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A Cordes Type Condition for non Linear non Variational Systems (**)

ABSTRACT. — We define a condition of Cordes type for non linear non variational operators $a(x, H(u))$, with $a(x, \xi)$ measurable in x and of class C^1 in ξ.

We then show that, $\forall f \in L^2(\Omega)$, the Dirichlet problem

$$u \in H^2 \cap H_0^1(\Omega),$$
$$a(x, H(u)) = f \qquad \text{in } \Omega,$$

has a unique solution.

Under the non linear condition of Cordes, we show that the solutions $u \in H^2(\Omega)$ of a quasi-basic system

$$a(x, H(u)) = 0 \qquad \text{in } \Omega,$$

belong to $H^{2,p}_{loc}(\Omega)$, for a $p > 2$, and as a consequence they satisfy the fundamental interior estimate. From these above facts, as is well known, we have the possibility of developping the theory of $\mathfrak{C}^{2,\lambda}$ regularity, and that of partial $-\mathfrak{C}^{2,\lambda}$ regularity for the vector Du for the solution $u \in H^2(\Omega)$ of the system

$$a(x, u, Du, H(u)) = b(x, u, Du).$$

Una condizione del tipo di Cordes per sistemi non lineari non variazionali

RIASSUNTO. — Si definisce una condizione di Cordes per operatori non lineari e non variazionali $a(x, H(u))$, con $a(x, \xi)$ misurabile in x e di classe C^1 in ξ.

Si dimostra quindi che, $\forall f \in L^2(\Omega)$, il problema di Dirichlet

$$u \in H^2 \cap H_0^1(\Omega),$$
$$a(x, H(u)) = f \qquad \text{in } \Omega,$$

ha una e una sola soluzione.

Sotto la condizione non lineare di Cordes, si dimostra che le soluzioni $u \in H^2(\Omega)$ di un sistema

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quasi-base

$$a(x, H(u)) = 0 \qquad \text{in } \Omega ,$$

appartengono ad $H^{2,p}_{loc}(\Omega)$, per un $p > 2$, e di conseguenza esse verificano la maggiorazione fondamentale all'interno.

Da tutti questi fatti segue, come è noto, la possibilità di sviluppare la teoria della regolarità $\mathfrak{L}^{2,\lambda}$, e regolarità $\mathfrak{L}^{2,\lambda}$-parziale, per il vettore Du di soluzioni $u \in H^2(\Omega)$ del sistema

$$a(x, u, Du, H(u)) = b(x, u, Du) .$$

1 – Introduction

Let Ω be a bounded open set in $\boldsymbol{R}^n$, $n > 2$, let $x = (x_1, \ldots, x_n)$ denote a point of Ω and $N > 1$ be an integer. A generic vector of $\boldsymbol{R}^{nN}$ is denoted by $p = (p_1, \ldots, p_n)$ where $p_i \in \boldsymbol{R}^N$. We shall denote by $\xi = \{\xi_{ij}\}$, $i, j = 1, \ldots, n$ where $\xi_{ij} \in \boldsymbol{R}^N$ a generic element of $\boldsymbol{R}^{n^2N}$. For a vector $u\colon \Omega \to \boldsymbol{R}^N$ we set

$$Du = (D_1 u, \ldots, D_n u) ,$$
$$H(u) = \{D_{ij} u\} , \qquad i, j = 1, \ldots, n .$$

We can identify $H(u)$ with DDu since the vector $D_i Du$, $i = 1, \ldots, n$ is precisely the i-th row of the matrix $H(u)$.

If $A = \{A_{ij}\}$, $i, j = 1, \ldots, n$, is an $n \times n$ matrix of $N \times N$ matrices, $\{A_{ij}\} = \{A^{hk}_{ij}\}$, $h, k = 1, \ldots, N$ and if $\xi = \{\xi_{ij}\}$, $i, j = 1, \ldots, n$ is an $n \times n$ matrix of vectors $\xi_{ij} \in \boldsymbol{R}^N$, we set

$$(A|B) = \sum_{ij}^{n} A_{ij} \xi_{ij} \in \boldsymbol{R}^N . \tag{1.1}$$

If $A = \{A_{ij}\}$ and $B = \{B_{ij}\}$, $i, j = 1, \ldots, m$, are $m \times m$ matrices of elements in $\boldsymbol{R}$, then we set

$$(A|B) = \sum_{ij=1}^{m} A_{ij} B_{ij} \in \boldsymbol{R} . \tag{1.2}$$

We shall denote by I the $nN \times nN$ identity matrix. Let $a(x, \xi)\colon \Omega \times \boldsymbol{R}^{n^2N} \to \boldsymbol{R}^N$ be a mapping which is measurable in x and of class C^1 in ξ. Suppose that $a(x, 0) = 0$ and consider the quasi-basic differential operator

$$a(x, H(u)) . \tag{1.3}$$

We set

$$\frac{\partial a(x, \xi)}{\partial \xi_{ij}} = \left\{ \frac{\partial a^h(x, \xi)}{\partial \xi^k_{ij}} \right\} , \qquad h, k = 1, \ldots, N , \tag{1.4}$$

$$\frac{\partial a(x, \xi)}{\partial \xi} = \left\{ \frac{\partial a(x, \xi)}{\partial \xi_{ij}} \right\} , \qquad i, j = 1, \ldots, n \tag{1.5}$$

and suppose that the vector $\xi \to a(x, \xi)$ is elliptic; that is, there exist two constants M and ν, $M > \nu > 0$, such that, for almost all $x \in \Omega$, $\forall \xi \in R^{n^2N}$, $\forall \eta \in R^N$ and $\forall \lambda \in R^n$, we have

$$\left\| \frac{\partial a(x, \xi)}{\partial \xi} \right\| < M, \tag{1.6}$$

$$\sum_{i,j=1}^{n} \lambda_i \lambda_j \left(\frac{\partial a(x, \xi)}{\partial \xi_{ij}} \eta | \eta \right) > \nu |\lambda|^2 \|\eta\|^2 . \tag{1.7}$$

We do not know whether the hypothesis that $\xi \to a(x, \xi)$ is strictly monotone would be sufficient for what we shall say in the following.

We observe that it follows, from the hypotehsis (1.6), that $\forall \xi \in R^{n^2N}$ and for almost all $x \in \Omega$ we have

$$\|a(x, \xi)\|_N < \|\xi\| . \tag{1.8}$$

In particular, the differential operator (1.3) maps $H^2(\Omega) \to L^2(\Omega)$.

We consider the Hilbert space

$$H(\Omega) = H^2 \cap H^1_0(\Omega) \tag{1.9}$$

with the norm

$$\|u\|_{H(\Omega)} = \|H(u)\|_{L^2(\Omega)}$$

and we consider the Dirichlet problem

$$u \in H(\Omega), \qquad a(x, H(u)) = f \in L^2(\Omega) . \tag{1.10}$$

If the operator (1.3) is linear

$$a(x, H(u)) = \sum_{ij=1}^{n} A_{ij}(x) D_{ij} u = (A(x) | H(u)) \tag{1.11}$$

where $A_{ij}(x)$ are $N \times N$ matrices of class $L^\infty(\Omega)$ and $A(x) = \{A_{ij}(x)\}$, $i, j = 1, \ldots, n$, the *« condition of Cordes »* is well known for a long time:

$\exists K \in (0, 1)$ such that for almost all $x \in \Omega$ we have

$$\frac{|(A(x)|I)|^2}{\|A(x)\|^2} > nN - K^2 = \|I\|^2 - K^2 . \tag{1.12}$$

This condition assures that, $\forall f \in L^2(\Omega)$, the Dirichlet problem (1.10) has a unique solution (see [3], [4]). It is also known that, if the condition (1.12) is not satisfied then the Dirichlet problem (1.10) may not have a unique solution (see [3], Appendix).

In analogy with this, we can introduce a « *nonlinear condition of Cordes* » also for operators $a(x, H(u))$.

For each $\eta, \tau \in R^{n^2N}$ and for almost all $x \in \Omega$ we define two $N \times N$ and $nN \times nN$ matrices, respectively, as follows:

$$(1.13) \qquad \frac{\partial \bar{a}(x, \eta, \tau)}{\partial \xi_{ij}} = \left\{ \int_0^1 \frac{\partial a^h(x, \eta + t\tau)}{\partial \xi^k_{ij}} \, dt \right\}, \qquad h, k = 1, \ldots, N,$$

$$(1.14) \qquad \frac{\partial \bar{a}(x, \eta, \tau)}{\partial \xi} = \left\{ \frac{\partial \bar{a}(x, \eta \tau)}{\partial \xi_{ij}} \right\}, \qquad i, j = 1, \ldots, n$$

and we define the function $\beta(x, \eta, \tau)$, which is measurable in x and continuous in η, τ as follows

$$(1.15) \qquad \beta(x, \eta, \tau) = \frac{\left(\frac{\partial \bar{a}(x, \eta, \tau)}{\partial \xi} \middle| I \right)}{\left\| \frac{\partial \bar{a}(x, \eta, \tau)}{\partial \xi} \right\|^2}.$$

In view of the ellipticity condition the function β is bounded in $\Omega \times R^{n^2N} \times \times R^{n^2N}$ and strictly positive

$$(1.16) \qquad \frac{\nu n N}{M^2} < l < \beta < L < \frac{1}{\nu}.$$

We shall say that the operator (1.3) (as well as the vector $a(x, \xi)$) satisfies the *non linear condition of Cordes* if $\exists K$, $0 < K < \sqrt{l/L}$ such that, $\forall \eta, \tau \in R^{n^2N}$ and for almost all $x \in \Omega$ we have

$$(1.17) \qquad \beta(x, \eta, \tau) \left(\frac{\partial \bar{a}(x, \eta, \tau)}{\partial \xi} \middle| I \right) > nN - K^2.$$

We shall show that the pointwise condition of Cordes (1.17) assures that the operator $a(x, H(u)) : H(\Omega) \to L^2(\Omega)$ is *near* the linear operator Δu which is well known to be a bijection $H(\Omega) \to L^2(\Omega)$.

It now follows, by a general result on mappings between Hilbert spaces, that $a(x, H(u))$ is also a bijection $H(\Omega) \to L^2(\Omega)$.

We observe that, while for non variational linear systems proving the fact, that the condition of Cordes implies the nearness of these operators to Δu, is an interesting optional choice but not really necessary, for non variational non linear operators it is, however, an essential fact because we do not know any other way to prove the existence and uniqueness theorem in this case.

Under the hypothesis (1.17) it is also possible to prove that for the solution $u \in H^2(\Omega)$ of the quasi-basic system

$$(1.18) \qquad a(x, H(u)) = 0 \qquad \text{in } \Omega$$

the following *fundamental interior estimate* holds (see section 3) and more over, every solution $u \in H^2(\Omega)$ of the system (1.18) really belongs to $H^{2,q}_{loc}(\Omega)$ for q sufficiently near 2 (see section 5).

The situation is thus completely analogous to that one has for the solution $u \in H^1(\Omega)$ of a basic variational system

$$\operatorname{div} a(Du) = 0 \qquad \text{in } \Omega$$

where $a(p)$ is strictly monotone in p (see [4]).

From this point onwards one can proceed with the study of $\mathfrak{L}^{2,\lambda}$ regularity, or of the partial-$\mathfrak{L}^{2,\lambda}$ regularity, of the vector Du for solutions $u \in H^2(\Omega)$ of a complete non variational system

$$a(x, u, Du, H(u)) = b(x, u, Du) \tag{1.19}$$

exactly as was done for quasi linear systems (1.19) in [3].

2. - An existence and uniqueness theorem for quasi-basic systems

Suppose that the bounded open set Ω is convex and is of class C^2. Then, by a theorem of Miranda and Talenti ([6], [7]) we get the following

LEMMA 2.1: *For any vector $u \in H(\Omega)$ the following estimate holds*

$$\|H(u)\|_{L^2(\Omega)} \leq \|\Delta u\|_{L^2(\Omega)} . \tag{2.1}$$

We can easily prove the following

LEMMA 2.2: *If the vector $a(x, \xi)$ satisfies the condition* (1.17) *then* $\forall \eta, \tau \in R^{n^2N}$ *and for almost all $x \in \Omega$ we have the following pointwise estimate*

$$\Big\| \sum_i \tau_{ii} - \beta(x, \eta, \tau)[a(x, \tau + \eta) - a(x, \eta)] \Big\|_N \leq K\|\tau\| \tag{2.2}$$

where $K^2 < l/L$ (¹)

PROOF: Since

$$\sum_i \tau_{ii} = (I|\tau)$$

and

$$a(x, \tau + \eta) - a(x, \eta) = \left(\frac{\partial \bar{a}(x, \eta, \tau)}{\partial \xi} \middle| \tau \right)$$

(¹) L and l are defined in (1.16).

we obtain

$$(2.3)\qquad \Big\|\sum_i \tau_{ii} - \beta(x,\eta,\tau)[a(x,\tau+\eta) - a(x,\eta)]\Big\|_N =$$

$$= \left\|\left(I - \beta(x,\eta,\tau)\frac{\partial \bar{a}(x,\eta,\tau)}{\partial \xi}\middle|\tau\right)\right\|_N < \left\|I - \beta(x,\eta,\tau)\frac{\partial \bar{a}(x,\eta,\tau)}{\partial \xi}\right\| \cdot \|\tau\| .$$

On the other hand, it follows, from the definition (1.15) of the function $\beta(x, \eta, \tau)$ and from the condition (1.17), that

$$\left\|I - \beta(x,\eta,\tau)\frac{\partial \bar{a}(x,\eta,\tau)}{\partial \xi}\right\|^2 = \|I\|^2 - \beta(x,\eta,\tau)\left(\frac{\partial \bar{a}(x,\eta,\tau)}{\partial \xi}\middle| I\right) < K^2 .$$

The estimate (2.2) is thus proved.

We shall list a few facts concerning Hilbert spaces.

Let H_1 and H_2 be two real Hilbert spaces, eventually of finite dimensions. Let A and B be two mappings $H_1 \to H_2$.

DEFINITION 2.1: *We say that A is monotone with respect to B if there exist two positive constants $M^* > \nu^* > 0$ such that $\forall u, v \in H_1$ we have*

$$(2.4)\qquad \|A(u) - A(v)\|_{H_2} < M^* \|B(u) - B(v)\|_{H_2} ,$$

$$(2.5)\qquad (A(u) - A(v) | B(u) - B(v))_{H_2} > \nu^* \|B(u) - B(v)\|^2_{H_2} .$$

DEFINITION 2.2: *We say that A is near B if there exist two positive constants α and K with $0 < K < 1$, such that $\forall u, v \in H_1$ we have*

$$(2.6)\qquad \|B(u) - B(v) - \alpha[A(u) - A(v)]\|_{H_2} < K \|B(u) - B(v)\|_{H_2} .$$

It is easy to show the following proposition

(*a*) *A is monotone with respect to B if and only if A is near B.*

and the following *existence and uniqueness theorem* can also be proved easily

THEOREM 2.1: *If B: $H_1 \to H_2$ is bijective and A: $H_1 \to H_2$ is near B with constants α and K, then A is also a bijection; that is, $\forall f \in H_2$ $\exists_1 u \in H_1$ such that*

$$(2.7)\qquad A(u) = f$$

and we have the following estimate

$$(2.8)\qquad \|B(u) - B(0)\|_{H_2} < \frac{\alpha}{1-K} \|f - A(0)\|_{H_2} .$$

We present here a proof for the convenience of the reader.

PROOF: Since B is a bijection, we can define a metric on H_1 by

$$d(u, v) = \|B(u) - B(v)\|_{H_2} \tag{2.9}$$

for which $\{H_1, d\}$ becomes a complete metric space. More over, solving the equation (2.7) is equivalent to finding a $u \in H_1$ such that

$$B(u) = B(u) - \alpha A(u) + \alpha f = F(u) . \tag{2.10}$$

For all $u \in H_1$, $F(u) \in H_2$ and hence, since B is a bijection, $\exists_1 U = \mathfrak{T}(u) \in H_1$ such that

$$B(U) = F(u) .$$

Thus we define a mapping $\mathfrak{T}\colon H_1 \to H_1$ which is a contraction. Infact, if $U = \mathfrak{T}(u)$ and $V = \mathfrak{T}(v)$, then

$$B(U) - B(V) = B(u) - B(v) - \alpha[A(u) - A(v)]$$

and, by the assumption that A is near B, we have

$$\|B(U) - B(V)\|_{H_2} < K\|B(u) - B(v)\|_{H_2}$$

which means, by (2.9), that

$$d(U, V) < Kd(u, v) , \qquad \forall u, v \in H_1 .$$

Hence $\exists_1 u \in H_1$ which solves (2.10), and hence $\exists_1' u \in H_1$ ohich solves (2.7). Finally the estimate (2.8) follows from the assumption that A is near B.

We shall now prove the following lemma

LEMMA 2.3: *If the vector $a(x, \xi)$ is elliptic and satifies the condition* (1.17) *then the operator $a(x, H(u))\colon H(\Omega) \to L^2(\Omega)$ is near the operator $\Delta u\colon H(\Omega) \to L^2(\Omega)$; that is, there exist two constants α and K^* with $K^* \in (0, 1)$ such that $\forall u, v \in H(\Omega)$ we have*

$$\|\Delta(u-v) - \alpha[a(x, H(u)) - a(x, H(v))]\|_{L^2(\Omega)} < K^* \|\Delta(u-v)\|_{L^2(\Omega)} . \tag{2.11}$$

PROOF: In view of the Proposition (*a*) it is sufficient to show that the operator $a(x, H(u))$ is monotone with respect to the operator $\Delta(u)$ and this is an easy consequence of the pointwise Lemma 2.2 and of the lemma of Miranda and Talenti 2.1.

In fact, it follows from the estimate (2.2) that $\forall \tau, \eta \in R^{n^2N}$ and for almost all $x \in \Omega$ we have

$$\beta(x, \eta, \tau) \| a(x, \tau + \eta) - a(x, \eta) \|_N < K \|\tau\| + \| \sum_i \tau_{ii} \|_N ,$$

$$l \| a(x, \tau + \eta) - a(x, \eta) \|_N < \|\tau\| + \| \sum_{ii} \tau_{ii} \|_N .$$

Hence, by the Lemma 2.1, we have, $\forall u, v \in H$

$$\| a(x, H(u)) - a(x, H(v)) \|_{L^2(\Omega)} < \frac{(K+1)}{l} \| \Delta(u - v \|_{L^2(\Omega)} . \tag{2.12}$$

It follows from (2.2) that $\forall \tau, \eta \in \mathbf{R}^{n^2 N}$ and for almost all $x \in \Omega$

$$K^2 \|\tau\|^2 > \| \sum_i \tau_{ii} - \beta(x, \eta, \tau) [a(x, \tau + \eta) - a(x, \eta)] \|_N^2 >$$
$$> \| \sum_i \tau_{ii} \|_N^2 - 2\beta \left(a(x, \tau + \eta) - a(x, \eta) | \sum_{ii} \tau_{ii} \right)_N$$

from which we get

$$\left(a(x, \tau + \eta) - a(x, \eta) | \sum_i \tau_{ii} \right)_N > \frac{\| \sum_{ii} \tau_{ii} \|_N^2}{2L} - \frac{K^2}{2l} \|\tau\|^2 . \tag{2.13}$$

Now it follows, from (2.13) and once again usint the Lemma 2.1, that $\forall u, v \in H(\Omega)$

$$\left(a(x, H(u)) - a(x, H(v)) | \Delta(u - v) \right)_{L^2(\Omega)} > \frac{1}{2} \left(\frac{1}{L} - \frac{K^2}{l} \right) \| \Delta(u - v) \|_{L^2(\Omega)}^2 \tag{2.14}$$

and here $1/L - K^2/l > 0$ because of the bound (2.2) on K.

The estimates (2.12) and (2.14) prove the monotonicity, and hence the nearness, of the operator $a(x, H(u))$ with respect to the operator $\Delta(u)$.

In virtue of Theorem 2.1 we can conclude with the following *existence and uniqueness theorem.*

THEOREM 2.2: *If the vector* $a(x, \xi)$ *is elliptic and satisfies the condition* (1.17) *then* $\forall f \in L^2(\Omega)$ *the Dirichlet problem*

$$u \in H(\Omega) , \qquad a(x, H(u)) = f \quad \text{ in } \Omega \tag{2.15}$$

has a unique solution u *and the following estimate holds*

$$\| H(u) \|_{L^2(\Omega)} < \frac{L}{1 - K} \| f \|_{L^2(\Omega)} \tag{2.16}$$

L *and* K *being the constants which occur in* (1.16) *and* (1.17).

It is interesting to observe that the constants appearing in the estimate (2.16) dipend only on the constants K and L, or l and L, which occur in the condition (1.17).

3. - A fundamental interior estimate

Let Ω be a bounded open set in R^n and $a(x, \xi)$ be a vector $\Omega \times R^{n^2N} \to R^N$, which is measurable in x and of class C^1 in ξ. Suppose that $a(x, 0) = 0$ in Ω and suppose that the vector $a(x, \xi)$ is elliptic and satisfies the condition (1.17).

Define the function $\beta(x, \eta, \tau)$ as in (1.15) and set, for the sake of simplicity,

$$\beta(x, \tau) = \beta(x, 0, \tau)\,. \tag{3.1}$$

It follows from the pointwise estimate (2.2) that, $\forall \tau \in R^{n^2N}$ and for almost all $x \in \Omega$

$$\Big\|\sum_i \tau_{ii} - \beta(x, \tau) a(x, \tau)\Big\|_N < K\|\tau\|$$

with $K \in (0, 1)$. We denote by $B(\sigma) = B(x^0, \sigma)$ the ball of center x^0 and radius σ.

Let $u \in H^2(\Omega)$ be a solution of the *quasi-basic* system

$$a(x, H(u)) = 0 \qquad \text{in } \Omega\,. \tag{3.3}$$

We shall prove the following *fundamental interior estimate* for the vector $H(u)$.

Theorem 3.1: *If $u \in H^2(\Omega)$ is a solution of the quasi basic system* (3.3), *then there exists a constant $c(K) \in (0, 1)$ such that, $\forall B(\sigma) \subset\subset \Omega$ and $\forall t \in (0, 1)$, we have*

$$\int_{B(t\sigma)} \|H(u)\|^2\, dx < c t^n \int_{B(\sigma)} \|H(u)\|^2\, dx \tag{3.4}$$

where c does not depend on x^0, t and σ.

Proof: Let us fix $B(\sigma) \subset\subset \Omega$. In $B(\sigma)$ $u = v + w$ where w is the solution of the Dirichlet problem

$$w \in H(B(\sigma))\,, \qquad \Delta w = \Delta u - \beta(x, H(u)) a(x, H(u)) \qquad \text{in } B(\sigma)\ (^*) \tag{3.5}$$

while $v \in H^2(B(\sigma))$ is a solution of the basic linear system

$$\Delta v = 0 \qquad \text{in } B(\sigma)\,. \tag{3.6}$$

(*) We note that the right hand side of (3.5) is a vector of R^N of class $L^2(B(\sigma))$ and hence w exists and is unique.

It is well known that, $\forall t \in (0, 1)$, the following fundamental estimate holds for the vector v.

$$\int_{B(t\sigma)} \|H(v)\|^2 dx < t^n \int_{B(\sigma)} \|H(v)\|^2 dx \tag{3.7}$$

(see [1] and the (3.1) in [5]). In view of the pointwise estimate (3.2), we have the following estimate for the vector w:

$$\int_{B(\sigma)} \|H(w)\|^2 dx < \int_{B(\sigma)} \|\Delta u - \beta(x, H(u)) a(x, H(u))\|^2 dx < K^2 \int_{B(\sigma)} \|H(u)\|^2 dx . \tag{3.8}$$

Finally, since $u = v + w$ in $B(\sigma)$, it follows from (3.7) and (3.8) that $\forall t \in (0, 1)$

$$\Big(\int_{B(t\sigma)} \|H(u)\|^2 dx\Big)^{1/2} < ((1+K)t^{n/2} + K)\Big(\int_{B(\sigma)} \|H(u)\|^2 dx\Big)^{1/2} .$$

The claim (3.4) follows from this last estimate and the Lemma 1.V, p. 2 of [1].

REMARK: In contrast with the variational case, we have proved the fundamental estimate for the second derivatives for solutions of a quasi-basic system instead of for those of a basic system.

This is due to the fact that in the variational case we start with solutions in $H^1(\Omega)$ and hence, we should obtain the existence before getting the estimate (3.4). In the non variational case, we instead start with solutions which are already in $H^2(\Omega)$ and we have only to obtain the estimate (3.4).

By the known Poincaré type estimates, it follows from (3.4) that, if $u \in H^2(\Omega)$ is a solution in Ω of the quasi-basic system $a(x, H(u)) = 0$, then

$$Du \in \mathfrak{L}^{2,2+m}_{loc}(\Omega)$$

and

$$u \in \mathfrak{L}^{2,4+m}_{loc}(\Omega) .$$

Hence

(3.9) Du is Hölder continuous in Ω if $n = 2$

and

(3.10) u is Hölder continuous in Ω if $n < 4$.

This result is in accordance with the result that one has for the solutions $u \in H^1(\Omega)$ of a basic variational system.

4. - An $\mathcal{L}^{2,\lambda}$ regularity result

We conserve the hypothesis on Ω and on the vector $a(x, \xi)$ made in the previous section. We observe that if $(x, \xi) \to a(x, \xi)$ is elliptic and satisfies the assumption (1.17) then, for any fixed vector $\bar{\xi} \in R^{n^2N}$ also the vector $(x, \xi) \to a(x, \xi + \bar{\xi})$ is elliptic and satisfies the assumption (1.17) with the same constants l, L and K. It is enough to make the trivial observation that, if the conditions (1.16) and (1.17) hold for $\eta, \tau \in R^{n^2N}$, then they hold for all pairs $(\eta + \bar{\xi})$ and τ with $\eta, \bar{\xi}, \tau \in R^{n^2N}$.

Let $f\colon \Omega \to R^N$ be a vector of class $L^2(\Omega)$ and let $u \in H^2(\Omega)$ be a solution of the system

$$a(x, H(u)) = f \qquad \text{in } \Omega . \tag{4.1}$$

We prove the following lemma

Lemma 4.1: *For any ball $B(\sigma) = B(x^0, \sigma) \subset\subset \Omega$ and $\forall t \in (0, 1)$ we have the estimate*

$$\int_{B(t\sigma)} \|H(u)\|^2 dx < c_1 t^n \int_{B(\sigma)} \|H(u)\|^2 dx + c_2 \int_{B(\sigma)} \|f\|^2 dx \tag{4.2}$$

where the constants c_1 and c_2 do not depend on x^0, t and σ.

Proof: In $B(\sigma)$ we have $u = v - w$ where w is the solution of the Dirichlet problem

$$w \in H(B(\sigma)) , \qquad a(x, H(w) + H(u)) = a(x, H(u)) - f \quad \text{in } B(\sigma) \tag{4.3}$$

while $v \in H^2(B(\sigma))$ is a solution of the system

$$a(x, H(v)) = 0 \qquad \text{in } B(\sigma) . \tag{4.4}$$

We note that w exists and is unique in view of the remark made in the beginning of this section and of the fact that $a(x, H(u)) - f \in L^2(\Omega)$. We have the following estimate for w

$$\|H(w)\|_{L^2(B(\sigma))} < \frac{L}{1-K} \|f\|_{L^2(B(\sigma))} . \tag{4.5}$$

In fact, it follows, from the pointwise estimate (2.2), that we have

$$\|\Delta w\| < \|\Delta w - \beta(x, H(u), H(w)) [a(x, H(w) + H(u)) - a(x, H(u))]\| +$$
$$+ \|\beta(x, H(u), H(w)) [a(x, H(w) + H(u)) - a(x, H(u))]\| < K\|H(w)\| + L\|f\|$$

and hence (4.5) follows in virtue of Lemma 2.1. We have the following fundamental estimate (3.4) for the vector v

$$(4.6) \qquad \|H(v)\|_{L^2(B(t\sigma))} \le c t^{\varepsilon n/2} \|H(v)\|_{L^2(B(\sigma))} \qquad \forall t \in (0,1)\,.$$

Since $u = v - w$ in $B(\sigma)$, the estimate (4.2) follows from (4.6) and (4.5).
The following $\mathfrak{L}^{2,\lambda}$ regularity theorem follows from the Lemma 4.1:

Theorem 4.1: *If $f \in \mathfrak{L}^{2,\lambda}(\Omega)$, $0<\lambda<\varepsilon n$ and $u \in H^2(\Omega)$ is a solution of the system*

$$(4.7) \qquad a(x, H(u)) = f \qquad \text{in } \Omega$$

then

$$(4.8) \qquad H(u) \in \mathfrak{L}^{2,\lambda}_{\mathrm{loc}}(\Omega)$$

and for every open subset $\Omega^* \subset\subset \Omega$ we have the estimate

$$(4.9) \qquad \|H(u)\|_{\mathfrak{L}^{2,\lambda}(\Omega^*)} \le c\{\|H(u)\|_{L^2(\Omega)} + \|f\|_{\mathfrak{L}^{2,\lambda}(\Omega)}\}$$

where c depends also on the distance of Ω^* from Ω.

Proof: It follows, from (4.2) and the hypothesis that $f \in \mathfrak{L}^{2,\lambda}(\Omega)$, that $\forall B(\sigma) \subset\subset \Omega$ and $\forall t \in (0,1)$ we have

$$\int_{B(t\sigma)} \|H(u)\|^2 dx \le c_1 t^{\varepsilon n} \int_{B(\sigma)} \|H(u)\|^2 dx + c_2 \sigma^\lambda \|f\|^2_{\mathfrak{L}^{2,\lambda}(\Omega)}\,.$$

From this the assertion follows in view of the Lemma 1.I, p. 7 of [1].
We remark that, from (4.8), we have the following corollary

$$(4.10) \qquad Du \in \mathfrak{L}^{2,2+\lambda}_{\mathrm{loc}}(\Omega) \quad \text{and} \quad u \in \mathfrak{L}^{2,4+\lambda}_{\mathrm{loc}}(\Omega)$$

and hence we again have

Du is Hölder continuous in Ω if $n = 2$,
u is Hölder continuous in Ω if $n < 4$.

5. - A $H^{2,q}_{\mathrm{loc}}$ regularity result with q near 2

Let $u \in H^2(\Omega)$ be a solution of the system

$$(5.1) \qquad a(x, H(u)) = 0 \qquad \text{in } \Omega\,.$$

We denote by $A_{ij}(x, \tau)$ and by $A(x, \tau)$ respectively the $N\times$ and $nN \times nNN$

matrices

$$A_{ij}(x,\tau)=\frac{\partial \hat{a}(x,0,\tau)}{\partial \xi_{ij}} \text{ and } A(x,\tau)=\{A_{ij}(x,\tau)\}\,,\ i,j=1,\dots,n\,.$$

The system (5.1) can also be written in the form

$$(5.2)\qquad a(x,H(u))=\sum_{ij}A_{ij}(x,H(u))\,D_{ij}u=\big(A(x,H(u))|H(u)\big)=0 \quad \text{in } \Omega$$

and hence becomes a linear system with coefficients $A_{i,j}\in L^{\infty}(\Omega)$, which is elliptic and satisfies the condition of Cordes (1.17) with $\eta=0$. Many results of local L^q and $\mathfrak{C}^{2,\lambda}$ regularity for solutions of the system (5.1) can be deduced from the theory of linear systems with bounded and measurable coefficients, which is well known in the literature.

We shall prove, as an example, the $H^{2,q}(\Omega)$ regularity for q near 2.

Analogous results for quasi-linear systems were obtained in section 3 of [3].

Let us fix a ball $B(2\sigma)\subset\subset\Omega$; let $\theta(x)$ be a $C_0^{\infty}(R)^n$ function with the following properties

$$0\le\theta\le1\,,\quad \theta=1 \quad \text{in } B(\sigma)\,,\qquad \theta=0 \quad \text{in } R^n\setminus B(2\sigma)\,,$$

$$|D^{\alpha}\theta|\le c\sigma^{-|\alpha|} \qquad \text{for each multi index } \alpha\,.$$

Let $P=(P_1,\dots,P_n)$ the polynomial vector of degree ≤1 such that

$$\int_{B(2\sigma)}D^{\alpha}(u-P)\,dx=0\,,\qquad \forall\alpha: |\alpha|\le1\,.$$

Let $U=\theta(u-P)$. $U\in H_0^2(B(2\sigma))$ and is a solution of the system

$$(5.3)\qquad \sum_{ij}A_{ij}(x,H(u))\,D_{ij}U=$$
$$=\sum_{ij}[A_{ij}(x,H(u))+A_{ji}(x,H(u))]\,D_i\theta D_j(u-P)+$$
$$+\sum_{ij}A_{ij}(x,H(u))\,D_{ij}\theta(u-P)=F(u-P)\,.$$

Hence we have the estimate (see (2.16) of Theorem 2.2)

$$(5.4)\qquad \int_{B(2\sigma)}\|H(U)\|^2dx\le c(L,K)\int_{B(2\sigma)}\|F(u-P)\|^2dx\,.$$

On the other hand, by the well known Poincaré inequality we have

$$(5.5)\qquad \int_{B(2\sigma)}\|F(u-P)\|^2dx\le c(M,n)\sigma^{-2}\int_{B(2\sigma)}\|Du-(Du)_{2\sigma}\|^2dx$$

where we have set, as usual, $w_{2\sigma}=\fint_{B(2\sigma)}w\,dx$.

Therefore, if $u \in H^2(\Omega)$ is a solution of the system (5.1) then $\forall B(2\sigma) \subset\subset \Omega$ we have the Caccioppoli type estimate

$$\int_{B(\sigma)} \|H(u)\|^2 dx \leq c(L, M, K, n)\left(\int_{B(2\sigma)} \|H(u)\|^{2n/(n+2)} dx\right)^{(n+2)/n}.$$

From this and by the lemma of Gehring - Giaquinta - G. Modica (see, for instance, Lemma 10.I, p. 100 of [1]) it follows that $\exists q > 2$ such that $H(u) \in \in L^q_{loc}(\Omega)$ and

$$\left(\fint_{B(\sigma)} \|H(u)\|^q dx\right)^{1/q} \leq C\left(\fint_{B(2\sigma)} \|H(u)\|^2 dx\right)^{1/2} \tag{5.6}$$

where c does not depend on σ and x_0.

Remark: The fundamental estimate (3.4) can also be obtained in this manner.

In fact if $u \in H^2(\Omega)$ is a solution of the system (5.1) then (5.6) holds and hence $\forall B(\sigma) \subset\subset \Omega$ and $t \in (0, \frac{1}{2})$ we have

$$\int_{B(t\sigma)} \|H(u)\|^2 dx \leq c(n)\left(\fint_{B(\sigma/2)} \|H(u)\|^q dx\right)^{2/q} t^{n(1-2/q)}\sigma^n \leq c t^{n(1-2/q)} \int_{B(\sigma)} \|H(u)\|^2 dx. \tag{5.7}$$

Finally the estimate (5.7) is trivially true for $\frac{1}{2} < t < 1$.

Thus there are two ways of expressing the exponent εn which appears in (3.4) and there are two ways to justify the result (3.9)-(3.10).

As already remarked in the introduction, one can proceed with the study of $\mathfrak{C}^{0,\lambda}$ regularity or of the partial-$\mathfrak{C}^{0,\lambda}$-regularity, of the vector Du for the solution $u \in H^2(\Omega)$ of a complete non-variational system

$$a(x, u, Du, H(u)) = G(x, u, Du) \tag{5.8}$$

exactly in the same way as was done in [3] for the system (5.8) of quasi-linear type.

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