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# Green's Function and Potential Theory for the Schrödinger Operator: a Nonprobabilistic Approach (**)

## Funzione di Green e teoria del potenziale per l'operatore di Schrödinger

SOMMARIO. — Si considera l'operatore di Schrödinger stazionario $Lu = Au + Vu$, con parte principale $A$ in forma di divergenza e potenziale $V$ in una classe di Kato, su domini lipschitziani. Con tecniche analitiche (non probabilistiche) viene ottenuta, sotto un'opportuna condizione integrale su $V$, la stima per la funzione di Green per $L$:

$$c_1 \cdot G_A(x, y) \leq G_L(x, y) \leq c_2 \cdot G_A(x, y) .$$

Si stabilisce poi un'analoga stima per la misura $L$-armonica, e se ne deducono risultati di teoria del potenziale per $L$: regolarità dei punti alla frontiera, comportamento alla frontiera per soluzioni positive, in particolare esistenza di limiti non tangenziali (quasi ovunque rispetto alla misura $L$-armonica).

### INTRODUCTION

In this paper we consider the Schrödinger operator (in steady state)

$$Lu = Au + Vu = -(a_{ij}u_{x_i})_{x_j} + V \cdot u$$

where $A$ is a uniformly elliptic operator in devergence form, with bounded measurable coefficients, $V$ is assumed in the Kato class $K(\Omega)$, and $L$ is defined on a bounded Lipschitz domain $\Omega$. (Precise definitions will be given later). This operator, or its particular case $-\Delta + V$, has been studied in recent years, especially with probabilistic methods: Aizenman-Simon [1] proved a Harnack's inequality and some subsolution estimates for $-\Delta + V$, using the Feynman-Kac formalism; Simon [9] developed a theory for the same operator from the point of view of semigroups of operators; Zhao [12] proved

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a fundamental estimate for the Green's function of $-\Delta + V$ on $C^{1,1}$ domains. More recently, Cranston-Fabes-Zhao [4], using the results of Caffarelli-Fabes-Mortola-Salsa [2], extended many of these results to the operator $A + V$ on Lipschitz domains, and developed a potential theory for it. On the other hand, Chiarenza-Fabes-Garofalo [3], with a non probabilistic approach, proved for $L$ a Harnack's inequality and the continuity of solutions of $Lu = 0$. Following this line, our aim is to develop, by typical methods of P.D.E., a theory for the operator $L$.

Since, in the usual variational theory, $V$ is assumed in $L^p(\Omega)$ with $p > n/2$, a smaller class than $K(\Omega)$, we shall at first prove—section 1—that Dirichlet's problem for $L$ can still be formulated, and is well posed, if $V$ is, in a proper sense, rather small.

Then—section 2—we shall investigate properties of the Green's function for $L$; particularly, we will show (Th. 2.6) that it is controlled from above and from below by the Green's function for $A$. A «weak theory» for the operator $L$ will be also developed, extending to $L$ some results of [6] about $A$.

In sections 3-4 we shall obtain some potential theoretical results, i.e. properties of solutions of $Lu = 0$. First we shall state—section 3— solvability of Dirichlet's problem for $L$ when the datum is a continuous function defined only on the boundary; from this fact will follow existence of the $L$-harmonic measure. Then we shall obtain an estimate about harmonic measures (Th. 3.8), analogue to the one involving the Green's functions for $L$ and $A$. From this fact a comparison principle for positive solutions of $Lu = 0$, vanishing on a part of the boundary, will follow (section 4). Then we shall state regularity of boundary points for $L$. Finally a «Fatou's theorem» will be obtained, i.e. existence of nontangential boundary limits (a.e. with respect to the $L$-harmonic measure) for positive solutions of $Lu = 0$.

Note that the two facts we have borrowed from [4] (Theorems 1.4 and 4.5) have analytical proofs. So our work relies on no probabilistic argument.

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## 1. - Some definitions and known results. Dirichelet's problem for $L$.

Let $\Omega$ be a bounded Lipschitz domain of $R^n$ ($n \geq 3$). This means that there exists a pair of positive numbers $r_0$ and $M$ such that for every $\zeta \in \partial\Omega$, local coordinates can be selected so that $B(\zeta, r_0) \cap \partial\Omega$ is the graph of a Lipschitz function $\varphi$ with $|D\varphi| \leq M$. The constants $r_0$ and $M$ determine what will be called the Lipschitz character of $\Omega$.

The operator $A$ is supposed to have bounded, measurable, real valued coefficients $a_{ij}$. We also suppose $a_{ij} = a_{ji}$ and $A$ uniformly elliptic. So there is a positive constant $\lambda$ such that

$$\lambda^{-1}|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \lambda|\xi|^2 \qquad \text{for every } \xi \in R^n\text{, for a.e. } x \in \Omega\,.$$

Let us now define the Kato class $K(\Omega)$.

$$K(\Omega)=\left\{f\in \mathfrak{L}^1_{loc}(\Omega):\lim_{r\downarrow 0}\sup_{x\in\mathbb{R}^n}\int_{\Omega\cap B(x,r)}\frac{|f(y)|}{|x-y|^{n-2}}\,dy=0\right\}.$$

If $V\in K(\Omega)$, put

$$\eta(r)=\sup_{x\in\mathbb{R}^n}\int_{\Omega\cap B(x,r)}\frac{|V(y)|}{|x-y|^{n-2}}\,dy\,.$$

Sometimes we shall call $\eta$ «the Kato norm of $V$». Now, all the informations we need about $L$ are contained in $\lambda$ and $\eta$.

Note that if $V\in\mathfrak{L}^p(\Omega)$ with $p>n/2$, then by Hölder's inequality, $V\in K(\Omega)$ and $\eta(r)<c\cdot\|V\|_p\cdot r^\alpha$ with $c$, $\alpha$ only depending on $n$ and $p$. So our assumptions generalize the case which is studied in standard variational approach.

If $V\in K(\Omega)$, it is easy to prove the following properties:

(i) $V\in\mathfrak{L}^1(\Omega)$;

(ii) $\eta(r)$ is finite for every $r$, monotone non decreasing;

(iii) $\|V\|_1<d^{n-2}\cdot\eta(d)$ where $d=\operatorname{diam}\Omega$;

(iv) $\sup_{\Omega}\int_{\Omega}\frac{|V(y)|}{|x-y|^{n-2}}\,dy<\eta(d)$;

(v) $\eta$ is bounded and definitively constant, $\eta(r)<\eta(2d)$ for every $r$;

(vi) if $f(x)=\int_{\Omega}\frac{|V(y)|}{|x-y|^{n-2}}\,dy$, $f$ is continuous in $\Omega$.

Note that, if $f\in\mathfrak{L}^1_{loc}(\Omega)$ and $\eta(r)<\infty$ for some $r$, then properties (i)-(v) hold, but $f$ must not necessarily belong to $K(\Omega)$. (A counterexample is given in [1]). So the crucial property in defining $K(\Omega)$ is that $\eta(r)\to 0$.

A fundamental result, due to Schechter (See [8], p. 138) is the following:

Theorem 1.1: If $V\in K(\Omega)$, there exists a constant $k=k(n)$ and, for every $\delta>0$, a constant $c_\delta=c(\delta,n)$, such that for all $\varphi\in W_0^{1,2}(\Omega)$:

$$\int_{\Omega}|V|\varphi^2<k\cdot\eta(\delta)\|\varphi\|^2_{W^{1,2}}+c_\delta\cdot\eta(1)\cdot\|\varphi\|^2_{\mathfrak{L}^2}\,.$$

Remark 1.2: From Theorem 1.1 it follows that, since $\Omega$ is Lipschitz, the bilinear form associated to $L$ is well defined and continuous on $W^{1,2}(\Omega)$, and it is coercive on $W_0^{1,2}(\Omega)$ provided that a condition

$$c_1(n,\lambda)\cdot\eta(2d)<1 \tag{1.1}$$

holds, with $d=\operatorname{diam}\Omega$.

Let us recall also two basic estimates regarding the Green's function $G$ for $A$.

THEOREM 1.3 (see [6]):

$$G(x,y) < \frac{c_2}{|x-y|^{n-2}} \quad \text{for all } x, y \in \Omega, \text{ for some constant } c_2(n, \lambda). \tag{1.2}$$

THEOREM 1.4 (see [4]): There exists a constant $c_3$ depending on $\lambda$, $n$ and the Lipschitz character of $\Omega$ such that:

$$\frac{G(x,y)G(y,z)}{G(x,z)} < c_3 \cdot \left\{ \frac{1}{|x-y|^{n-2}} + \frac{1}{|y-z|^{n-2}} \right\} \quad \text{for all } x, y, z \in \Omega. \tag{1.3}$$

Now, put $c = \max(c_1, c_2, 2c_3)$ where $c_1, c_2, c_3$ are as in (1.1), (1.2), (1.3). Note that $c = c(\lambda, n, r_0, M)$. Put:

$$\delta = c \cdot \eta(2d). \tag{1.4}$$

Henceforth we shall suppose the Kato norm of $V$ so small to have:

$$\delta < \tfrac{1}{2}. \tag{1.5}$$

Then (1.1) holds, and Lax-Milgram's lemma implies:

THEOREM 1.5: The problem

$$\begin{cases} Lu = T & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

for $T \in W^{-1,2}$ and $g \in W^{1,2}$ assigned, is well posed. The constants in the continuous dependence estimate depend on $n$, $\lambda$, $\eta$, $r_0$, $M$.

(If $g = 0$, the constant only depends on $n$, $\lambda$, $\eta$.)

Let us state also a maximum principle for $L$. From coerciveness of the bilinear form associated to $L$ it follows:

THEOREM 1.6: If $u \in W^{1,2}(\Omega)$ is a supersolution for $L$ and $u \geq 0$ on $\partial\Omega$ (in sense $W^{1,2}$) then $u \geq 0$ a.e. in $\Omega$.

## 2. - GREEN'S FUNCTION FOR $L$

In the following, we shall indicate with $G$ and $G_L$ the Green's functions for $A$ and $L$, respectively. Existence of $G_L$ will follow from theorem 2.3. First, we state a lemma regarding $G$.

LEMMA 2.1. Let $f\in \mathfrak{L}^1(\Omega)$ such that $\sup_x\int_\Omega G(x,y)|f(y)|\,dy<c<\infty$ for some constant $c$. Then there exists a unique $u\in W_0^{1,2}(\Omega)$ satisfying $Au=f$ in $\Omega$, in the sense that

$$\int_\Omega a_{ij}u_{x_i}\varphi_{x_j}\,dx=\int_\Omega f\varphi\,dx \qquad \text{for all } \varphi\in C_0^\infty(\Omega)\,.$$

Moreover $u(x)=\int_\Omega G(x,y)f(y)\,dy$.

PROOF: Put

$$f_n(x)=\begin{cases} n & \text{if } f(x)>n\,,\\ f(x) & \text{if } |f(x)|<n\,,\\ -n & \text{if } f(x)<-n\,,\end{cases}$$

and let $u_n$ be the solution of

$$\begin{cases} Au_n=f_n & \text{in } \Omega\,,\\ u_n\in W_0^{1,2}(\Omega)\,.\end{cases}$$

Then

$$u_n(x)=\int_\Omega G(x,y)f_n(y)\,dy \quad \text{and} \quad \|u_n\|_\infty<c\,.$$

From the equation $Au_n=f_n$ it follows:

$$\int_\Omega |Du_n|^2<\lambda\int_\Omega a_{ij}(u_n)_{x_i}(u_n)_{x_j}\,dx=\lambda\int_\Omega u_nf_n<\lambda\|f_n\|_1\|u_n\|_\infty<\lambda c\|f\|_1\,.$$

Therefore $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$ and there exists a subsequence $u_n$ converging to some $u$ weakly in $W_0^{1,2}$ and strongly in $\mathfrak{L}^2$. So, taking limits in:

$$\int a_{ij}(u_n)_{x_i}\varphi_{x_j}\,dx=\int f_n\varphi$$

for a fixed $\varphi\in C_0^\infty(\Omega)$, we have:

$$\int a_{ij}u_{x_i}\varphi_{x_j}\,dx=\int f\varphi\,.$$

On the other hand, choosing a subsequence $f_n\to f$ a.e., since

$$G(x,\cdot)|f_n|<G(x,\cdot)|f|\in\mathfrak{L}^1\,,$$

by Lebesgue's theorem one has:

$$u_n(x) \to \int_\Omega G(x,y) f(y)\, dy$$

that is

$$u(x) = \int_\Omega G(x,y) f(y)\, dy . \qquad //$$

REMARK 2.2: When $f = V \in K(\Omega)$, by (1.3) and definition of $K(\Omega)$ we have:

$$\int_\Omega G(x,y)\,|V(y)|\,dy < c_1 \int_\Omega \frac{|V(y)|}{|x-y|^{n-2}}\,dy < c_1 \eta(d) < \delta , \tag{2.1}$$

and the previous lemma holds. Moreover, by Theorem 1.1, the functional $\varphi \to \int V\varphi$ is continuous on $W_0^{1,2}$. So we have:

$$\int_\Omega a_{ij} u_{x_i} \varphi_{x_j} = \int_\Omega V\varphi \qquad \text{for all } \varphi \in W_0^{1,2} .$$

Next theorem is taken from [3].

THEOREM 2.3: If $u \in W_0^{1,2}$ is the solution of $Lu = f$ with $f \in \mathfrak{L}^p(\Omega)$, $p > n/2$, then:

$$\|u\|_\infty < c \|f\|_p \qquad \text{with } c = \frac{c(n, \lambda, p, |\Omega|)}{1-\delta} .$$

REMARK 2.4: As a consequence of Theorem 2.3, there exists the Green's function $G_L$ for $L$, i.e. for every $f \in \mathfrak{L}^p$ $(p > n/2)$, the solution of:

$$\left|\begin{array}{l} Lu = f , \\ u \in W_0^{1,2} , \end{array}\right.$$

is given by:

$$u(x) = \int_\Omega G_L(x,y) f(y)\, dy \tag{2.2}$$

with:

$$\sup_x \|G_L(x,\cdot)\|_q < c(n, \lambda, q, |\Omega|, \delta) \qquad \text{for all } q < \frac{n}{n-2} . \tag{2.3}$$

Moreover $G_L$ is symmetric and, by the maximum principle (Theorem 1.6), nonnegative.

The maximum principle also implies the following:

COROLLARY 2.5: Let $V_1, V_2 \in K(\Omega)$ (both $V_i$ satisfying our assumption $\delta < \frac{1}{2}$) and let $G_{L_1}$, $G_{L_2}$ be the Green's functions for $A + V_1$, $A + V_2$, respectively. If $V_1 < V_2$, $G_{L_1} > G_{L_2}$.

Now we can state our basic result:

THEOREM 2.6 (Comparison between $G$ and $G_L$):

$$\left(\frac{1-2\delta}{1-\delta}\right)\cdot G(x,y) < G_L(x,y) < \frac{1}{1-\delta}\cdot G(x,y) \qquad \text{for a.e. } x, y \in \Omega . \tag{2.4}$$

PROOF: Using the representation formula (2.2) one can find the following identity:

$$G_L(x,y) = G(x,y) - \int_\Omega G_L(x,w)V(w)G(w,y)\,dw \qquad \text{for a.e. } x, y \in \Omega .$$

Now, let us consider the space $B$ defined by:

$$B = \left\{ f\colon \Omega\times\Omega \to \mathbf{R},\ f \text{ measurable such that } \|f\|_B = \sup_x \int_\Omega |f(x,y)|\,dy < +\infty \right\}.$$

$B$ is a Banach space. If we define the operator $T$ as:

$$Tf(x,y) = \int_\Omega f(w,y)V(w)G(x,w)\,dw$$

it is easy to verify that $T$ is a well defined, linear continuous operator from $B$ to $B$, with $\|T\|_{\mathfrak{L}(B)} < \delta$. Let us consider the integral equation:

$$f + Tf = G \tag{2.6}$$

where the unknown function $f$ is sought in $B$. Then $(I + T)$ can be inverted by Neumann series:

$$(I + T)^{-1} = \sum_0^\infty (-)^n T^n \tag{2.7}$$

(where the series converges in $\mathfrak{L}(B)$). Since, by (2.5), the solution of (2.6) is $G_L$, (2.7) gives:

$$G_L = \sum_0^\infty (-)^n T^n G \tag{2.8}$$

(where the series converges in $B$). Using Theorem 1.4 we have:

$$|TG(x,y)| = \left|\int_\Omega G(w,y)\,V(w)\,G(x,w)\,dw\right| <$$

$$< G(x,y)\cdot\int_\Omega \frac{G(x,w)\,G(w,y)}{G(x,y)}\,|V(w)|\,dw < G(x,y)\cdot c_2\cdot$$

$$\cdot\left\{\int_\Omega \frac{|V(w)|}{|y-w|^{n-2}}\,dw + \int_\Omega \frac{|V(w)|}{|x-w|^{n-2}}\,dw\right\} < G(x,y)\cdot c_2\cdot 2\eta(d) < \delta\cdot G(x,y)\,.$$

By iteration:

$$|T^n G(x,y)| < \delta^n\cdot G(x,y)\,. \tag{2.9}$$

Since convergence in $B$ implies convergence a.e. of a subsequence, it follows from (2.8) and (2.9) that:

$$G_L(x,y) < \frac{1}{1-\delta}\cdot G(x,y) \qquad \text{for a.e. } x, y \in \Omega\,, \tag{2.10}$$

and so we have the right hand inequality in (2.4).

On the other hand, again from (2.5) we have:

$$G_L(x,y) = G(x,y) - \int_\Omega G_L(w,y)V(w)G(x,w)\,dw =$$

$$= G(x,y)\cdot\left\{1 - \int_\Omega \frac{G_L(w,y)G(x,w)}{G(x,y)}\,V(w)\,dw\right\} > \qquad \text{(by (2.10))}$$

$$G(x,y)\cdot\left\{1 - \frac{1}{1-\delta}\int_\Omega \frac{G(w,y)G(x,w)}{G(x,y)}\,|V(w)|\,dw\right\} > \qquad \text{(by Theorem 1.4)}$$

$$> G(x,y)\cdot\left\{1 - \frac{1}{1-\delta}\cdot c_2\cdot 2\eta(d)\right\} > \frac{1-2\delta}{1-\delta}\cdot G(x,y)$$

and the proof is complete. //

Let us see some consequences of Theorem 2.6. Combining this fact with results in [6], we have:

THEOREM 2.7: Let $\Sigma$ be a simply connected, bounded Lipschitz domain which can be mapped smoothly onto a sphere, and let $G_L$, $g$ be the Green's functions for $L$ and $-\Delta$, respectively, in $\Sigma$. Then, for any compact subset $C$ of $\Sigma$, there exists a constant $k$ only depending on $C$, $\Sigma$ and $\delta$, such that:

$$k^{-1}\cdot g(x,y) < G_L(x,y) < k\cdot g(x,y) \qquad \text{for a.e. } x, y \in C\,.$$

COROLLARY 2.8:

$$G_L(x, y) < \frac{c(\delta)}{|x-y|^{n-2}} \quad \text{for a.e. } x, y \in \Omega . \tag{2.11}$$

REMARK 2.9: We can obtain from (2.11) an integrability property of $G_L$ (and $G$). Let us recall that $f \in \mathfrak{L}^{p,\infty}(\Omega)$ if, by definition, $f \in \mathfrak{L}^1_{loc}(\Omega)$ and:

$$\|f\|_{p,\infty} = \sup_K \frac{\int_K |f(x)|\,dx}{|K|^{1/q}} < +\infty$$

where the sup is taken among all compact subsets of $\Omega$ and $p^{-1} + q^{-1} = 1$. It is clear that $\mathfrak{L}^p(\Omega)$ is continuously embedded in $\mathfrak{L}^{p,\infty}(\Omega)$. We already know that $G_L(x, \cdot) \in \mathfrak{L}^q$ for $q < n/(n-2)$. Since it is known that the function $f_x(y) = |x-y|^{2-n}$ belongs to $\mathfrak{L}^{n/(n-2),\infty}$, uniformly in $x$, by (2.11) the same is true for $G_L$:

$$\sup_x \|G_L(x, \cdot)\|_{n/(n-2),\infty} < c(\delta) .$$

REMARK 2.10: Let $V = V^+ - V^-$, let $\eta$ be the Kato norm of $V^-$, $\eta$ satisfying our assumptions (1.4)-(1.5), and $V^+ \in K(\Omega)$. Then the bilinear form associated to $L$ is still coercive, and there exists the Green's function $G_L$. By Corollary 2.5, $G_L < G_{A-V^-}$ while $G_{A-V^-}$ clearly satisfies Theorem 2.7. So it is still true that

$$G_L(x, y) < \frac{1}{1-\delta} \cdot G(x, y) .$$

Green's function can be seen also as the solution of $LG_L(x, \cdot) = \delta_x$ (where $\delta_x$ is the Dirac mass concentrated in $x$) in a weak sense, by developing a «weak theory» for the equation $Lu = \mu$ (where $\mu$ is a measure). This can be done, following [6], by transferring to $L$ some properties which are known to hold for $A$.

THEOREM 2.11: If $u$ is the solution of

$$\begin{cases} Lu = (f_i)_{x_i} & \text{in } \Omega , \\ u = b & \text{on } \partial\Omega , \end{cases} \tag{2.12}$$

with $f_i \in \mathfrak{L}^p(\Omega)$, $b \in W^{1,p}(\Omega)$ and $p > n$, then $u$ is continuous on $\bar{\Omega}$.

THEOREM 2.12. Under the same assumptions of the previous theorem, if $b = 0$ then

$$\max_{\Omega} |u(x)| < c(n, \lambda, p, |\Omega|, \delta) \cdot \|f_i\|_p .$$

PROOF: Theorems 2.11-2.12 hold for $A$; these results are due to Stampacchia (see [6] for references). Since $u$ is bounded, $Vu \in K(\Omega)$. So, by Lemma 2.1 and Remark 2.2, $u$ can be seen as the sum of a function satisfying (2.12) with $L$ substituted by $A$, and a function expressed by:

$$\tilde{u}(x) = -\int_\Omega G(x,y)\, V(y)\, u(y)\, dy .$$

Now, $\tilde{u}$ is continuous, by definition of Kato class and (1.2), so Theorem 2.11 is proved. Moreover:

$$\|u\|_\infty < c \cdot \|f_1\|_p + \delta \cdot \|u\|_\infty$$

and Theorem 2.12 follows. //

DEFINITION 2.13: For a measure $\mu$ of bounded variation on $\Omega$, we say that $u \in \mathfrak{L}^1(\Omega)$ is a *weak solution of the equation $Lu = \mu$ vanishing at the boundary* $\partial\Omega$ if it satisfies

$$\int_\Omega u \cdot L\varphi \, dx = \int_\Omega \varphi \, d\mu$$

for every $\varphi \in W_0^{1,2}(\Omega) \cap C(\bar{\Omega})$ such that $L\varphi \in C(\bar{\Omega})$.

Once one knows the results of Theorems 2.11-2.12, the arguments contained in sections 5-6 of [6] can be repeated. We summarize the main results in the following:

THEOREM 2.14: For any measure $\mu$ of bounded variation, a unique solution $u$ of $Lu = \mu$ vanishing at $\partial\Omega$ exists, and lies in $W_0^{1,p'}(\Omega)$ for every $p' < n/(n-1)$; moreover $u$ satisfies

$$\|u\|_{W_0^{1,p'}} < c(n, \lambda, p', |\Omega|, \delta) \int_\Omega |d\mu|$$

and $u$ is assigned by the integral (a.e. converging)

$$u(x) = \int_\Omega G_L(x,y)\, d\mu(y) .$$

Finally, $G_L(x, \cdot)$ is the weak solution vanishing at $\partial\Omega$ of $Lu = \delta_x$.

REMARK 2.15: If $u$ is the weak solution of $Lu = \mu$ ($Au = \mu$) vanishing at $\partial\Omega$, then $Vu \in \mathfrak{L}^1(\Omega)$ and

$$\|Vu\|_1 < c \int_\Omega |d\mu|$$

with $\varepsilon = \delta/(1-\delta)$ $(\varepsilon = \delta)$. In fact we have:

$$\int |Vu| < \int |V(x)|\,dx \int G_L(x,y)|d\mu(y)| = \int |d\mu(y)| \int G_L(x,y)|V(x)|\,dx < \frac{\delta}{1-\delta}\int |d\mu|$$

where we have used (2.1) and (2.4).

Let us point out also the following regularity properties of $G_L$:

THEOREM 2.16:

(i) $G_L(x,\cdot) \in W_0^{1,p'}(\Omega)$ for every $p' < n/(n-1)$;

(ii) $G_L(x,\cdot) \in W_{\mathrm{loc}}^{1,2}(\Omega - \{x\})$ and is a local solution of $Lu = 0$ in $\Omega - \{x\}$;

(iii) $G_L(x,\cdot) \in C(\Omega - \{x\})$.

PROOF: (i) follows from 2.14, while (ii)-(iii) can be stated as in [6], using two results about $L$ (a «Caccioppoli's inequality» and the continuity of solutions of $Lu = 0$) contained in [3]. //

## 3. - DIRICHLET'S PROBLEM WITH CONTINUOUS BOUNDARY DATA. $L$-HARMONIC MEASURE

In order to define the concept of $L$-harmonic measure and develop a potential theory for $L$, we need a sharper version of maximum principle.

THEOREM 3.1: If $u$ is a supersolution (subsolution) for $L$ in $\Omega$, then, respectively:

$$\text{(3.1)}\qquad \begin{aligned} &a)\ \min_{\Omega} u > \frac{1}{1-\delta}\cdot \min_{\partial\Omega} u_- ,\\ &b)\ \max_{\Omega} u < \frac{1}{1-\delta}\cdot \max_{\partial\Omega} u^+ . \end{aligned}$$

If $u$ is a solution of $Lu = 0$ in $\Omega$, then

$$\text{(3.2)}\qquad \max_{\Omega} |u| < \frac{1}{1-\delta}\cdot \max_{\partial\Omega} |u| .$$

PROOF: Let $u > k$ on $\partial\Omega$ for some $k < 0$. Then

$$\begin{aligned} L(u-k) &= Lu - Vk > -Vk &&\text{in } \Omega ,\\ (u-k) &> 0 &&\text{on } \partial\Omega . \end{aligned}$$

Let $w$ be the solution of

$$\begin{cases} Lw = -Vk & \text{in } \Omega\,, \\ w = 0 & \text{on } \partial\Omega\,. \end{cases}$$

Then $w(x) = -k\int_\Omega G_k(x, y)V(y)\,dy$ and, by (2.1) and (2.4).

$$|w(x)| < -\frac{k}{1-\delta}\int_\Omega G(x, y)|V(y)|\,dy < -k\cdot\frac{\delta}{1-\delta}\,.$$

Put $v = (u-k) - w$. By the maximum principle (Th. 1.6), $v > 0$ in $\Omega$ and so

$$u(x) > k - |w(x)| > \frac{k}{1-\delta}\,. \tag{3.3}$$

Now, $\min_{\partial\Omega} u = \sup\{k\colon u > k \text{ on } \partial\Omega\}$. If $\min_{\partial\Omega} u > 0$, (3.1.$a$) holds by Theorem 1.6 while if $\min u < 0$, (3.1.$a$) follows from (3.3).

Changing $u$ in $-u$ it follows (3.1.$b$), and combining (3.1.$a$) with (3.1.$b$) one has (3.2). //

Remark 3.2: The constant $\delta$ in (3.1)-(3.2) actually depends only on $V^-$, so that when $V > 0$ one has:

$$\max_\Omega |u| < \max_{\partial\Omega} |u|\,. \tag{3.4}$$

In the general case one cannot expect (3.4) to be true. To see this, it is sufficient to consider $\Omega$ the unit ball, $V$ a small negative constant, $A = -\Delta$: then the solution $u$ with boundary value 1 is a positive function assuming a strong maximum at the origin.

Now, using Theorem 2.1 and Caccioppoli's inequality of [3], one can repeat an argument used in [6] and give sense to Dirichelet's problem for $L$ when the datum is a continuous function defined on $\partial\Omega$. Namely, the following holds:

Theorem 3.3: There exists a mapping $B$ which to any continuous function $f$ defined on $\partial\Omega$ associates a local solution of $Lu = 0$ (which is continuous by [3]) such that whenever $f$ is the trace of a $C^1(\bar\Omega)$ function, $Bf$ coincides with the variational solution of

$$\begin{cases} Lu = 0 & \text{in } \Omega\,, \\ u = f & \text{on } \partial\Omega\,. \end{cases}$$

Moreover, if $u = Bf$, one has:

$$\sup_{K \subset\subset \Omega} [\operatorname{dist}(K, \partial\Omega) \cdot \|Du\|_{C^0(K)}] + \max_{\Omega} |u| < +\infty ;$$

$$\max_{\Omega} |u| < \frac{1}{1-\delta} \cdot \max_{\partial\Omega} |f| .$$

This theorem makes it possible to give the following:

Definition 3.4: A point $y \in \partial\Omega$ is said to be *regular* for $L$ iff for every $h \in C(\partial\Omega)$ one has:

$$\lim_{\substack{x \to y \\ x \in \Omega}} Bh(x) = h(y) .$$

Definition 3.5: For a fixed $x \in \Omega$, let us consider the functional $f \to Bf(x)$ defined on $C(\partial\Omega)$. By Riesz' theorem there exists a positive regular Borel measure $\nu_L^x$ representing it:

$$Bf(x) = \int_{\partial\Omega} f(y)\, d\nu_L^x(y) .$$

We call $\nu_L^x$ the $L$-harmonic measure evaluated at $x$. By the Harnack principle of [3] it follows that, given $x_1, x_2 \in \Omega$, there exists a constant $c$ such that:

$$\nu_L^{x_1}(E) < c \cdot \nu_L^{x_2}(E) \qquad \text{for any Borel set } E \subset \partial\Omega .$$

We shall indicate with $\nu_A^x$ the Borel measure obtained with the same construction for the operator $A$ ($A$-harmonic measure evaluated at $x$).

The main result we shall prove in this section is the following: there exist constants $c_1$, $c_2$ such that for any Borel set $E$ and $x \in \Omega$:

$$c_1 \nu_A^x(E) < \nu_L^x(E) < c_2 \nu_A^x(E) . \tag{3.5}$$

From this fact and the results contained in [2] it will follow potential theory for $L$. To obtain (3.5) two facts are used: the estimate (2.4) involving the Green's functions for $A$ and $L$; the notion of kernel function for $A$ and some relative results contained in [2].

Definition 3.6: Fix $x \in \Omega$, $z \in \partial\Omega$. A function $K_A^x(w, z)$ defined in $\Omega$ is called a kernel function at $z$ for the operator $A$, normalized at $x$, if:

(i) $K_A^x(\cdot, z)$ is a solution of $Au = 0$ in $\Omega$;

(ii) $K_A^x(\cdot, z) \in C(\bar{\Omega} - \{z\})$ and $\lim_{\substack{w \to z' \neq z \\ z' \in \partial\Omega}} K_A^x(w, z) = 0$;

(iii) $K_A^x(w, z) > 0$ for each $w \in \Omega$ and $K_A^x(x, z) = 1$.

For $x$ and $z$ fixed, there exists one and only one kernel function $K_A^x(\cdot, z)$ and it is:

$$K_A^x(w, z) = \frac{d\mu_A^w}{d\mu_A^x}(z) \tag{3.6}$$

(Radon-Nikodym derivative of the $A$-harmonic measures).

It can be proved also that, for any $w \in \Omega$, $z \in \partial\Omega$, there exists

$$\lim_{\substack{y \to z \in \partial\Omega \\ y \in \Omega}} \frac{G(w, y)}{G(x, y)} = K_A^x(w, z). \tag{3.7}$$

THEOREM 3.7: For $x \in \Omega$, $z \in \partial\Omega$, there exists

$$F(x, z) = \lim_{\substack{y \to z \\ y \in \Omega}} \frac{G_L(x, y)}{G(x, y)}. \tag{3.8}$$

Moreover $F$ is continuous on $\Omega \times \partial\Omega$, and:

$$F(x, z) = 1 - \int_\Omega K_A^x(w, z) V(w) G_L(x, w)\, dw. \tag{3.9}$$

PROOF: Let us consider (2.5), written as:

$$\frac{G_L(x, y)}{G(x, y)} = 1 - \int_\Omega \frac{G(w, y)}{G(x, y)} \cdot G_L(x, w) V(w)\, dw. \tag{3.10}$$

By (3.7), what we have to prove is that in (3.10) the limit can be taken under the integral sign. Put:

$$f_r(y) = \int_{|w-z|>r} G_L(x, w) \cdot \frac{G(w, y)}{G(x, y)} \cdot V(w)\, dw.$$

By Lebesgue's theorem, Theorems 2.4 and 2.6, one has:

$$\lim_{y \to z} f_r(y) = \int_{|w-z|>r} K_A^x(w, z) G_L(x, w)\, V(w)\, dw$$

and, by definition of Kato class,

$$\lim_{r \to 0} \lim_{y \to z} f_r(y) = \int_\Omega K_A^x(w, z) G_L(x, w) V(w)\, dw \qquad \text{uniformly in } z.$$

On the other hand, for $r \to 0$, $f_r(y)$ converges uniformly to:

$$\int_{\Omega} \frac{G(w, y)}{G(x, y)} \cdot G_L(x, w) V(w)\, dw \,.$$

So, exchanging the limits it follows (3.9) and the continuity of $F$. //

THEOREM 3.8 (Comparison between harmonic measures): For each $x \in \Omega$, $\zeta \in \partial\Omega$, one has:

$$d\omega_L^x(\zeta) = F(x, \zeta)\, d\omega_A^x(\zeta) \,. \tag{3.11}$$

Moreover, there exist constants $c_1$, $c_2$ depending on $\delta$ such that:

$$c_1 \cdot \omega_A^x(E) < \omega_L^x(E) < c_2 \cdot \omega_A^x(E) \qquad \text{for every Borel set } E\,,\ x \in \Omega \,. \tag{3.12}$$

PROOF: Let $f \in C(\partial\Omega)$. By (3.9):

$$\int_{\partial\Omega} f(\zeta) F(x, \zeta)\, d\omega_A^x(\zeta) = \int_{\partial\Omega} f(\zeta)\, d\omega_A^x(\zeta) - \int_{\partial\Omega} f(\zeta)\, d\omega_A^x(\zeta) \int_{\Omega} K_A^x(w, \zeta) V(w) G_L(x, w)\, dw =$$

$$\text{(by Tonelli and (3.6))} = \int_{\partial\Omega} f(\zeta)\, d\omega_A^x(\zeta) - \int_{\Omega} G_L(x, w) V(w)\, dw \int_{\partial\Omega} f(\zeta)\, d\omega_A^w(\zeta) \,.$$

Now, let $v$, $u$ be the solutions of

$$\begin{cases} Lv = 0 & \text{in } \Omega\,, \\ v = f & \text{on } \partial\Omega\,, \end{cases} \qquad \begin{cases} Au = 0 & \text{in } \Omega\,, \\ u = f & \text{on } \partial\Omega\,. \end{cases}$$

Then the previous identity becomes:

$$\int_{\partial\Omega} f(\zeta) F(x, \zeta)\, d\omega_A^x(\zeta) = u(x) - \int_{\Omega} G_L(x, w) V(w) u(w)\, dw = v(x) = \int_{\partial\Omega} f(\zeta)\, d\omega_L^x(\zeta) \,.$$

Since this is true for every $f \in C(\partial\Omega)$, it follows (3.11). Now, by Theorem 3.7, (2.4) implies:

$$\frac{1 - 2\delta}{1 - \delta} < F(x, \zeta) < \frac{1}{1 - \delta} \qquad \text{for all } x \in \Omega\,,\ \zeta \in \partial\Omega \,.$$

So (3.12) follows from (3.11). //

#### 4. - POTENTIAL THEORY FOR $L$

THEOREM 4.1 (Boundary Harnack principle): Let $\zeta_0 \in \partial\Omega$, $r > 0$, $x_r \in \Omega$ such that $|x_r - \zeta_0| = r$ and $\operatorname{dist}(x_r, \partial\Omega) \simeq r$. If $v$ is a positive solution of

$Lv = 0$ in $\Omega$, vanishing continuously on $\partial\Omega \cap B(\zeta_0, 2r)$, then

$$\sup_{B(\zeta_0, r)} v < c \cdot v(x_r)$$

for some constant $c$ depending on $n$, $\lambda$, $\delta$ and the Lipschitz character of $\Omega$.

PROOF: Let $\Omega' = \Omega \cap B(\zeta_0, 2r)$. Then $v \in C(\bar{\Omega}')$. Let us call $\omega_L^x$, $\omega_\Delta^x$ the harmonic measures for $\Omega'$. Then, by the boundary Harnack principle for $\Delta$ (see [2]) and (3.12), one has, for $x \in B(\zeta_0, r)$:

$$v(x) = \int_{\partial\Omega'} v(\zeta)\, d\omega_L^x(\zeta) < c_2 \int_{\partial\Omega'} v(\zeta)\, d\omega_\Delta^x(\zeta) < \text{const} \int_{\partial\Omega'} v(\zeta)\, d\omega_\Delta^{x_r}(\zeta) <$$

$$< \text{const} \int_{\partial\Omega'} v(\zeta)\, d\omega_L^{x_r}(\zeta) = c \cdot v(x_r). \qquad //$$

In the same way the following can be obtained:

THEOREM 4.2 (Comparison principle): Let $u$, $v$ be positive solutions of $Lw = 0$ in $\Omega$, vanishing continuously on $\partial\Omega \cap B(\zeta, 2r)$. Then

$$\sup_{B(\zeta, r)} \frac{u}{v} < c \cdot \frac{u}{v}(x_r).$$

THEOREM 4.3 (Comparison between solutions of $L$ and $\Delta$): Let $u$, $v$ be positive solutions of $Lv = 0$, $\Delta u = 0$ in $\Omega$, vanishing continuously on $\partial\Omega \cap$ $\cap B(\zeta, 2r)$. Then

$$\sup_{B(\zeta, r)} \frac{u}{v} < c \cdot \frac{u}{v}(x_r).$$

Once one knows these results, one can repeat the arguments contained in section 3 of [2]: the kernel function for the operator $L$ can be defined, and its existence and uniqueness can be proved. Moreover, from the formula

$$K_L^x(w, \zeta) = \frac{d\omega_L^w}{d\omega_L^x}(\zeta)$$

it follows, by (3.11), that

$$K_L^x(w, \zeta) = \frac{F(w, \zeta)\, d\omega_\Delta^w(\zeta)}{F(x, \zeta)\, d\omega_\Delta^x(\zeta)} = \frac{F(w, \zeta) K_\Delta^x(w, \zeta)}{F(x, \zeta)}.$$

Hence we have that $K_L^x(w, \cdot) \in C(\partial\Omega)$ and:

$$c_1 K_\Delta^x(w, \zeta) < K_L^x(w, \zeta) < c_2 K_\Delta^x(w, \zeta)$$

for all $x, w \in \Omega$, $\zeta \in \partial\Omega$, for some constants $c_1$, $c_2$ depending on $\delta$.

Now we are interested in stating regularity of boundary points for $L$. From this fact and the previous theorems, by the same arguments of [2], some results about boundary behavior of nonnegative solutions will follow. First, we want to point out a consequence of comparison theorem.

THEOREM 4.4: Let $u$, $v$ be positive solutions of $Lu = 0$ in $\Omega$, vanishing continuously on $\partial\Omega \cap B(z_0, 2r_0)$ (for a fixed $z_0 \in \partial\Omega$, $r_0 > 0$). Then the quotient $u/v$ can be extended as a Holder continuous function on $\bar{\Omega} \cap B(z_0, 2r_0)$. (Note that, by [3], the solutions $u$, $v$ are in general continuous but not Holder).

PROOF: For every $r > 0$, put $M_r = \sup_{B_r} u/v$, $m_r = \inf_{B_r} u/v$. Since $v$ and $M_{2r} \cdot v - u$ are positive solutions in $B_{2r}$, by comparison theorem we have:

$$\sup_{B_r}\left(M_{2r} - \frac{u}{v}\right) < c \cdot \inf_{B_r}\left(M_{2r} - \frac{u}{v}\right)$$

that is

(4.1) $$M_{2r} - m_r < c \cdot (M_{2r} - M_r)\,.$$

Considering now $u - m_{2r} v$, with the same reasoning one obtains

(4.2) $$M_r - m_{2r} < c \cdot (m_r - m_{2r})$$

and from (4.1), (4.2), with a standard technique (see [7]) Holder continutiy of $u/v$ follows. //

The following lemma is taken from [4].

LEMMA 4.5: For any $z \in \bar{\Omega}$, $w \in \Omega$, $w \neq z$

(4.3) $$\lim_{\substack{x,y \to z \\ x,y \in \Omega}} \frac{G(x, w)\, G(w, y)}{G(x, y)} = 0\,.$$

Also, if $z, z' \in \partial\Omega$, then

(4.4) $$\lim_{x \to z} \frac{G(x, w)\, G(w, y)}{G(x, y)} = K(z, w, z')$$

exists and is a continuous function of $(z, z')$ on $\partial\Omega \times \partial\Omega$.

Let us now consider $F(x, y) = G_L(x, y)/G(x, y)$. We have already proved that $F$ can be extended continuously on $\Omega \times \bar{\Omega}$. We now state the following:

LEMMA 4.6: $F$ can be extended continuously on $\bar{\Omega} \times \bar{\Omega}$. For $z, z' \in \partial\Omega$, it is

(4.5) $$F(z, z') = 1 - \int_{\Omega} F(w, z')\, K(z, w, z')\, V(w)\, dw\,.$$

(Note that, since $K(z, w, z) = 0$, $F(z, z) = 1$.)

PROOF: By (3.9):

$$\lim_{x\to z'} F(x,z) = 1 - \lim_{x\to z'} \int_\Omega \lim_{y\to z} \frac{G(w,y)}{G(x,y)} \cdot G_L(x,w) V(w)\, dw .$$

Note that

$$\lim_{\substack{y\to z\\ x\to z'}} \frac{G(w,y)}{G(x,y)} \cdot G_L(x,w) = \lim_{\substack{y\to z\\ x\to z'}} \frac{G_L(x,w)}{G(x,w)} \cdot \frac{G(x,w)G(w,y)}{G(x,y)} =$$

$$= F(w,z')K(z,w,z') \qquad \text{by (3.8) and (4.4)}.$$

Also, by (1.3), we have that $\int_\Omega F(w,z')K(z,w,z')V(w)\,dw$ converges. Put:

$$f_r(x) = \int_{|w-z|>r} K_A^x(w,z) G_L(x,w) V(w)\, dw .$$

By Lebesgue's theorem, (1.2), (1.3) we see that

$$\lim_{x\to z'} f_r(x) = \int_{|w-z'|>r} F(w,z')K(z,w,z')V(w)\, dw$$

and the right hand side is a continuous function of $z'$, uniformly converging, when $r\to 0$, to

$$\int_\Omega F(w,z')K(z,w,z')V(w)\, dw .$$

So this is a continuous function. Furthermore, when $r\to 0$:

$$f_r(x) \to \int_\Omega K_A^x(w,z) G_L(x,w) V(w)\, dw$$

uniformly in $x$ (again by (1.2), (1.3) and definition of Kato class). So (4.5) holds. //

THEOREM 4.7 (Regularity of boundary points): For every $f\in C(\partial\Omega)$, $z_0\in\partial\Omega$, when $x\to z_0$ $(x\in\Omega)$

$$\int_{\partial\Omega} f(z)\, d\omega_L^x(z) \to f(z_0) .$$

PROOF: Let $\{x_n\}$ be a sequence in $\Omega$ converging to $z_0$. $g_n(z) = f(z)\cdot F(x_n, z)$. Then $g_n \in C(\partial\Omega)$ and $\|g_n\|_\infty < c\cdot\|f\|_\infty$. By (3.11):

$$\int_{\partial\Omega} f(z)\, d\omega_L^{x_n}(z) = \int_{\partial\Omega} g_n(z)\, d\omega_A^{x_n}(z) =$$

$$= \int_{\partial\Omega} [g_n - f\cdot F(z_0,\cdot)](z)\, d\omega_A^{x_n}(z) + \int_{\partial\Omega} f(z) F(z_0,z)\, d\omega_A^{x_n}(z) .$$

The first term tends to zero by uniform continuity of $F(\cdot,\cdot)$ on $\partial\Omega\times\partial\Omega$, while the second term, by regularity of boundary points for $A$ (see [6]) converges to $f(\zeta_0)F(\zeta_0,\zeta_0)=f(\zeta_0)$ (by (4.5)). So we are done. //

From the facts we have stated up to this point, the arguments contained in section 4 of [2] can be repeated for the operator $L$. Let $\Sigma$ be the unit ball in $R^n$, $K_L(\cdot,\cdot)$ the kernel function for $L$ in $\Sigma$ evaluated at the origin. Then the following hold:

THEOREM 4.8: Let $u$ be a nonnegative solution of $Lu=0$ in $\Sigma$. Then there exists a finite Borel measure $\nu$ on $\partial\Sigma$ such that:

$$u(x)=\int_{\partial\Sigma}K_L(x,\zeta)\,d\nu(\zeta)\,. \tag{4.6}$$

THEOREM 4.9 (Existence of nontangential limits): Let $u$ be a nonnegative solution of $Lu=0$ in $\Sigma$. Then almost everywhere on $\partial\Sigma$ with respect to the $L$-harmonic measure $w_L=w_L^0$, the nontangential limit of $u$ exists.

If $\nu$ is as in (4.6), let us consider the Lebesgue decomposition of $\nu$ with respect to $w_L$:

$$d\nu=d\nu_s+f\,dw_L\,.$$

Then the limit is given by $f$. Moreover, if $f$ is bounded, the following representation holds:

$$u(x)=\int_{\partial\Sigma}K_L(x,\zeta)f(\zeta)\,dw_L(\zeta)=\int_{\partial\Sigma}f(\zeta)\,dw_L^x(\zeta)\,.$$

As in [2], Theorems 4.8-4.9 still hold when $\Sigma$ is a bounded Lipschitz star-shaped domain.

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