ed una sola soluzione periodica.

1. - INTRODUCTION

In note [2], a mathematical problem is considered, established for the Oberbeck-Boussinesq equations (see [3]), which, under suitable hypotheses, describes the motion of a viscous, compressible fluid contained in a bounded, 2-dimensional rectangular basin, with a side subject to heating. Relative to such a model and under suitable conditions, a result has been shown regarding the existence and uniqueness of the solution of a Dirichlet-Neumann boundary

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conditions problem both in the evolution case, with initial conditions, and in the stationary case.

The present work intends to continue the study of the above mentioned model, considering again a Dirichlet-Neumann boundary value problem but with data *periodic* in time with given period $\mathcal{T}$. In this work we also take into account (see [10]) the fact that the validity of the Navier-Stokes equations, and consequently also that of the Oberbeck-Boussinesq model, is bound by the condition that the velocity of the fluid is always maintained substantially below a constant c (in particular the velocity of light: otherwise, in fact, the Navier-Stokes equations should be substituted by suitable relativistic equations). Following these observations, the model which is here examined partly differs (in the weak formulation) from the one considered in [2], due to the presence of an *inequality* that, on the other hand, is associated in a natural way to the Navier-Stokes equations. Furthermore, the « natural » functional spaces for the study of the periodic solutions are slightly different from the ones introduced in [2], where the Cauchy problem was considered; the definition of solution must therefore be accordingly modified.

The present work consists of five paragraphs, in addition to this first introductory paragraph; the second paragraph is devoted to a more precise presentation of the problem examined and of the notations used. The third and fourth paragraphs examine, respectively, the periodic solutions of the Navier-Stokes inequalities and of a particular linear « heat » equation. Finally, the fifth and sixth paragraphs study the existence and uniqueness of the periodic solution.

2. - The periodic problem: notations and definition of the solution

Proceeding now to the precise formulation of the problem indicated in the previous paragraph, we assume for the sake of simplicity that Ω is the open square $\{(x_1, x_2) \in \mathbb{R}^2 \mid -1 < x_i < 1,\ (i = 1, 2)\}$, Γ its boundary, Γ_1, Γ_2, Γ_3, Γ_4 its four sides, ordered counterclock from $\Gamma_1 = \{(x_1, -1) \mid -1 < x_1 < 1\}$.

We assume, moreover, that the external force is constituted by the force of gravity. The Oberbeck-Boussinesq model which governs the motion of a fluid in the above described conditions therefore presents the form:

$$\varrho_0 \frac{\partial \boldsymbol{u}}{\partial t} - \mu \Delta \boldsymbol{u} + \varrho_0 (\boldsymbol{u} \cdot \nabla) \boldsymbol{u} + \nabla p = \varrho_0 \{1 - \alpha (T - T_0)\} \boldsymbol{g}\,, \tag{2.1}$$

$$\nabla \cdot \boldsymbol{u} = 0\,, \tag{2.2}$$

$$\frac{\partial T}{\partial t} - \gamma \Delta T + \boldsymbol{u} \cdot \nabla T = 0\,, \tag{2.3}$$

where $\boldsymbol{u} = (u_1, u_2)$ represents the velocity of the fluid, p the pressure, T the temperature, and where ϱ_0 and T_0 are reference values (constant) respectively for pressure and temperature; in addition $\boldsymbol{g}$ is the acceleration of gravity, μ the viscosity, $\gamma = K^*/\varrho_0 C_v$ the thermic diffusivity, with K^* coefficient of thermic conductivity, C_v the specific heat, and α is a constant equal to $R^* P_R/g$, with R^* the Rayleigh thermic number, P_R the Prandtl number, $g = |\boldsymbol{g}|$.

We consider furthermore, the following boundary conditions

$$(2.4) \qquad \boldsymbol{u}|_{\Gamma\times(0,\mathcal{T})} = 0\,,$$

$$(2.5) \qquad T|_{\Gamma_1\times(0,\mathcal{T})} = T_1\,, \quad T|_{\Gamma_2\times(0,\mathcal{T})} = T_2\,, \quad \left.\frac{\partial T}{\partial n}\right|_{(\Gamma_3\cup\Gamma_4)\times(0,\mathcal{T})} = 0\,,$$

where $(0, \mathcal{T})$ is a generic time interval, $\boldsymbol{n}$ is the outward normal to Γ, T_1 and T_2 are given *functions of time independent of space variables*, with $T_2 > T_1$. The *periodic* problem in question is therefore constituted by (2.1)-(2.5), with the associated periodicity condition (writing, $\forall t \in [0, \mathcal{T}]$, $\boldsymbol{u}(t) = \{\boldsymbol{u}(x_1, x_2, t) \mid (x_1, x_2) \in \Omega\}$ and $T(t) = \{T(x_1, x_2, t) \mid (x_1, x_2) \in \Omega\}$)

$$(2.6) \qquad \boldsymbol{u}(0) = \boldsymbol{u}(\mathcal{T})\,, \qquad T(0) = T(\mathcal{T})\,,$$

under the assumption that T_1 and T_2 are *periodic* functions of time with assigned period $\mathcal{T}$.

For the study of our problem, it is convenient to substitute (2.1)-(2.6) with the following *equivalent* equations in the unknown functions $\boldsymbol{u}$ and θ (see also [2]):

$$(2.7) \qquad \frac{\partial \boldsymbol{u}}{\partial t} - \frac{\mu}{\varrho_0}\Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \frac{1}{\varrho_0}\nabla p = \{1 - \alpha(\theta + d - T_0)\}\boldsymbol{g}\,,$$

$$(2.8) \qquad \nabla\cdot\boldsymbol{u} = 0\,,$$

$$(2.9) \qquad \frac{\partial\theta}{\partial t} - \gamma\Delta\theta + \boldsymbol{u}\cdot\nabla\theta = \frac{T_2 - T_1}{2}u_1 - \frac{\partial d}{\partial t}\,,$$

with the homogeneous boundary conditions

$$(2.10) \qquad \boldsymbol{u}|_{\Gamma\times(0,\mathcal{T})} = 0\,,$$

$$(2.11) \qquad \theta|_{(\Gamma_1\cup\Gamma_2)\times(0,\mathcal{T})} = 0\,, \quad \left.\frac{\partial\theta}{\partial n}\right|_{(\Gamma_3\cup\Gamma_4)\times(0,\mathcal{T})} = 0\,,$$

and the condition of periodicity

$$(2.12) \qquad \boldsymbol{u}(0) = \boldsymbol{u}(\mathcal{T})\,, \qquad \theta(0) = \theta(\mathcal{T})\,,$$

where d: $[-1, 1]\times[0, \mathcal{T}] \to \mathbf{R}$ is a known function defined by:

$$(2.13) \qquad d(x_1, t) = \frac{T_2(t) - T_1(t)}{2}x_1 + \frac{T_1(t) + T_2(t)}{2}\,.$$

and θ is a new unknown related to T by the relationship

$$T = \theta + d . \tag{2.14}$$

In order to give a precise *definition of solution* of problem (2.7)-(2.12), the following notations are introduced:

$$\mathcal{V} = \{\boldsymbol{v} = (v_1, v_2) \mid v_j \in \mathcal{D}(\Omega),\ (j = 1, 2),\ \nabla \cdot \boldsymbol{v} = 0\} ; \tag{2.15}$$

$$V = \{\boldsymbol{v} \in (H_0^1(\Omega))^2 \mid \nabla \cdot \boldsymbol{v} = 0\} = \overline{\mathcal{V}}^{H^1} ; \tag{2.16}$$

$$H = \{\boldsymbol{v} \in (L^2(\Omega))^2 \mid \nabla \cdot \boldsymbol{v} = 0,\ \gamma_n \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{n}|_\Gamma = 0\} = \overline{\mathcal{V}}^{L^2} ; \tag{2.17}$$

$$\bar{H}_0^1(\Omega) = \{\varphi \in H^1(\Omega) \mid \varphi|_{\Gamma_1 \cup \Gamma_2} = 0\} ; \tag{2.18}$$

$$K = \{\boldsymbol{v} \in H \mid |\boldsymbol{v}| < r,\ \text{a.e. in } \Omega\}, \quad \text{(the set } K \text{ is closed and convex)} ; \tag{2.19}$$

$$b_1(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \sum_{i,j=1}^{2} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \, d\Omega ; \tag{2.20}$$

$$b_2(\boldsymbol{u}, \theta, \varphi) = \sum_{i=1}^{2} \int_\Omega u_i \frac{\partial \theta}{\partial x_i} \varphi \, d\Omega ; \tag{2.21}$$

$$\langle \, , \rangle = \text{duality between } \bar{H}_0^1(\Omega) \text{ and } (\bar{H}_0^1(\Omega))' . \tag{2.22}$$

Assuming that T_1 and T_2 are periodic functions with period $\mathcal{T}$, we shall say now that $(\boldsymbol{u}, \theta)$ is a « *periodic solution of period $\mathcal{T}$ of the problem* (2.7)-(2.12) » if the following conditions are satisfied:

$$\boldsymbol{u} \in L^2(0, \mathcal{T}; K \cap V) , \tag{2.23}$$

$$\theta \in L^2(0, \mathcal{T}; \bar{H}_0^1(\Omega)) \cap L^\infty(0, \mathcal{T}; L^2(\Omega)) \cap H^1(0, \mathcal{T}; (\bar{H}_0^1(\Omega))') \quad \text{with } \theta(0) = \theta(\mathcal{T}) ; \tag{2.24}$$

$$\begin{aligned} \int_0^{\mathcal{T}} \Big\{ & (\boldsymbol{v}'(\eta), \boldsymbol{u}(\eta) - \boldsymbol{v}(\eta))_{L^2(\Omega)} + \frac{\mu}{\varrho_0} (\boldsymbol{u}(\eta), \boldsymbol{u}(\eta) - \boldsymbol{v}(\eta))_{H_0^1(\Omega)} + \\ & + b_1(\boldsymbol{u}(\eta), \boldsymbol{u}(\eta), \boldsymbol{u}(\eta) - \boldsymbol{v}(\eta)) + \alpha\big(\boldsymbol{g}(\theta(\eta) + d(\eta)), \boldsymbol{u}(\eta) - \boldsymbol{v}(\eta)\big)_{L^2(\Omega)} \Big\} \, d\eta < 0 , \\ & \forall \boldsymbol{v} \in L^2(0, \mathcal{T}; K \cap V) \cap H^1(0, \mathcal{T}; H) ,\ \boldsymbol{v}(0) = \boldsymbol{v}(\mathcal{T}) ; \end{aligned} \tag{2.25}$$

$$\begin{aligned} \int_0^{\mathcal{T}} \Big\{ & \langle \theta'(\eta), \varphi(\eta) \rangle + \gamma(\theta(\eta), \varphi(\eta))_{H_0^1(\Omega)} + \\ & + b_2(\boldsymbol{u}(\eta), \theta(\eta), \varphi(\eta)) - \left(\frac{T_2(\eta) - T_1(\eta)}{2} u_1(\eta) - \frac{\partial d}{\partial \eta}(\eta), \varphi(\eta) \right)_{L^2(\Omega)} \Big\} \, d\eta = 0 , \\ & \forall \varphi \in L^2(0, \mathcal{T}; \bar{H}_0^1(\Omega)) \cap L^4(0, \mathcal{T}; L^4(\Omega)) . \end{aligned} \tag{2.26}$$

For the justification of (2.25) see [10]: in particular, by b'_2) of [10] (with $t = T$, $\varphi(T) = \varphi(0)$ arbitrary), it follows that, if u is continuous, then $u(T) = u(0)$ (see Lemma 3.2).

3. - The periodic solutions of the Navier-Stokes inequalities

The periodic problem for the Navier-Stokes equations and inequalities has already been subjected to study by many authors among which G. Prodi [8], V. T. Judovic [4], G. Prouse [9], A. Zaretti [12] and T. Collini Lanzi [1].

The aim of the present paragraph is therefore to recall some results which are already known and to present the proof of others regarding the uniqueness of the periodic solution of the Navier-Stokes inequalities. For a complete discussion of the physical meaning of the introduction of an inequality in place of (2.7), see [10], as yet observed.

Theorem 3.1: *Assume ν constant and $f \in L^2_{loc}(-\infty, +\infty; L^2(\Omega))$ periodic of period $\mathcal{T}$ (in the sense of distributions:*

$$f(t) = \sum_{n=-\infty}^{+\infty} a_n \exp\left[int\frac{2\pi}{\mathcal{T}}\right], \qquad a_n = \frac{1}{\mathcal{T}}\int_0^{\mathcal{T}} f(t)\exp\left[-int\frac{2\pi}{\mathcal{T}}\right] dt,$$

$$\sum_{-\infty}^{+\infty} \|a_n\|^2_{L^2(\Omega)} < +\infty)$$

Then there exists at least a function u periodic such that, $\forall t_0 \in \mathbb{R}$:

(3.1) $\qquad u \in L^2_{loc}(-\infty, +\infty; K \cap V)$,

(3.2) $$\int_{t_0}^{t_0+\mathcal{T}} \{(v'(\eta), u(\eta) - v(\eta))_{L^2(\Omega)} + \nu(u(\eta), u(\eta) - v(\eta))_{H^1_0(\Omega)} + b_1(u(\eta), u(\eta), u(\eta) - v(\eta)) - (f(\eta), u(\eta) - v(\eta))_{L^2(\Omega)}\}\, d\eta < 0,$$

$$\forall v \in L^2_{loc}(-\infty, +\infty; K \cap V) \cap H^1_{loc}(-\infty, +\infty; H) \text{ periodic of period } \mathcal{T}.$$

Proof: For the proof of this theorem, which is also valid under more general assumptions on Ω and on the boundary conditions, we refer to [1]. ■

Consequently, such a function u will be indicated from now on as « *$\mathcal{T}$-periodic solution of the Navier-Stokes inequalities* ».

Obs. 3.1: Regarding the periodic solution u of the Navier-Stokes inequalities, the following bounds can be proven, reasoning as in [9, p. 446] or also in [1, p. 24]:

$$(3.3)\qquad \|\mathbf{u}\|_{L^\infty(0,\mathcal{T};H)} \leq M_1 = (1-\exp[-\nu\mathcal{T}/\sigma^2])^{-1}\int_0^{\mathcal{T}}\|f(\eta)\|_{L^2(\Omega)}\,d\eta = $$

$$= (1-\exp[-\nu\mathcal{T}/\sigma^2])^{-1}\|f\|_{L^1(0,\mathcal{T};L^2(\Omega))},$$

$$(3.4)\qquad \|\mathbf{u}\|_{L^2(0,\mathcal{T};H_0^1(\Omega))} \leq [\nu(1-\exp[-\nu\mathcal{T}/\sigma^2])]^{-1}\|f\|_{L^1(0,\mathcal{T};L^2(\Omega))},$$

where σ is the embedding constant of $H_0^1(\Omega)$ into $L^2(\Omega)$. ■

LEMMA 3.1: *Assume that the hypotheses of Theorem* 3.1 *hold and* $\mathbf{u}$ *is a periodic solution of the Navier-Stokes inequalities, in the sense specified in Theorem* 3.1. *Then for every pair of points* (t_1, t_2) *that fall out of a certain set of measure zero, with* $t_1, t_2 \in \mathbf{R}$, $t_1 < t_2$ *and* $|t_1 - t_2| \leq \mathcal{T}$:

$$(3.5)\qquad \tfrac{1}{2}\|\mathbf{u}(t_2)-\mathbf{v}(t_2)\|^2_{L^2(\Omega)} - \tfrac{1}{2}\|\mathbf{u}(t_1)-\mathbf{v}(t_1)\|^2_{L^2(\Omega)} +$$

$$+\int_{t_1}^{t_2}\{(\mathbf{v}'(\eta),\mathbf{u}(\eta)-\mathbf{v}(\eta))_{L^2(\Omega)} + \nu(\mathbf{u}(\eta),\mathbf{u}(\eta)-\mathbf{v}(\eta))_{H_0^1(\Omega)} +$$

$$+ b_1(\mathbf{u}(\eta),\mathbf{u}(\eta),\mathbf{u}(\eta)-\mathbf{v}(\eta)) - (f(\eta),\mathbf{u}(\eta)-\mathbf{v}(\eta))_{L^2(\Omega)}\}\,d\eta \leq 0,$$

$\forall \mathbf{v} \in L^2(t_1, t_2; K \cap V) \cap H^1(t_1, t_2; H)$, *such that* $\mathbf{v}(t_1) = \mathbf{v}(t_2)$ *if* $t_2 = t_1 + \mathcal{T}$.

PROOF: If $t_2 = t_1 + \mathcal{T}$, (3.5) obviously holds. If $t_2 - t_1 < \mathcal{T}$, then let $t_0 \in \mathbf{R}$ be such that $t_0 < t_1 < t_2 < t_0 + \mathcal{T}$ and such that $\mathbf{u}$ is defined in t_0; then (3.2) holds. Furthermore, let us denote by J_ε ($\varepsilon \in \mathbf{R}^+$) a Friedrichs regularisation operator (mollifier) operating only on the variable time. For a precise description of J_ε and its properties, we refer to G. Prodi [7]. We here limit ourselves to observe that J_ε transforms periodic functions in periodic functions.

Given then:

$$(3.6)\qquad \mathbf{u}' = \mathbf{u}(t_0),$$

$$(3.7)\qquad \mathbf{u}'' = \mathbf{u} - \mathbf{u}',$$

the function $\mathbf{u}''$ is defined a.e. in $\mathbf{R}$ and is periodic of period $\mathcal{T}$ and such are also $J_\varepsilon^2\mathbf{u}''$ and $\mathbf{u}' + J_\varepsilon^2\mathbf{u}''$.

Taking $\delta > 0$ such that $t_0 < t_1 - \delta < t_1 < t_2 < t_2 + \delta < t_0 + \mathcal{T}$, and given:

$$(3.8)\qquad \mathbf{u}_\varepsilon = \mathbf{u}' + J_\varepsilon^2\mathbf{u}'' = \mathbf{u}(t_0) + J_\varepsilon^2\mathbf{u}'',$$

$$(3.9)\qquad \mathbf{v}_\delta(\eta) = \begin{cases} \mathbf{v}(\eta), & \text{for } \eta \in [t_1, t_2], \\ \mathbf{u}_\varepsilon(\eta), & \text{for } \eta \in [t_0, t_1-\delta] \cup [t_2+\delta, t_0+\mathcal{T}], \\ \mathbf{v}(t_1) + \dfrac{t_1-\eta}{\delta}(\mathbf{u}_\varepsilon(t_1-\delta)-\mathbf{v}(t_1)), & \text{for } \eta \in [t_1-\delta, t_1], \\ \mathbf{v}(t_2) + \dfrac{\eta-t_2}{\delta}(\mathbf{u}_\varepsilon(t_2+\delta)-\mathbf{v}(t_2)), & \text{for } \eta \in [t_2, t_2+\delta], \end{cases}$$

with $v \in L^2(t_1, t_2; K \cap V) \cap H^1(t_1, t_2; H)$, it follows that v_0 can be extended outside $[t_0, t_0 + \mathcal{T}]$ to a periodic function belonging to

$$L^2_{loc}(-\infty, +\infty; K \cap V) \cap H^1_{loc}(-\infty, +\infty; H).$$

Substituting therefore v with v_0 in (3.2), it follows:

$$\text{(3.10)} \quad \int_{t_0}^{t_0+\mathcal{T}} \left\{ \left(\frac{\partial v_0}{\partial \eta}(\eta), u(\eta) - v_0(\eta) \right)_{L^2(\Omega)} + \nu(u(\eta), u(\eta) - v_0(\eta))_{H^1_0(\Omega)} + \right.$$
$$\left. + b_1(u(\eta), u(\eta), u(\eta) - v_0(\eta)) - (f(\eta), u(\eta) - v_0(\eta))_{L^2(\Omega)} \right\} d\eta \leq 0 .$$

The integral that appears in (3.10) is equal to the sum of the integrals of the same integrand on the subintervals

$$[t_0, t_1 - \delta], \quad [t_1 - \delta, t_1], \quad [t_1, t_2], \quad [t_2, t_2 + \delta], \quad [t_2 + \delta, t_0 + \mathcal{T}] .$$

Let us proceed then to the calculation of the limit value (for δ and $\varepsilon \to 0$), of each single integral, starting from the one relative to the interval $[t_0, t_1 - \delta]$.

We have

$$\text{(3.11)} \quad \int_{t_0}^{t_1-\delta} \left\{ \left(\frac{\partial v_0}{\partial \eta}(\eta), u(\eta) - v_0(\eta) \right)_{L^2(\Omega)} + \nu(u(\eta), u(\eta) - v_0(\eta))_{H^1_0(\Omega)} + \right.$$
$$\left. + b_1(u(\eta), u(\eta), u(\eta) - v_0(\eta)) - (f(\eta), u(\eta) - v_0(\eta))_{L^2(\Omega)} \right\} d\eta =$$
$$= \int_{t_0}^{t_1-\delta} \left\{ \left(\frac{\partial}{\partial \eta} J_\varepsilon^2 u^\varepsilon(\eta), u^\varepsilon(\eta) - J_\varepsilon^2 u^\varepsilon(\eta) \right)_{L^2(\Omega)} + \nu(u(\eta), u^\varepsilon(\eta) - J_\varepsilon^2 u^\varepsilon(\eta))_{H^1_0(\Omega)} + \right.$$
$$\left. + b_1(u(\eta), u(\eta), u^\varepsilon(\eta) - J_\varepsilon^2 u^\varepsilon(\eta)) - (f(\eta), u^\varepsilon(\eta) - J_\varepsilon^2 u^\varepsilon(\eta))_{L^2(\Omega)} \right\} d\eta .$$

In particular, as J_ε is selfadjoint and commutes with the derivation operator:

$$\text{(3.12)} \quad \int_{t_0}^{t_1-\delta} \left(\frac{\partial}{\partial \eta} J_\varepsilon^2 u^\varepsilon(\eta), u^\varepsilon(\eta) - J_\varepsilon^2 u^\varepsilon(\eta) \right)_{L^2(\Omega)} d\eta = \int_{t_0}^{t_1-\delta} \left\{ \left(\frac{\partial}{\partial \eta} J_\varepsilon u^\varepsilon(\eta), J_\varepsilon u^\varepsilon(\eta) \right)_{L^2(\Omega)} + \right.$$
$$\left. - \left(\frac{\partial}{\partial \eta} J_\varepsilon^2 u^\varepsilon(\eta), J_\varepsilon^2 u^\varepsilon(\eta) \right)_{L^2(\Omega)} \right\} d\eta = \frac{1}{2} \int_{t_0}^{t_1-\delta} \left\{ \frac{\partial}{\partial \eta} \| J_\varepsilon u^\varepsilon(\eta) \|^2_{L^2(\Omega)} - \frac{\partial}{\partial \eta} \| J_\varepsilon^2 u^\varepsilon(\eta) \|^2_{L^2(\Omega)} \right\} d\eta =$$
$$= \frac{1}{2} \{ \| J_\varepsilon u^\varepsilon(t_1 - \delta) \|^2_{L^2(\Omega)} - \| J_\varepsilon u^\varepsilon(t_0) \|^2_{L^2(\Omega)} - \| J_\varepsilon^2 u^\varepsilon(t_1 - \delta) \|^2_{L^2(\Omega)} + \| J_\varepsilon^2 u^\varepsilon(t_0) \|^2_{L^2(\Omega)} \} .$$

For the continuity of $J_\varepsilon u^\varepsilon$ and $J_\varepsilon^2 u^\varepsilon$ and thanks to

$$\text{(3.13)} \quad \| u^\varepsilon - J_\varepsilon^2 u^\varepsilon \|_{L^2(t_0, t_0 + \mathcal{T}; L^2(\Omega))} \to 0 , \qquad \text{for } \varepsilon \to 0 ,$$

$$\text{(3.14)} \quad \| u^\varepsilon - J_\varepsilon u^\varepsilon \|_{L^2(t_0, t_0 + \mathcal{T}; L^2(\Omega))} \to 0 , \qquad \text{for } \varepsilon \to 0 ,$$

$$\text{(3.15)} \quad \left\| \frac{\partial u^\varepsilon}{\partial x_i} - \frac{\partial}{\partial x_i} J_\varepsilon^2 u^\varepsilon \right\|_{L^2(t_0, t_0 + \mathcal{T}; L^2(\Omega))} \to 0 , \qquad \text{for } \varepsilon \to 0 ,$$

(for their validity see [7]), it follows:

$$(3.16)\qquad \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\int_{t_0}^{t_1-\delta}\left(\frac{\partial}{\partial\eta}J_\varepsilon^2 u'(\eta),\, u'(\eta)-J_\varepsilon^2 u'(\eta)\right)_{L^2(\Omega)} d\eta = 0\,,$$

and therefore also

$$(3.17)\qquad \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\int_{t_0}^{t_1-\delta}\Big\{\left(\frac{\partial}{\partial\eta}J_\varepsilon^2 u'(\eta),\, u'(\eta)-J_\varepsilon^2 u'(\eta)\right)_{L^2(\Omega)} + $$
$$+\,\nu\big(u(\eta),\, u'(\eta)-J_\varepsilon^2 u'(\eta)\big)_{H_0^1(\Omega)} + b_1\big(u(\eta), u(\eta), u'(\eta)-J_\varepsilon^2 u'(\eta)\big) +$$
$$-\big(f(\eta),\, u'(\eta)-J_\varepsilon^2 u'(\eta)\big)_{L^2(\Omega)}\Big\}\, d\eta = 0\,.$$

We proceed now to the examination of the integral relative to the interval $[t_1-\delta, t_1]$, namely:

$$(3.18)\qquad \int_{t_1-\delta}^{t_1}\Big\{\left(\frac{\partial}{\partial\eta}v_\delta(\eta),\, u(\eta)-v_\delta(\eta)\right)_{L^2(\Omega)} +$$
$$+\,\nu\big(u(\eta), u(\eta)-v_\delta(\eta)\big)_{H_0^1(\Omega)} + b_1\big(u(\eta), u(\eta), u(\eta)-v_\delta(\eta)\big) - \big(f(\eta), u(\eta)-v_\delta(\eta)\big)_{L^2(\Omega)}\Big\}\, d\eta\,.$$

Relative to the first term of (3.18), we deduce

$$(3.19)\qquad \int_{t_1-\delta}^{t_1}\left(\frac{\partial}{\partial\eta}v_\delta(\eta),\, u(\eta)-v_\delta(\eta)\right)_{L^2(\Omega)} d\eta =$$
$$=\frac{1}{\delta}\int_{t_1-\delta}^{t_1}\left(v(t_1)-u_\varepsilon(t_1-\delta),\, u(\eta)-v(t_1)+\frac{\eta-t_1}{\delta}\big(u_\varepsilon(t_1-\delta)-v(t_1)\big)\right)_{L^2(\Omega)} d\eta\,.$$

Observe now that, obviously (by the continuity of the scalar product)

$$(3.20)\qquad \lim_{\delta\to 0}\frac{1}{\delta}\int_{t_1-\delta}^{t_1}\big(v(t_1), u(\eta)\big)_{L^2(\Omega)} d\eta = \lim_{\delta\to 0}\left(v(t_1), \frac{1}{\delta}\int_{t_1-\delta}^{t_1} u(\eta)\, d\eta\right)_{L^2(\Omega)} = \big(v(t_1), u(t_1)\big)_{L^2(\Omega)}\,,$$

and, moreover,

$$(3.21)\qquad \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\frac{1}{\delta}\int_{t_1-\delta}^{t_1}\left(v(t_1), \frac{\eta-t_1}{\delta}u_\varepsilon(t_1-\delta)\right)_{L^2(\Omega)} d\eta =$$
$$=\lim_{\varepsilon\to 0} -\frac{1}{2}\big(v(t_1), u_\varepsilon(t_1)\big)_{L^2(\Omega)} = -\frac{1}{2}\big(v(t_1), u(t_1)\big)_{L^2(\Omega)}\,;$$

$$(3.22)\qquad \lim_{\delta\to 0}\frac{1}{\delta}\int_{t_1-\delta}^{t_1}\left(v(t_1), \frac{t_1-\eta}{\delta}v(t_1)\right)_{L^2(\Omega)} d\eta = \frac{1}{2}\big(v(t_1), v(t_1)\big)_{L^2(\Omega)}\,;$$

$$\text{(3.23)}\qquad \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\frac{1}{\delta}\int_{t_1-\delta}^{t_1} -(\mathbf{u}_\varepsilon(t_1-\delta),\mathbf{u}(\eta))_{L^2(\Omega)}\,d\eta = -(\mathbf{u}(t_1),\mathbf{u}(t_1))_{L^2(\Omega)}\,;$$

$$\text{(3.24)}\qquad \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\frac{1}{\delta}\int_{t_1-\delta}^{t_1} (\mathbf{u}_\varepsilon(t_1-\delta),\mathbf{v}(t_1))_{L^2(\Omega)}\,d\eta = (\mathbf{u}(t_1),\mathbf{v}(t_1))_{L^2(\Omega)}\,;$$

$$\text{(3.25)}\qquad \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\frac{1}{\delta}\int_{t_1-\delta}^{t_1} -(\mathbf{u}_\varepsilon(t_1-\delta),\mathbf{u}_\varepsilon(t_1-\delta))_{L^2(\Omega)}\frac{\eta-t_1}{\delta}\,d\eta =$$

$$= \lim_{\varepsilon\to 0}\frac{1}{2}(\mathbf{u}_\varepsilon(t_1),\mathbf{u}_\varepsilon(t_1))_{L^2(\Omega)} = \frac{1}{2}(\mathbf{u}(t_1),\mathbf{u}(t_1))_{L^2(\Omega)}\,;$$

$$\text{(3.26)}\qquad \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\frac{1}{\delta}\int_{t_1-\delta}^{t_1}\left(\mathbf{u}_\varepsilon(t_1-\delta),\frac{\eta-t_1}{\delta}\mathbf{v}(t_1)\right)_{L^2(\Omega)}d\eta =$$

$$= \lim_{\varepsilon\to 0} -\frac{1}{2}(\mathbf{u}_\varepsilon(t_1),\mathbf{v}(t_1))_{L^2(\Omega)} = -\frac{1}{2}(\mathbf{u}(t_1),\mathbf{v}(t_1))_{L^2(\Omega)}\,.$$

From (3.19)-(3.26) it follows:

$$\text{(3.27)}\qquad \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\int_{t_1-\delta}^{t_1}\left(\frac{\partial}{\partial\eta}\mathbf{v}_\delta(\eta),\mathbf{u}(\eta)-\mathbf{v}_\delta(\eta)\right)_{L^2(\Omega)}d\eta =$$

$$= (\mathbf{v}(t_1),\mathbf{u}(t_1)-\mathbf{v}(t_1))_{L^2(\Omega)} - \frac{1}{2}(\mathbf{v}(t_1),\mathbf{u}(t_1)-\mathbf{v}(t_1))_{L^2(\Omega)} +$$

$$-(\mathbf{u}(t_1),\mathbf{u}(t_1)-\mathbf{v}(t_1))_{L^2(\Omega)} + \frac{1}{2}(\mathbf{u}(t_1),\mathbf{u}(t_1)-\mathbf{v}(t_1))_{L^2(\Omega)} = -\frac{1}{2}\|\mathbf{u}(t_1)-\mathbf{v}(t_1)\|^2_{L^2(\Omega)}\,.$$

The other terms which appear in (3.18) all vanish and, consequently:

$$\text{(3.28)}\qquad \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\int_{t_1-\delta}^{t_1}\left\{\left(\frac{\partial}{\partial\eta}\mathbf{v}_\delta(\eta),\mathbf{u}(\eta)-\mathbf{v}_\delta(\eta)\right)_{L^2(\Omega)} + \nu(\mathbf{u}(\eta),\mathbf{u}(\eta)-\mathbf{v}_\delta(\eta))_{H^1_0(\Omega)} +\right.$$

$$\left. + b_1(\mathbf{u}(\eta),\mathbf{u}(\eta),\mathbf{u}(\eta)-\mathbf{v}_\delta(\eta)) - (\mathbf{f}(\eta),\mathbf{u}(\eta)-\mathbf{v}_\delta(\eta))_{L^2(\Omega)}\right\}d\eta = -\frac{1}{2}\|\mathbf{u}(t_1)-\mathbf{v}(t_1)\|^2_{L^2(\Omega)}\,.$$

Analogously to (3.17) and (3.28), it follows that the integrals of the same integrand, extended to the intervals $[t_2+\delta, t_0+\mathcal{T}]$ and $[t_2, t_2+\delta]$ converge respectively to 0 and to $\frac{1}{2}\|\mathbf{u}(t_2)-\mathbf{v}(t_2)\|^2_{L^2(\Omega)}$. Since the remaining integral on the interval $[t_1, t_2]$ coincides with the expression

$$\text{(3.29)}\qquad \int_{t_1}^{t_2}\left\{\left(\frac{\partial}{\partial\eta}\mathbf{v}(\eta),\mathbf{u}(\eta)-\mathbf{v}(\eta)\right)_{L^2(\Omega)} + \nu(\mathbf{u}(\eta),\mathbf{u}(\eta)-\mathbf{v}(\eta))_{H^1_0(\Omega)} +\right.$$

$$\left. + b_1(\mathbf{u}(\eta),\mathbf{u}(\eta),\mathbf{u}(\eta)-\mathbf{v}(\eta)) - (\mathbf{f}(\eta),\mathbf{u}(\eta)-\mathbf{v}(\eta))_{L^2(\Omega)}\right\}d\eta\,,$$

the thesis follows from the above mentioned considerations. ■

LEMMA 3.2: *If u is a periodic solution of the Navier-Stokes inequalities, then $u \in C^0(-\infty, +\infty; H)$.*

PROOF: Reasoning as in [10, Nota I, Obs. 1], after having chosen arbitrarily a point $t \in \mathbb{R}$ in which u is defined, let us fix $t_1 \in \mathbb{R}$ with $t < t_1 + \mathcal{T}$ in such a way that the pair (t_1, t) verifies the required conditions of Lemma 3.1. In addition, let $\{u_j\}$ be a regularising sequence associated to u such that (see [10]):

$$u_j \in H^1(t_1, t_1 + \mathcal{T}; V); \tag{3.30}$$

$$u_j(t_1) = u(t_1); \tag{3.31}$$

$$\|u_j - u\|_{L^2(t_1, t_1 + \mathcal{T}; V)} \to 0, \qquad j \to +\infty; \tag{3.32}$$

$$\int_{t_1}^{t} (u_j'(\eta), u(\eta) - u_j(\eta))_{L^2(\Omega)}\, d\eta \geq 0, \qquad \forall t \in [t_1, t_1 + \mathcal{T}]; \tag{3.33}$$

$$|u_j(t, x)| \leq c, \qquad \text{a.e. in } [t_1, t_1 + \mathcal{T}] \times \Omega. \tag{3.34}$$

Observe that such u_j can be obtained as solution of the problem (see [5, Ch. 2, Th. 9.1])

$$\frac{1}{j} u_j' + u_j = u, \qquad u_j(t_1) = u(t_1).$$

Applying now Lemma 3.1 and taking u_j as test function, remembering (3.31) and (3.34), we obtain

$$\begin{aligned} \|u(t) - u_j(t)\|^2_{L^2(\Omega)} \leq 2\int_{t_1}^{t} \{ -\nu(u(\eta), u(\eta) - u_j(\eta))_{H_0^1(\Omega)} + \\ - b_1(u(\eta), u(\eta), u(\eta) - u_j(\eta)) + (f(\eta), u(\eta) - u_j(\eta))_{L^2(\Omega)}\}\, d\eta . \end{aligned} \tag{3.35}$$

Since the integral in (3.35) converges uniformly to zero on every bounded interval when $j \to +\infty$ from the continuity of u_j it follows that

$$u \in C^0(-\infty, +\infty; H). \qquad \blacksquare$$

LEMMA 3.3: *Let $u^{(1)}$ and $u^{(2)}$ be two periodic solutions of period $\mathcal{T}$ of the Navier-Stokes inequalities, corresponding respectively to known terms, which are also periodic, f_1 and $f_2 \in L^2_{loc}(-\infty, +\infty; L^2(\Omega))$.*

Setting $w = u^{(1)} - u^{(2)}$, we have, for every pair (t_1, t_2) with $|t_1 - t_2| < \mathcal{T}$ and $t_1 < t_2$:

$$\begin{aligned} \tfrac{1}{2}\|w(t_2)\|^2_{L^2(\Omega)} - \tfrac{1}{2}\|w(t_1)\|^2_{L^2(\Omega)} + \\ + \int_{t_1}^{t_2} \{\nu\|w(\eta)\|^2_{H_0^1(\Omega)} - b_1(w(\eta), w(\eta), u^{(2)}(\eta)) - (f_1(\eta) - f_2(\eta), w(\eta))_{L^2(\Omega)}\}\, d\eta \leq 0 . \end{aligned} \tag{3.36}$$

PROOF: Setting, as in [10], $z = (u^{(1)} + u^{(2)})/2$, let $\{z_j\}$ be a sequence of regularising functions of z, in the sense that z_j verifies with respect to z relationships analogous to (3.30)-(3.34). Furthermore, observe that (by Lemma 3.2) Lemma 3.1 is valid for every pair of points (t_1, t_2)

with $|t_1 - t_2| < \mathcal{T}$ and $t_1 < t_2$. Considering then (3.5) with $v = z_j$, written now for $u^{(1)}$ and $u^{(2)}$, and adding the two relationships so obtained, we have:

$$(3.37)\quad \tfrac{1}{2}\|u^{(1)}(t_2) - z_j(t_2)\|^2_{L^2(\Omega)} - \tfrac{1}{2}\|u^{(1)}(t_1) - z_j(t_1)\|^2_{L^2(\Omega)} + \tfrac{1}{2}\|u^{(2)}(t_2) - z_j(t_2)\|^2_{L^2(\Omega)} -$$

$$- \tfrac{1}{2}\|u^{(2)}(t_1) - z_j(t_1)\|^2_{L^2(\Omega)} + \int_{t_1}^{t_2}\{2(z_j'(\eta), z(\eta) - z_j(\eta))_{L^2(\Omega)} + \nu(u^{(1)}(\eta), u^{(1)}(\eta) - z_j(\eta))_{H^1_0(\Omega)} +$$

$$+ \nu(u^{(2)}(\eta), u^{(2)}(\eta) - z_j(\eta))_{H^1_0(\Omega)} + b_1(u^{(1)}(\eta), u^{(1)}(\eta), u^{(1)}(\eta) - z_j(\eta)) +$$

$$+ b_1(u^{(2)}(\eta), u^{(2)}(\eta), u^{(2)}(\eta) - z_j(\eta)) +$$

$$-(f_1(\eta), u^{(1)}(\eta) - z_j(\eta))_{L^2(\Omega)} - (f_2(\eta), u^{(2)}(\eta) - z_j(\eta))_{L^2(\Omega)}\}\, d\eta \le 0\,.$$

Letting $j \to +\infty$ in (3.37), we obtain

$$(3.38)\quad \tfrac{1}{2}\|u^{(1)}(t_2) - u^{(2)}(t_2)\|^2_{L^2(\Omega)} - \tfrac{1}{2}\|u^{(1)}(t_1) - u^{(2)}(t_1)\|^2_{L^2(\Omega)} +$$

$$+ \tfrac{1}{2}\|u^{(2)}(t_2) - u^{(1)}(t_2)\|^2_{L^2(\Omega)} - \tfrac{1}{2}\|u^{(2)}(t_1) - u^{(1)}(t_1)\|^2_{L^2(\Omega)} +$$

$$+ \int_{t_1}^{t_2}\{\nu\|u^{(1)}(\eta) - u^{(2)}(\eta)\|^2_{H^1_0(\Omega)} + b_1(u^{(1)}(\eta), u^{(1)}(\eta), u^{(1)}(\eta) - u^{(2)}(\eta)) +$$

$$+ b_1(u^{(2)}(\eta), u^{(2)}(\eta), u^{(2)}(\eta) - u^{(1)}(\eta)) - (f_1(\eta) - f_2(\eta), u^{(1)}(\eta) - u^{(2)}(\eta))_{L^2(\Omega)}\}\, d\eta \le 0\,.$$

From (3.38) it follows also that

$$(3.39)\quad \tfrac{1}{2}\|w(t_2)\|^2_{L^2(\Omega)} - \tfrac{1}{2}\|w(t_1)\|^2_{L^2(\Omega)} + \int_{t_1}^{t_2}\{\nu\|w(\eta)\|^2_{H^1_0(\Omega)} + b_1(u^{(1)}(\eta), u^{(1)}(\eta), w(\eta)) -$$

$$- b_1(u^{(2)}(\eta), u^{(2)}(\eta), w(\eta)) - (f_1(\eta) - f_2(\eta), w(\eta))_{L^2(\Omega)}\}\, d\eta \le 0\,.$$

Since

$$(3.40)\quad b_1(u^{(1)}, u^{(1)}, w) - b_1(u^{(2)}, u^{(2)}, w) =$$

$$= b_1(u^{(1)}, u^{(1)}, w) - b_1(u^{(2)}, u^{(1)}, w) + b_1(u^{(2)}, u^{(1)}, w) - b_1(u^{(2)}, u^{(2)}, w) =$$

$$= b_1(w, u^{(1)}, w) + b_1(u^{(2)}, w, w) = b_1(w, u^{(1)}, w) = -b_1(w, w, u^{(1)}),$$

from (3.39) the assert follows. ∎

Theorem 3.2: *If there exists a periodic solution* u *(of period* $\mathcal{T}$*) of the Navier-Stokes inequalities and if* u *verifies the additional condition:*

$$(3.41)\quad \max_{\eta\in[0,\mathcal{T}]}\|u(\eta)\|^{\frac{1}{2}}_{L^2(\Omega)}\left\{\int_0^{\mathcal{T}}\|u(\eta)\|^2_{H^1_0(\Omega)}\,d\eta\right\}^{\frac{1}{2}} < \nu\min\left\{\sqrt[4]{\frac{(1-\bar{a})}{8\nu}}, \frac{\sqrt[4]{\bar{a}\mathcal{T}}}{\sqrt{2\sigma}}\right\},$$

with $0 < \bar{a} < 1$ *and* σ *embedding constant of* $H^1_0(\Omega)$ *in* $L^4(\Omega)$*, then no other periodic solutions exist for the Navier-Stokes inequalities.*

Proof: The main line of the proof follows that given in [11, Theorem 1.7].

Let u and v be two periodic solutions, of period $\mathcal{T}$, corresponding to the same known term f. Setting $w = u - v$, and being $w \in C^0(-\infty, +\infty; H)$ by Lemma 3.2, let t' and t'' be respectively the points where (referring to any interval of size $\mathcal{T}$) the continuous function $t \to \|w(t)\|_{L^2(\Omega)}$ takes maximum and minimum values. By periodicity, it can be assumed that $t' - \mathcal{T} < t'' < t'$.

By Lemma 3.3 with $t_2 = t'$ and $t_1 = t' - \mathcal{T}$, it follows:

$$(3.42)\qquad \|w(t'-\mathcal{T})\|^2_{L^2(\Omega)} - \|w(t')\|^2_{L^2(\Omega)} \geq 2\nu \int_{t'-\mathcal{T}}^{t'} \|w(\eta)\|^2_{H^1_0(\Omega)}\, d\eta - 2\int_{t'-\mathcal{T}}^{t'} b_1(w(\eta), w(\eta), u(\eta))\, d\eta\,.$$

Setting

$$(3.43)\qquad \Gamma = \max_{\eta\in[0,\mathcal{T}]} \|u(\eta)\|^{\frac{1}{2}}_{L^2(\Omega)} \left\{\int_0^{\mathcal{T}} \|u(\eta)\|^2_{H^1_0(\Omega)}\, d\eta\right\}^{\frac{1}{2}},$$

it follows, from (3.42) (see [11, p. 201])

$$(3.44)\qquad \|w(t'-\mathcal{T})\|^2_{L^2(\Omega)} \geq \|w(t')\|^2_{L^2(\Omega)} + 2\nu \int_{t'-\mathcal{T}}^{t'} \|w(\eta)\|^2_{H^1_0(\Omega)}\, d\eta + \\ - 2\sqrt{2}\,\Gamma \|w(t')\|^{\frac{1}{2}}_{L^2(\Omega)} \left\{\int_{t'-\mathcal{T}}^{t'} \|w(\eta)\|^2_{H^1_0(\Omega)}\, d\eta\right\}^{\frac{3}{4}}.$$

We now show that it exists $\lambda > 0$ (dependent on Γ, ν, Ω, but independent of w), such that

$$(3.45)\qquad \int_{t'-\mathcal{T}}^{t'} \|w(\eta)\|^2_{H^1_0(\Omega)}\, d\eta \geq \frac{1}{\lambda\nu} \max_{[t'-\mathcal{T},t']} \|w(\eta)\|^2_{L^2(\Omega)} = \frac{1}{\lambda\nu} \|w(t')\|^2_{L^2(\Omega)}\,.$$

In fact, if $t'' = t'$, we have

$$(3.46)\qquad \int_{t'-\mathcal{T}}^{t'} \|w(\eta)\|^2_{H^1_0(\Omega)} \geq \frac{1}{\sigma^2} \int_{t'-\mathcal{T}}^{t'} \|w(\eta)\|^2_{L^2(\Omega)} \geq \frac{\mathcal{T}}{\sigma^2} \|w(t'')\|^2_{L^2(\Omega)} = \frac{\mathcal{T}}{\sigma^2} \|w(t')\|^2_{L^2(\Omega)},$$

and (3.45) follows assuming

$$(3.47)\qquad \lambda \geq \frac{\sigma^2}{\delta\nu\mathcal{T}}, \qquad 0 < \delta < 1\,.$$

If instead $t' > t''$, we procede ad absurdum. Assume that, for every λ,

$$(3.48)\qquad \int_{t'-\mathcal{T}}^{t'} \|w(\eta)\|^2_{H^1_0(\Omega)}\, d\eta < \frac{1}{\lambda\nu} \|w(t')\|^2_{L^2(\Omega)}\,,$$

and consequently

$$(3.49)\qquad \int_{t''}^{t'} \|w(\eta)\|^2_{H^1_0(\Omega)}\, d\eta < \frac{1}{\lambda\nu} \|w(t')\|^2_{L^2(\Omega)}\,.$$

Using Lemma 3.3 with $t_2 = t'$ and $t_1 = t''$, it follows from (3.49):

$$\text{(3.50)}\qquad \|w(t'')\|^2_{L^2(\Omega)} > \|w(t')\|^2_{L^2(\Omega)} - \frac{2\sqrt{2}}{(\lambda\nu)^{\frac{3}{4}}}\Gamma'\|w(t')\|^2_{L^2(\Omega)} = \left(1 - \frac{2\sqrt{2}\Gamma'}{(\lambda\nu)^{\frac{3}{4}}}\right)\|w(t')\|^2_{L^2(\Omega)}.$$

If λ is chosen in such a way that

$$1 - \frac{2\sqrt{2}\Gamma'}{(\lambda\nu)^{\frac{3}{4}}} > \bar{a}, \qquad (0 < \bar{a} < 1),$$

that is

$$\text{(3.51)}\qquad \lambda > \frac{1}{\nu}\left(\frac{2\sqrt{2}\Gamma'}{1-\bar{a}}\right)^{\frac{4}{3}},$$

then

$$\text{(3.52)}\qquad \|w(t'')\|^2_{L^2(\Omega)} > \bar{a}\,\|w(t')\|^2_{L^2(\Omega)}.$$

Therefore in the case $t' > t''$, if λ verifies the condition:

$$\text{(3.53)}\qquad \lambda > \frac{1}{\nu}\max\left\{\left(\frac{2\sqrt{2}\Gamma'}{1-\bar{a}}\right)^{\frac{4}{3}}, \frac{\sigma^2}{\bar{a}\bar{\tau}}\right\},$$

then

$$\int_{t'-\bar{\tau}}^{t'} \|w(\eta)\|^2_{H^1_0(\Omega)}\,d\eta > \frac{1}{\lambda\nu}\|w(t')\|^2_{L^2(\Omega)},$$

contrary to (3.48).

It follows by (3.44) and (3.53),

$$\text{(3.54)}\qquad \|w(t'-\bar{\tau})\|^2_{L^2(\Omega)} > \|w(t')\|^2_{L^2(\Omega)} + 2(\nu - \sqrt{2}\,\Gamma'\lambda^{\frac{1}{4}}\nu^{\frac{1}{4}})\int_{t'-\bar{\tau}}^{t'} \|w(\eta)\|^2_{H^1_0(\Omega)}\,d\eta.$$

Assume now, besides (3.53),

$$\text{(3.55)}\qquad \lambda < \frac{1}{\nu}\left(\frac{\nu-\beta}{\sqrt{2}\Gamma'}\right)^4, \qquad (0 < \beta < \nu).$$

We have then, by (3.54),

$$\text{(3.56)}\qquad \|w(t'-\bar{\tau})\|^2_{L^2(\Omega)} > \|w(t')\|^2_{L^2(\Omega)} + 2\beta\int_{t'-\bar{\tau}}^{t'} \|w(\eta)\|^2_{H^1_0(\Omega)}\,d\eta.$$

Observe that by (3.41) (where the left hand side coincides with Γ') it follows that:

$$\text{(3.57)}\qquad \max\left\{\left(\frac{2\sqrt{2}\Gamma'}{1-\bar{a}}\right)^{\frac{4}{3}}, \frac{\sigma^2}{\bar{a}\bar{\tau}}\right\} < \left(\frac{\nu-\beta}{\sqrt{2}\Gamma'}\right)^4.$$

Therefore it is possible to choose the parameter λ in order to verify (3.53) and (3.55) simultaneously, thus assuring the validity of (3.56) and also of

$$\|w(t'-\mathfrak{T})\|^2_{L^2(\Omega)} \geq \|w(t')\|^2_{L^2(\Omega)} + \frac{2\beta}{\sigma^2}\int_{t'-\mathfrak{T}}^{t'} \|w(\eta)\|^2_{L^2(\Omega)}\, d\eta . \tag{3.58}$$

Since w is periodic of period $\mathfrak{T}$, from (3.58), we have

$$\int_{t'-\mathfrak{T}}^{t'} \|w(\eta)\|^2_{L^2(\Omega)}\, d\eta \leq 0 ,$$

that implies the uniqueness. ■

An important consequence is the following *uniqueness theorem* corresponding to a *small known term*.

THEOREM 3.3: *If* $f \in L^2_{loc}(0, \mathfrak{T}; L^2(\Omega))$ *is periodic of period* $\mathfrak{T}$ *and verifies*

$$\|f\|_{L^2(0,\mathfrak{T};L^2(\Omega))} \leq (1-\exp[-\nu\mathfrak{T}/\sigma^2])^{\frac{1}{2}}\cdot \nu^{\frac{3}{2}} \min\left\{\sqrt[4]{\frac{(1-\bar{a})}{8\nu}}, \frac{\sqrt[4]{\bar{a}\mathfrak{T}}}{\sqrt{2\sigma}}\right\} \tag{3.59}$$

with $0 < \bar{a} < 1$, *then the periodic solution of the Navier-Stokes inequalities is unique and depends continuously on* f.

PROOF: Let u be the periodic solution corresponding to f. Following Obs. 3.1, we have

$$\|u\|^{\frac{1}{2}}_{L^\infty(0,\mathfrak{T};H)}\|u\|^{\frac{1}{2}}_{L^2(0,\mathfrak{T};H^1_0(\Omega))} \leq (1-\exp[-\nu\mathfrak{T}/\sigma^2])^{-\frac{1}{4}}\|f\|_{L^2(0,\mathfrak{T};L^2(\Omega))}\cdot \frac{(1-\exp[-\nu\mathfrak{T}/\sigma^2])^{-\frac{1}{4}}}{\nu^{\frac{1}{2}}} = \frac{(1-\exp[-\nu\mathfrak{T}/\sigma^2])^{-\frac{1}{2}}}{\nu^{\frac{1}{2}}}\|f\|_{L^2(0,\mathfrak{T};L^2(\Omega))} . \tag{3.60}$$

If (3.59) is valid, we still have:

$$\|u\|^{\frac{1}{2}}_{L^\infty(0,\mathfrak{T};H)}\|u\|^{\frac{1}{2}}_{L^2(0,\mathfrak{T};H^1_0(\Omega))} \leq \nu \min\left\{\sqrt[4]{\frac{(1-\bar{a})}{8\nu}}, \frac{\sqrt[4]{\bar{a}\mathfrak{T}}}{\sqrt{2\sigma}}\right\} ; \tag{3.61}$$

therefore (3.41) holds and, by Theorem 3.2 the uniqueness of u follows. The continuous dependence of u on f can be proved in the same way. ■

Therefore, we conclude the present paragraph with

THEOREM 3.4: *In the hypotheses of Theorem* 3.3, *the periodic solution of Navier-Stokes inequalities exists, is unique and depends continuously on the data.* ■

4. - THE PERIODIC SOLUTIONS OF A LINEAR EQUATION OF THE « HEAT » TYPE

Let us consider the equation:

$$\frac{\partial\theta}{\partial t} - \gamma\,\Delta\theta + u\cdot\nabla\theta = q , \tag{4.1}$$

where γ is a positive constant, u and q are *known functions* with $u \in L^2_{loc}(-\infty, +\infty; K \cap V)$ and $q \in L^2_{loc}(-\infty, +\infty; L^2(\Omega))$. Assuming that u and q are periodic of period $\mathcal{T}$, and associating to (4.1) the boundary conditions (2.11), we will say that a function θ is a « *periodic solution of period* $\mathcal{T}$ *of* (4.1) *satisfying* (2.11) » if the following conditions are verified (see (2.21)):

$$\theta \in L^2_{loc}(-\infty, +\infty; \tilde{H}^1_0(\Omega)) \cap L^\infty(-\infty, +\infty; L^2(\Omega)) \cap \cap H^1_{loc}(-\infty, +\infty; (\tilde{H}^1_0(\Omega))') , \tag{4.2}$$

(4.3) θ is periodic of period $\mathcal{T}$,

$$\int_{t_0}^{t_0+\mathcal{T}} \{\langle \theta'(\eta), \varphi(\eta)\rangle + \gamma(\theta(\eta), \varphi(\eta))_{H^1_0(\Omega)} + b_2(u(\eta), \theta(\eta), \varphi(\eta)) - (q(\eta), \varphi(\eta))_{L^2(\Omega)}\}\, d\eta = 0 , \tag{4.4}$$

$\forall t_0 \in \mathbb{R}$ and $\forall \varphi \in L^2_{loc}(-\infty, +\infty; \tilde{H}^1_0(\Omega)) \cap L^4_{loc}(-\infty, +\infty; L^4(\Omega))$.

With regard to the periodic solutions of (4.1) the following result holds:

THEOREM 4.1: *Let* $\gamma > 0$ *and*

$$u \in L^2_{loc}(-\infty, +\infty; K \cap V), \qquad q \in L^2_{loc}(-\infty, +\infty; L^2(\Omega)),$$

both periodic with period $\mathcal{T}$. *Then a « periodic solution of* (4.1) *» exists unique, in the sense previously stated.*

PROOF: For the proof of this theorem, see [6, Ch. 3, § 4.7 and Th. 6.1], bearing in mind that $\operatorname{div} u = 0$, which implies $b_2(u, \theta, \theta) = 0$. ■

OBS. 4.1: Proceeding as in [9], we have:

$$\|\theta\|_{L^\infty(0,\mathcal{T}; L^2(\Omega))} < M_2 = \left(1 - \exp\left[-\frac{\gamma\mathcal{T}}{\sigma^2}\right]\right)^{-1} \int_0^{\mathcal{T}} \|q(\eta)\|_{L^2(\Omega)}\, d\eta . \tag{4.5}$$ ■

OBS. 4.2: If $t_1 < t_2$ and $|t_1 - t_2| < \mathcal{T}$ and if θ is a « periodic solution of (4.1) » then the relationship

$$\int_{t_1}^{t_2} \{\langle \theta'(\eta), \varphi(\eta)\rangle + \gamma(\theta(\eta), \varphi(\eta))_{H^1_0(\Omega)} + b_2(u(\eta), \theta(\eta), \varphi(\eta)) - (q(\eta), \varphi(\eta))_{L^2(\Omega)}\}\, d\eta = 0 , \quad \forall \varphi \in L^2(t_1, t_2; \tilde{H}^1_0(\Omega)) \cap L^4(t_1, t_2; L^4(\Omega)) , \tag{4.6}$$

is also valid.

To obtain (4.6), it is sufficient to observe that (having chosen t_0 such that $t_0 < t_1 < t_2 < t_0 + \mathcal{T}$) (4.4) holds, and to substitute in (4.4) a particular test function $\bar{\varphi}$ periodic and such that

$$\bar{\varphi}(t) = \begin{cases} 0 & \text{for } t \in [t_0, t_1] \cup [t_2, t_0 + \mathcal{T}], \\ \varphi & \text{for } t \in [t_1, t_2], \end{cases}$$

with $\varphi \in L^2(t_1, t_2; \tilde{H}_0^1(\Omega)) \cup L^4(t_1, t_2; L^4(\Omega))$. ■

Lemma 4.1: *Let θ_1 and θ_2 be two «periodic solutions of period $\mathcal{T}$ of (4.1)» corresponding respectively to the known terms $u^{(1)}$, q_1 and $u^{(2)}$, q_2. Setting then $w = u^{(1)} - u^{(2)}$, $\delta_\theta = \theta_1 - \theta_2$, for every pair of points (t_1, t_2) with $t_1 < t_2$ and $|t_1 - t_2| < \mathcal{T}$ outside a certain set of measure zero, we have:*

$$(4.7) \qquad \tfrac{1}{2}\|\delta_\theta(t_2)\|^2_{L^2(\Omega)} - \tfrac{1}{2}\|\delta_\theta(t_1)\|^2_{L^2(\Omega)} + \\ + \int_{t_1}^{t_2} \{\gamma \|\delta_\theta(\eta)\|^2_{H_0^1(\Omega)} + b_2(w(\eta), \theta_1(\eta), \delta_\theta(\eta)) - (q_1(\eta) - q_2(\eta), \delta_\theta(\eta))_{L^2(\Omega)}\} \, d\eta = 0 .$$

Proof: Following Obs. 4.2, for $\theta = \theta_1$, $u = u^{(1)}$, $q = q_1$, we have:

$$(4.8) \quad \int_{t_1}^{t_2} \{\langle \theta_1'(\eta), \varphi(\eta)\rangle + \gamma(\theta_1(\eta), \varphi(\eta))_{H_0^1(\Omega)} + b_2(u^{(1)}(\eta), \theta_1(\eta), \varphi(\eta)) - (q_1(\eta), \varphi(\eta))_{L^2(\Omega)}\} \, d\eta = 0 ,$$

$\forall \varphi \in L^2(t_1, t_2; \tilde{H}_0^1(\Omega)) \cap L^4(t_1, t_2; L^4(\Omega))$; furthermore, an analogous relationship is valid for θ_2, $u^{(2)}$, q_2. Subtracting the latter from (4.8), it follows:

$$\int_{t_1}^{t_2} \{\langle \delta_\theta'(\eta), \varphi(\eta)\rangle + \gamma(\delta_\theta(\eta), \varphi(\eta))_{H_0^1(\Omega)} + b_2(w(\eta), \theta_1(\eta), \varphi(\eta)) + b_2(u^{(2)}(\eta), \delta_\theta(\eta), \varphi(\eta)) - \\ - (q_1(\eta) - q_2(\eta), \varphi(\eta))_{L^2(\Omega)}\} \, d\eta = 0 , \qquad \forall \varphi \in L^2(t_1, t_2; \tilde{H}_0^1(\Omega)) \cap L^4(t_1, t_2; L^4(\Omega)) .$$

Setting then $\varphi = \delta_\theta$, we have

$$\int_{t_1}^{t_2} \{\langle \delta_\theta'(\eta), \delta_\theta(\eta)\rangle + \gamma \|\delta_\theta(\eta)\|^2_{H_0^1(\Omega)} + b_2(w(\eta), \theta_1(\eta), \delta_\theta(\eta)) - (q_1(\eta) - q_2(\eta), \delta_\theta(\eta))_{L^2(\Omega)}\} \, d\eta = 0 ,$$

from which, since δ_θ is (by the assumptions made) defined in t_1, t_2, the lemma follows. ■

5. - Existence of a solution of the original problem

We remember firstly that the problem examined here, described in the second paragraph, consists in the research of a pair (u, θ) satisfying (2.23), (2.24), (2.25), (2.26), and we observe that (2.25) and (2.26) coincide respectively with (3.2) and (4.4) setting $\nu = \mu/\varrho_0$, $t_0 = 0$,

$$(5.1)\qquad f(x_1, x_2, t) = -\alpha g\big(\theta(x_1, x_2, t) + (T_2(t) - T_1(t))x_2/2\big),$$

$$(5.2)\qquad q = \frac{T_2 - T_1}{2}u_1 + \frac{\partial d}{\partial t}.$$

From these and from (4.5) it follows that, if (u, θ) is a solution to the problem (2.7)-(2.12) and $T_1, T_2 \in H^1(0, \mathcal{T})$, then

$$(5.3)\qquad \|f\|_{L^2(0,\mathcal{T};L^2(\Omega))} < \alpha g\|\theta\|_{L^2(0,\mathcal{T};L^2(\Omega))} + \frac{\alpha g}{\sqrt{3}}\|T_2 - T_1\|_{L^2(0,\mathcal{T})},$$

$$(5.4)\qquad \|q\|_{L^2(0,\mathcal{T};L^2(\Omega))} < \varepsilon\mathcal{T}\|T_2 - T_1\|_{L^\infty(0,\mathcal{T})} + \left\|\frac{\partial d}{\partial t}\right\|_{L^2(0,\mathcal{T};L^2(\Omega))},$$

$$(5.5)\qquad \|\theta\|_{L^\infty(0,\mathcal{T};L^2(\Omega))} < M_3 = \\ = (1 - \exp[-\gamma\mathcal{T}/\sigma^2])^{-1}\left(\varepsilon\mathcal{T}\|T_2 - T_1\|_{L^\infty(0,\mathcal{T})} + \left\|\frac{\partial d}{\partial t}\right\|_{L^2(0,\mathcal{T};L^2(\Omega))}\right)$$

and therefore

$$(5.6)\qquad \|f\|_{L^2(0,\mathcal{T};L^2(\Omega))} < \alpha g\mathcal{T}(1 - \exp[-\gamma\mathcal{T}/\sigma^2])^{-1}\cdot \\ \cdot\left\{\left(\varepsilon\mathcal{T} + \frac{1 - \exp[-\gamma\mathcal{T}/\sigma^2]}{\sqrt{3}}\right)\|T_2 - T_1\|_{L^\infty(0,\mathcal{T})} + \left\|\frac{\partial d}{\partial t}\right\|_{L^2(0,\mathcal{T};L^2(\Omega))}\right\}.$$

The following notations are now introduced:

$$(5.7)\qquad H^0_{\mathcal{T}} = \{\varphi \in L^2_{\mathrm{loc}}(-\infty, +\infty; L^2(\Omega))\,|\,\varphi \text{ periodic of period } \mathcal{T}\},$$

$$(5.8)\qquad K_{\mathcal{T}} = \{\varphi \in H^0_{\mathcal{T}}\,|\,\varphi \in L^\infty(-\infty, +\infty; L^2(\Omega)) \text{ and } \|\varphi\|_{L^\infty(0,\mathcal{T};L^2(\Omega))} < M_3\}.$$

Furthermore, let $A\colon H^0_{\mathcal{T}} \to K_{\mathcal{T}}$ be the transformation defined by

$$(5.9)\qquad A(\varphi) = \tilde{\varphi},$$

where, denoting by $D_\varphi \subset \mathrm{R}$ the set of points where φ is defined (meas $(\mathrm{R}\setminus D_\varphi) = 0$), the function $\tilde{\varphi}$ also is defined on D_φ by the following law:

$$\tilde{\varphi}(t) = \begin{cases} \varphi(t), & \text{if } \|\varphi(t)\|_{L^2(\Omega)} < M_3, \\[2mm] \dfrac{\varphi(t)}{\|\varphi(t)\|_{L^2(\Omega)}}\cdot M_3, & \text{if } \|\varphi(t)\|_{L^2(\Omega)} > M_3. \end{cases}$$

Assume now that the data satisfy the following condition:

$$(5.10)\qquad \left(\varepsilon\mathcal{T} + \frac{1 - \exp[-\gamma\mathcal{T}/\sigma^2]}{\sqrt{3}}\right)\|T_2 - T_1\|_{L^\infty(0,\mathcal{T})} + \left\|\frac{\partial d}{\partial t}\right\|_{L^2(0,\mathcal{T};L^2(\Omega))} < \\ < \frac{\mu^{5/4}}{\alpha g\mathcal{T}\varrho_0^{5/4}}(1 - \exp[-\gamma\mathcal{T}/\sigma^2])(1 - \exp[-\mu\mathcal{T}/\varrho_0\sigma^2])^{3/4}\cdot \\ \cdot\min\left\{\sqrt[4]{\varrho_0\frac{(1-\bar{\lambda})}{8\mu}},\ \sqrt[4]{\bar{\lambda}\mathcal{T}}/\sqrt{2\sigma}\right\}.$$

Then, a further transformation can be defined, such that

$$S\colon K_{\mathcal{T}} \to L^2_{\mathrm{loc}}(-\infty, +\infty; \hat{H}^1_0(\Omega)) \cap K_{\mathcal{T}} \cap H^1_{\mathrm{loc}}(-\infty, +\infty; H^{-1}(\Omega)),$$

in the following way. Let $\bar{\varphi} \in K_{\mathcal{T}}$: by substituting $\bar{\varphi}$ to θ in (2.25), the known term f of (2.25) satisfies condition (5.6). On the other hand, by (5.10), f satisfies condition (3.59) and, therefore, by Theorem 3.4, there exists $\bar{u}$, unique periodic solution of (2.25). Substitute then such $\bar{u}$ in (2.26); by Theorem 4.1, the existence and uniqueness of a

$$\theta \in L^2_{\mathrm{loc}}(-\infty, +\infty; \hat{H}^1_0(\Omega)) \cap K_{\mathcal{T}} \cap H^1_{\mathrm{loc}}(-\infty, +\infty; H^{-1}(\Omega))$$

and satisfying (2.26) can be assured.

Let S be the transformation defined setting $S(\bar{\varphi}) = \theta$. We finally define a trasformation $F\colon H^{\beta}_{\mathcal{T}} \to H^{\beta}_{\mathcal{T}}$, according to the following law of composition:

$$F = S \circ A. \tag{5.11}$$

The following theorem holds.

THEOREM 5.1: *Let $T_1, T_2 \in H^1_{\mathrm{loc}}(-\infty, +\infty)$ periodic of period $\mathcal{T}$ and satisfying condition* (5.10); *then there exists at least one periodic solution of problem* (2.7)-(2.12), *in the sense specified in* (2.23)-(2.26).

PROOF: For the proof of the present theorem it is sufficient to prove that the transformation F, defined by (5.11), has at least a fixed point: for this, the Leray-Schauder fixed point theorem will be used. Observe that F transforms $H^{\beta}_{\mathcal{T}}$ in a periodic functions space that is a subset of (see [6, Ch. 1, §§ 9 and 12])

$$H^{\varepsilon}_{\mathrm{loc}}(-\infty, +\infty; H^{1-2\varepsilon}(\Omega)) \underset{c.c.}{\subseteq} L^2_{\mathrm{loc}}(-\infty, +\infty; L^2(\Omega)), \qquad 0 < \varepsilon < 1;$$

bearing in mind the continuous dependence results given in Theorems 3.4 and 4.1, F is therefore completely continuous from $H^{\beta}_{\mathcal{T}}$ in itself.

Let then $\{S_\lambda\}$, $\lambda \in [0, 1]$ be a family of transformations defined $\forall\lambda$ in a similar way to S, referring to the following equations:

$$\frac{\partial u}{\partial t} - \frac{\mu}{\varrho_0}\Delta u + (u \cdot \nabla)u + \frac{1}{\varrho_0}\nabla p = g\{1 - \alpha(\theta + d - T_0)\}, \tag{5.12}$$

$$\nabla \cdot u = 0, \tag{5.13}$$

$$\frac{\partial \theta}{\partial t} - \gamma\,\Delta\theta + u \cdot \nabla\theta = \lambda\frac{T_2 - T_1}{2}u_3 - \lambda\frac{\partial d}{\partial t}, \tag{5.14}$$

with the associated boundary conditions (2.10), (2.11).

Let $\{F_\lambda\}$, $\lambda \in [0, 1]$, be the family of transformations defined by

$$F_\lambda = S_\lambda \circ A, \qquad \forall\lambda \in [0, 1]. \tag{5.15}$$

Hence, $F_1 = F$, and by (5.14), $F_0 = 0$. If then θ_λ is a fixed point of F_λ, it follows:

$$\|\theta_\lambda\|_{L^\infty(0,\mathcal{T};L^2(\Omega))} < (1 - \exp[-\gamma\mathcal{T}/\sigma^2])^{-1} \cdot \left(\sqrt{\mathcal{T}}\lambda\|T_2 - T_1\|_{L^\infty(0,\mathcal{T})} + \lambda\left\|\frac{\partial d}{\partial t}\right\|_{L^2(0,\mathcal{T};L^2(\Omega))}\right) < M_2 . \tag{5.16}$$

The fixed point θ_λ presents therefoee in $L^2(0, \mathcal{T}; L^2(\Omega))$ an a peiori bound independent of λ. From Leray-Schauder theorem, the transformation F has at least at fixed point θ, which associated to the corresponding u in the definition of S, gives rise to a pair (u, θ) that, according to what has been said before, is a periodic solution of problem (2.7)-(2.12). ■

6. - The uniqueness of the solution

It is possible to prove the uniqueness of the periodic solution (u, θ) of problem (2.7)-(2.12) in the case in which the data and other physical quantities that intervene in the problem are sufficiently «small». More precisely, the following theorem holds:

Theorem 6.1: *Let the assumption of Theorem* 5.1 *hold. In this case, if the quantities*

$$\|T_2 - T_1\|_{L^\infty(0,\mathcal{T})}, \qquad \left\|\frac{\partial d}{\partial t}\right\|_{L^2(0,\mathcal{T};L^2(\Omega))}, \qquad \alpha,$$

are «sufficiently small», the periodic solution (u, θ) *of problem* (2.7)-(2.12) *is unique.*

Proof: Under the assumpeioas of Theorem 5.1 the existence of at least one solution of problem (2.7)-(2.12) can be assured. Suppose now that two of them exist $(u^{(1)}, \theta_1)$ and $(u^{(2)}, \theta_2)$ and define $w = u^{(1)} - u^{(2)}$, $\delta_\theta = \theta_1 - \theta_2$.

Following Lemmas 3.3 and 4.1, the following relationships hold:

$$\frac{1}{2}\|w(t_2)\|^2_{L^2(\Omega)} - \frac{1}{2}\|w(t_1)\|^2_{L^2(\Omega)} + \int_{t_1}^{t_2}\left\{\frac{\mu}{\theta_0}\|w(\eta)\|^2_{H^1_0(\Omega)} - b_1(w(\eta), w(\eta), u^{(1)}(\eta)) + \alpha(\delta_\theta(\eta)g, w(\eta))_{L^2(\Omega)}\right\} d\eta < 0 , \tag{6.1}$$

for every pair of points (t_1, t_2), with $t_1 < t_2$, $|t_1 - t_2| < \mathcal{T}$ and

$$\frac{1}{2}\|\delta_\theta(t_2)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\delta_\theta(t_1)\|^2_{L^2(\Omega)} + \int_{t_1}^{t_2}\left\{\gamma\|\delta_\theta(\eta)\|^2_{H^1_0(\Omega)} + b_2(w(\eta), \theta_1(\eta), \delta_\theta(\eta)) + \right. \left. - \frac{T_2(\eta) - T_1(\eta)}{2}(w_1(\eta), \delta_\theta(\eta))_{L^2(\Omega)}\right\} d\eta = 0 , \tag{6.2}$$

for every pair of points (t_1, t_2), $t_1 < t_2$, $|t_1 - t_2| < \mathcal{T}$ outside a certain set of measure zero. By Lemma 3.2, $w \in C_0(-\infty, +\infty; H)$ and therefore $t \to \|w(t)\|^2_{L^2(\Omega)}$ has a minimum t' and a maximum t'' in any interval of length $\mathfrak{T}$. By virtue of the periodicity, it can be assumed that $t' - \mathfrak{T} < < t'' < t'$. If $t' = t''$, then $t \to \|\mathbf{w}(t)\|^2_{L^2(\Omega)}$ is constant, therefore there exists certainly t^* close to t' in which δ_θ is defined and, in whichever way it is chosen, the following relationship holds

$$(6.3)\qquad \int_{t^*-\mathfrak{T}}^{t^*} \|\mathbf{w}(\eta)\|^2_{H^1_0(\Omega)}\, d\eta > \int_{t^*-\mathfrak{T}}^{t^*} \frac{1}{\sigma^2}\|\mathbf{w}(\eta)\|^2_{L^2(\Omega)} = \frac{\|\mathbf{w}(t'')\|^2_{L^2(\Omega)}}{\sigma^2}\,\mathfrak{T}\,.$$

For a suitable choice of $\lambda > 0$, it will also be:

$$(6.4)\qquad \int_{t^*-\mathfrak{T}}^{t^*} \|\mathbf{w}(\eta)\|^2_{H^1_0(\Omega)}\, d\eta > \frac{\vartheta_0}{\lambda\mu}\|\mathbf{w}(t'')\|^2_{L^2(\Omega)} - \alpha g \int_{t^*-\mathfrak{T}}^{t^*} \|\delta_\theta(\eta)\|_{L^2(\Omega)}\|\mathbf{w}(\eta)\|_{L^2(\Omega)}\, d\eta\,.$$

If $t' > t''$, there exists t^*, $t' < t^* < t'' + \mathfrak{T}$ (and consequently $t^* - \mathfrak{T} < t'' < t' < t^*$), in which δ_θ is defined; moreover, analogously to what has been done in Theorem 3.2 to prove (3.45), we can prove that, also in this case, a quantity $\lambda > 0$ exists such that (6.4) holds.

It follows that there exists t^* such that $t^* - \mathfrak{T} < t'' < t' < t^*$, δ_θ is defined in t^*, and that there exists $\lambda > 0$ for which (6.4) holds, and consequently, $\forall \varepsilon > 0$:

$$(6.5)\qquad \int_{t^*-\mathfrak{T}}^{t^*} \|\mathbf{w}(\eta)\|^2_{H^1_0(\Omega)}\, d\eta > \frac{\vartheta_0}{\lambda\mu}\|\mathbf{w}(t'')\|^2_{L^2(\Omega)} +$$

$$- \alpha g \varepsilon \int_{t^*-\mathfrak{T}}^{t^*} \|\mathbf{w}(\eta)\|^2_{L^2(\Omega)} - \frac{\alpha g}{4\varepsilon}\int_{t^*-\mathfrak{T}}^{t^*} \|\delta_\theta(\eta)\|^2_{L^2(\Omega)}\, d\eta > \frac{\vartheta_0}{\lambda\mu}\|\mathbf{w}(t'')\|^2_{L^2(\Omega)} +$$

$$- \alpha g \sigma^2 \varepsilon \int_{t^*-\mathfrak{T}}^{t^*} \|\mathbf{w}(\eta)\|^2_{H^1_0(\Omega)}\, d\eta - \frac{\alpha g}{4\varepsilon}\int_{t^*-\mathfrak{T}}^{t^*} \|\delta_\theta(\eta)\|^2_{L^2(\Omega)}\, d\eta\,.$$

It also follows therefore

$$(6.6)\qquad (1 + \alpha g \sigma^2 \varepsilon)\int_{t^*-\mathfrak{T}}^{t^*} \|\mathbf{w}(\eta)\|^2_{H^1_0(\Omega)}\, d\eta > \frac{\vartheta_0}{\lambda\mu}\|\mathbf{w}(t'')\|^2_{L^2(\Omega)} - \frac{\alpha g}{4\varepsilon}\int_{t^*-\mathfrak{T}}^{t^*} \|\delta_\theta(\eta)\|^2_{L^2(\Omega)}\, d\eta\,.$$

The latter relationship also provides, in turn

$$(6.7)\qquad \|\mathbf{w}(t'')\|^2_{L^2(\Omega)} < \frac{\lambda\mu}{\vartheta_0}\left\{(1 + \alpha g \sigma^2 \varepsilon)\int_{t^*-\mathfrak{T}}^{t^*} \|\mathbf{w}(\eta)\|^2_{H^1_0(\Omega)}\, d\eta + \frac{\alpha g}{4\varepsilon}\int_{t^*-\mathfrak{T}}^{t^*} \|\delta_\theta(\eta)\|^2_{L^2(\Omega)}\, d\eta\right\},$$

and therefore

$$(6.8)\qquad \|\mathbf{w}(t'')\|_{L^2(\Omega)} < \left(\frac{\lambda\mu}{\vartheta_0}\right)^{\frac{1}{2}}\left\{(1 + \alpha g \sigma^2 \varepsilon)^{\frac{1}{2}}\left(\int_{t^*-\mathfrak{T}}^{t^*} \|\mathbf{w}(\eta)\|^2_{H^1_0(\Omega)}\, d\eta\right)^{\frac{1}{2}} + \left(\frac{\alpha g}{4\varepsilon}\right)^{\frac{1}{2}}\left(\int_{t^*-\mathfrak{T}}^{t^*} \|\delta_\theta(\eta)\|^2_{L^2(\Omega)}\, d\eta\right)^{\frac{1}{2}}\right\}.$$

We now find a bound for the term

$$\left|\int_{t^*-\mathfrak{T}}^{t^*} h_1(\mathbf{w}(\eta), \mathbf{w}(\eta), \mathbf{u}^{(1)}(\eta))\, d\eta\right|.$$

Reasoning as in [11, Theorem 1.7], we have

$$(6.9)\qquad \left|\int_{t^*-\mathfrak{T}}^{t^*} b_1(w(\eta), w(\eta), u^{(1)}(\eta))\, d\eta\right| < \sqrt{2} \max_{\eta\in[t^*-\mathfrak{T},\, t^*]} \|u^{(1)}(\eta)\|^{\frac12}_{L^2(\Omega)} \|w(t')\|^{\frac12}_{L^2(\Omega)} \cdot$$

$$\cdot\left\{\int_{t^*-\mathfrak{T}}^{t^*} \|u^{(1)}(\eta)\|^2_{H^1_0(\Omega)}\, d\eta\right\}^{\frac12} \cdot \left\{\int_{t^*-\mathfrak{T}}^{t^*} \|w(\eta)\|^2_{H^1_0(\Omega)}\, d\eta\right\}^{\frac12}.$$

Setting then

$$\Gamma_1 = \max_{\eta\in[t^*-\mathfrak{T},\, t^*]} \|u^{(1)}(\eta)\|^{\frac12}_{L^2(\Omega)} \left\{\int_{t^*-\mathfrak{T}}^{t^*} \|u^{(1)}(\eta)\|^2_{H^1_0(\Omega)}\, d\eta\right\}^{\frac12},$$

and remembering (6.8), it follows:

$$(6.10)\qquad \left|\int_{t^*-\mathfrak{T}}^{t^*} b_1(w(\eta), w(\eta), u^{(1)}(\eta))\, d\eta\right| < \sqrt{2}\,\Gamma_1 \left(\frac{\lambda\mu}{\theta_0}\right)^{\frac12} \cdot$$

$$\cdot\left\{(1+\alpha g\sigma^2 c)^{\frac12} \cdot \int_{t^*-\mathfrak{T}}^{t^*} \|w(\eta)\|^2_{H^1_0(\Omega)}\, d\eta + \left(\frac{\alpha g}{4c}\right)^{\frac12} \varepsilon_1 \int_{t^*-\mathfrak{T}}^{t^*} \|\delta_\theta(\eta)\|^2_{L^2(\Omega)}\, d\eta \right. +$$

$$\left. + \left(\frac{\alpha g}{4c}\right)^{\frac12} \mathfrak{T}_{\varepsilon_1} \int_{t^*-\mathfrak{T}}^{t^*} \|w(\eta)\|^2_{H^1_0(\Omega)}\, d\eta\right\} = \sqrt{2}\,\Gamma_1 \left(\frac{\lambda\mu}{\theta_0}\right)^{\frac12} \left\{\left[(1+\alpha g\sigma^2 c)^{\frac12} + \left(\frac{\alpha g}{4c}\right)^{\frac12} \mathfrak{T}_{\varepsilon_1}\right] \cdot\right.$$

$$\left. \cdot \int_{t^*-\mathfrak{T}}^{t^*} \|w(\eta)\|^2_{H^1_0(\Omega)}\, d\eta + \left(\frac{\alpha g}{4c}\right)^{\frac12} \varepsilon_1 \int_{t^*-\mathfrak{T}}^{t^*} \|\delta_\theta(\eta)\|^2_{L^2(\Omega)}\, d\eta\right\},$$

where ε_1 is an arbitrary positive constant and $\mathfrak{T}_{\varepsilon_1}$ a suitable constant. Examing now the term

$$\left|\int_{t^*-\mathfrak{T}}^{t^*} b_2(w(\eta), \delta_\theta(\eta), \theta_1(\eta))\, d\eta\right|,$$

we have

$$(6.11)\qquad \left|\int_{t^*-\mathfrak{T}}^{t^*} b_2(w(\eta), \delta_\theta(\eta), \theta_1(\eta))\, d\eta\right| < \int_{t^*-\mathfrak{T}}^{t^*} \|w(\eta)\|_{L^4(\Omega)} \|\delta_\theta(\eta)\|_{H^1_0(\Omega)} \|\theta_1(\eta)\|_{L^4(\Omega)}\, d\eta <$$

$$< \sigma'^2 \int_{t^*-\mathfrak{T}}^{t^*} \|w(\eta)\|^{\frac12}_{L^2(\Omega)} \|w(\eta)\|^{\frac12}_{H^1_0(\Omega)} \|\delta_\theta(\eta)\|_{H^1_0(\Omega)} \|\theta_1(\eta)\|^{\frac12}_{L^2(\Omega)} \|\theta_1(\eta)\|^{\frac12}_{H^1_0(\Omega)}\, d\eta <$$

$$< \sigma'^2 \|w(t')\|^{\frac12}_{L^2(\Omega)} \|\theta_1\|^{\frac12}_{L^\infty(0,\mathfrak{T};\, L^2(\Omega))} \int_{t^*-\mathfrak{T}}^{t^*} \|w(\eta)\|^{\frac12}_{H^1_0(\Omega)} \|\theta_1(\eta)\|^{\frac12}_{H^1_0(\Omega)} \|\delta_\theta(\eta)\|_{H^1_0(\Omega)}\, d\eta,$$

with σ' embedding constant of $H^1_0(\Omega)$ in $L^4(\Omega)$.

Hence, by Hölder's inequality:

$$\left|\int_{t^*-\mathfrak{T}}^{t^*} b_2(\boldsymbol{w}(\eta), \delta_\varphi(\eta), \theta_1(\eta))\, d\eta\right| < \sigma'^2 \|\boldsymbol{w}(t')\|^{\frac{1}{2}}_{L^2(\Omega)} \|\theta_1\|^{\frac{1}{2}}_{L^\infty(0,\mathfrak{T}; L^2(\Omega))} \left\{\int_{t^*-\mathfrak{T}}^{t^*} \|\theta_1(\eta)\|^2_{H^1_0(\Omega)}\, d\eta\right\}^{\frac{1}{4}} \cdot \left\{\int_{t^*-\mathfrak{T}}^{t^*} \|\boldsymbol{w}(\eta)\|^{\frac{3}{2}}_{H^1_0(\Omega)} \|\delta_\varphi(\eta)\|^{\frac{3}{2}}_{H^1_0(\Omega)}\, d\eta\right\}^{\frac{2}{3}}.$$

Setting

$$\Gamma_2 = \|\theta_1\|^{\frac{1}{2}}_{L^\infty(0,\mathfrak{T}; L^2(\Omega))} \left\{\int_{t^*-\mathfrak{T}}^{t^*} \|\theta_1(\eta)\|^2_{H^1_0(\Omega)}\, d\eta\right\}^{\frac{1}{4}},$$

and again by Hölder's inequality:

$$\text{(6.12)}\qquad \left|\int_{t^*-\mathfrak{T}}^{t^*} b_2(\boldsymbol{w}(\eta), \delta_\varphi(\eta), \theta_1(\eta))\, d\eta\right| <$$

$$< \sigma'^2 \Gamma_2 \cdot \|\boldsymbol{w}(t')\|^{\frac{1}{2}}_{L^2(\Omega)} \left(\int_{t^*-\mathfrak{T}}^{t^*} \|\boldsymbol{w}(\eta)\|^2_{H^1_0(\Omega)}\, d\eta\right)^{\frac{1}{2}} \left(\int_{t^*-\mathfrak{T}}^{t^*} \|\delta_\varphi(\eta)\|^2_{H^1_0(\Omega)}\, d\eta\right)^{\frac{1}{2}} <$$

$$< \sigma'^2 \Gamma_2 \|\boldsymbol{w}(t')\|^{\frac{1}{2}}_{L^2(\Omega)} \left[\varepsilon_2 \left(\int_{t^*-\mathfrak{T}}^{t^*} \|\boldsymbol{w}(\eta)\|^2_{H^1_0(\Omega)}\, d\eta\right)^{\frac{1}{2}} + \mathfrak{T}_{\varepsilon_2} \left(\int_{t^*-\mathfrak{T}}^{t^*} \|\delta_\varphi(\eta)\|^2_{H^1_0(\Omega)}\, d\eta\right)^{\frac{1}{2}}\right],$$

with ε_2 arbitrary positive constant and $\mathfrak{T}_{\varepsilon_2} > 0$ suitable constant.

Substituting for $\|\boldsymbol{w}(t')\|^{\frac{1}{2}}_{L^2(\Omega)}$ in (6.12), the expression (6.8), we have:

$$\text{(6.13)}\qquad \left|\int_{t^*-\mathfrak{T}}^{t^*} b_2(\boldsymbol{w}(\eta), \delta_\varphi(\eta), \theta_1(\eta))\, d\eta\right| <$$

$$< \sigma'^2 \Gamma_2 \left(\frac{\lambda\mu}{\vartheta_0}\right)^{\frac{1}{4}} \left\{\varepsilon_2 (1 + \varkappa g \sigma^2 \varepsilon)^{\frac{1}{4}} \int_{t^*-\mathfrak{T}}^{t^*} \|\boldsymbol{w}(\eta)\|^2_{H^1_0(\Omega)}\, d\eta + \mathfrak{T}_{\varepsilon_2} \left(\frac{\varkappa g \sigma^2}{4\varepsilon}\right)^{\frac{1}{4}} \int_{t^*-\mathfrak{T}}^{t^*} \|\delta_\varphi(\eta)\|^2_{H^1_0(\Omega)}\, d\eta +\right.$$

$$+ \varepsilon_2 \left(\frac{\varkappa g \sigma^2}{4\varepsilon}\right)^{\frac{1}{4}} \left(\int_{t^*-\mathfrak{T}}^{t^*} \|\delta_\varphi(\eta)\|^2_{H^1_0(\Omega)}\, d\eta\right)^{\frac{1}{2}} \left(\int_{t^*-\mathfrak{T}}^{t^*} \|\boldsymbol{w}(\eta)\|^2_{H^1_0(\Omega)}\, d\eta\right)^{\frac{1}{2}} +$$

$$\left. + (1 + \varkappa g \sigma^2 \varepsilon)^{\frac{1}{4}} \mathfrak{T}_{\varepsilon_2} \left(\int_{t^*-\mathfrak{T}}^{t^*} \|\boldsymbol{w}(\eta)\|^2_{H^1_0(\Omega)}\, d\eta\right)^{\frac{1}{2}} \left(\int_{t^*-\mathfrak{T}}^{t^*} \|\delta_\varphi(\eta)\|^2_{H^1_0(\Omega)}\, d\eta\right)^{\frac{1}{2}}\right\} <$$

$$< \sigma'^2 \Gamma_2 \left(\frac{\lambda\mu}{\vartheta_0}\right)^{\frac{1}{4}} \left\{\varepsilon_2 (1 + \varkappa g \sigma^2 \varepsilon)^{\frac{1}{4}} \int_{t^*-\mathfrak{T}}^{t^*} \|\boldsymbol{w}(\eta)\|^2_{H^1_0(\Omega)}\, d\eta + \mathfrak{T}_{\varepsilon_2} \left(\frac{\varkappa g \sigma^2}{4\varepsilon}\right)^{\frac{1}{4}} \|\delta_\varphi(\eta)\|^2_{H^1_0(\Omega)}\, d\eta +\right.$$

$$+\,\varepsilon_2\left(\frac{\varkappa g\sigma^2}{4\varepsilon}\right)^{\frac12}\varepsilon_3\int_{t^*-\mathfrak{T}}^{t^*}\|w(\eta)\|^2_{H^1_0(\Omega)}\,d\eta+\varepsilon_3\left(\frac{\varkappa g\sigma^2}{4\varepsilon}\right)^{\frac12}\mathfrak{T}_{\varepsilon_3}\int_{t^*-\mathfrak{T}}^{t^*}\|\vartheta_\theta(\eta)\|^2_{H^1_0(\Omega)}\,d\eta\,+$$

$$+\,(1+\varkappa g\sigma^2\varepsilon)^{\frac12}\mathfrak{T}_{\varepsilon_2}\varepsilon_4\int_{t^*-\mathfrak{T}}^{t^*}\|w(\eta)\|^2_{H^1_0(\Omega)}\,d\eta+(1+\varkappa g\sigma^2\varepsilon)^{\frac12}\mathfrak{T}_{\varepsilon_2}\mathfrak{T}_4\int_{t^*-\mathfrak{T}}^{t^*}\|\vartheta_\theta(\eta)\|^2_{H^1_0(\Omega)}\,d\eta\Bigg\}=$$

$$=\sigma'^2\Gamma_2\left(\frac{\lambda\mu}{\varrho_0}\right)^{\frac12}\Bigg\{\left[\varepsilon_2(1+\varkappa g\sigma^2\varepsilon)^{\frac12}+\varepsilon_2\varepsilon_3\left(\frac{\varkappa g\sigma^2}{4\varepsilon}\right)^{\frac12}+(1+\varkappa g\sigma^2\varepsilon)^{\frac12}\varepsilon_4\mathfrak{T}_{\varepsilon_2}\right]\int_{t^*-\mathfrak{T}}^{t^*}\|w(\eta)\|^2_{H^1_0(\Omega)}\,d\eta\,+$$

$$+\left[\mathfrak{T}_{\varepsilon_2}\left(\frac{\varkappa g\sigma^2}{4\varepsilon}\right)^{\frac12}+\varepsilon_3\left(\frac{\varkappa g\sigma^2}{4\varepsilon}\right)^{\frac12}\mathfrak{T}_{\varepsilon_3}+(1+\varkappa g\sigma^2\varepsilon)^{\frac12}\mathfrak{T}_{\varepsilon_2}\mathfrak{T}_{\varepsilon_4}\right]\int_{t^*-\mathfrak{T}}^{t^*}\|\vartheta_\theta(\eta)\|^2_{H^1_0(\Omega)}\,d\eta\Bigg\}.$$

In addition we observe that:

$$(6.14)\qquad\left|\varkappa\int_{t^*-\mathfrak{T}}^{t^*}(\vartheta_\theta(\eta)\mathbf{g},w(\eta))_{L^2(\Omega)}\,d\eta\right|<\varkappa g\int_{t^*-\mathfrak{T}}^{t^*}\|\vartheta_\theta(\eta)\|_{L^2(\Omega)}\|w(\eta)\|_{L^2(\Omega)}\,d\eta<$$

$$<\varkappa g\sigma^2\int_{t^*-\mathfrak{T}}^{t^*}\|\vartheta_\theta(\eta)\|_{H^1_0(\Omega)}\|w(\eta)\|_{H^1_0(\Omega)}\,d\eta<\frac{\varkappa g\sigma^2}{2}\int_{t^*-\mathfrak{T}}^{t^*}(\|\vartheta_\theta(\eta)\|^2_{H^1_0(\Omega)}+\|w(\eta)\|^2_{H^1_0(\Omega)})\,d\eta\,,$$

and

$$(6.15)\qquad\left|\int_{t^*-\mathfrak{T}}^{t^*}\frac{T_2(\eta)-T_1(\eta)}{2}(w_2(\eta),\vartheta_\theta(\eta))_{L^2(\Omega)}\,d\eta\right|<\frac12\|T_2-T_1\|_{L^\infty(0,\mathfrak{T})}\cdot$$

$$\cdot\int_{t^*-\mathfrak{T}}^{t^*}\|w(\eta)\|_{L^2(\Omega)}\|\vartheta_\theta(\eta)\|_{L^2(\Omega)}\,d\eta<\frac{\sigma^2}{2}\|T_2-T_1\|_{L^\infty(0,\mathfrak{T})}\int_{t^*-\mathfrak{T}}^{t^*}\|w(\eta)\|_{H^1_0(\Omega)}\|\vartheta_\theta(\eta)\|_{H^1_0(\Omega)}\,d\eta<$$

$$<\frac{\sigma^2}{4}\|T_2-T_1\|_{L^\infty(0,\mathfrak{T})}\int_{t^*-\mathfrak{T}}^{t^*}(\|w(\eta)\|^2_{H^1_0(\Omega)}+\|\vartheta_\theta(\eta)\|^2_{H^1_0(\Omega)})\,d\eta\,.$$

Going back now to the definitions of w and ϑ_θ and to their properties, writing (6.1) and (6.2) with $t_2=t^*$, $t_1=t^*-\mathfrak{T}$ and adding them, by periodicity of w and ϑ_θ, we have:

$$(6.16)\qquad\int_{t^*-\mathfrak{T}}^{t^*}\left\{\frac{\mu}{\varrho_0}\|w(\eta)\|^2_{H^1_0(\Omega)}+\gamma\|\vartheta_\theta(\eta)\|^2_{H^1_0(\Omega)}\right\}d\eta<$$

$$<\int_{t^*-\mathfrak{T}}^{t^*}\Bigg\{b_1(w(\eta),w(\eta),u^{(1)}(\eta))-b_2(w(\eta),\vartheta_\theta(\eta),\theta_1(\eta))-\varkappa(\vartheta_\theta(\eta)\mathbf{g},w(\eta))_{L^2(\Omega)}\,+$$

$$+\,\frac{T_2(\eta)-T_1(\eta)}{2}(w_2(\eta),\vartheta_\theta(\eta))_{L^2(\Omega)}\Bigg\}\,d\eta$$

where, for the various terms that appear under the integral sign, on the right hand side of the inequality, starting from $+b_1(w, w, u^{(1)})$ the inequalities (6.10), (6.13), (6.14), (6.15) respectively hold.

We then observe that if $\|T_2 - T_1\|_{L^\infty(0,\mathcal{T})}$ and $\|\partial d/\partial t\|_{L^1(0,\mathcal{T};L^2(\Omega))}$ are sufficiently «small», by (5.5), $\|\theta_1\|_{L^\infty(0,\mathcal{T};L^2(\Omega))}$ and, consequently, Γ_2 are small. In addition it follows that $\|f\|_{L^1(0,\mathcal{T};L^2(\Omega))}$ with f defined by (5.1), is small and consequently, by (3.3), Γ_1 is small too. In conclusion, if we assume that the values $\|T_2 - T_1\|_{L^\infty(0,\mathcal{T})}$, $\|\partial d/\partial t\|_{L^1(0,\mathcal{T};L^2(\Omega))}$ and $\alpha = R^* P_B/g$ are sufficiently small, then it is possible to control the four terms which appear on the right hand side of (6.16), and also to obtain a relationship of the type

$$\int_{t^*-\mathcal{T}}^{t^*} \left\{\beta_1 \|w(\eta)\|^2_{H_0^1(\Omega)} + \beta_2 \|\theta(\eta)\|^2_{H_0^1(\Omega)}\right\} d\eta < 0\,,$$

with positive constants β_1 and β_2, from which the uniqueness follows. ■

BIBLIOGRAPHY

[1] T. Collini Lanzi, *Soluzioni periodiche per le disequazioni di Navier-Stokes con condizioni al contorno non lineari*, Riv. Mat. Univ. Parma (4), 12 (1986), pp. 19-29.

[2] A. Iannelli, *Su un modello di Oberbeck-Boussinesq relativo al moto di un fluido viscoso soggetto a riscaldamento*, Istit. Lombardo (Rend. Sc.) A, 121 (1987), pp. 145-190.

[3] D. D. Joseph, *Stability of Fluid Motions - II*, Springer-Verlag (1976).

[4] V. T. Judovic, *Periodic motions of a viscous incompressible fluid*, Dokl. Akad. Nauk SSSR, 130 (1960), pp. 38-52.

[5] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Paris (1969).

[6] J. L. Lions - E. Magenes, *Problèmes aux limites non homogènes et applications*, Vol. I, Dunod, Paris (1968).

[7] G. Prodi, *Rassegna di ricerche intorno alle equazioni di Navier-Stokes*, Istituto di Matematica dell'Università di Trieste, Quaderno n. 2 (1959).

[8] G. Prodi, *Qualche risultato riguardo alle equazioni di Navier-Stokes nel caso bidimensionale*, Ann. Mat. Pura Appl., 48 (1959), pp. 1-15.

[9] G. Prouse, *Soluzioni periodiche dell'equazione di Navier-Stokes*, Atti Acc. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., 35 (1963), pp. 443-447.

[10] G. Prouse, *On an inequality related to the motion, in any dimension, of a viscous incompressible fluid*, (I), (II), Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., 67 (1979), pp. 191-196, pp. 282-288.

[11] G. Prouse, *Soluzioni quasi-periodiche dell'equazione differenziale di Navier-Stokes in due dimensioni*, Rend. Sem. Mat. Univ. Padova, 33 (1963), pp. 186-212.

[12] A. Zaretti, *Soluzioni periodiche di un problema misto non lineare per le equazioni di Navier-Stokes*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., 51 (1971), pp. 84-91.

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