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WOJCIECH ZYGMUNT (*)

On the Scorza-Dragoni's Type Property of the Real Function Semicontinuous to the Second Variable (**)

ABSTRACT. — The equivalence of four properties which can have a real function $f(t, x)$ measurable in t and semicontinuous in x is proved. They are Scorza-Dragoni's type property, Baire's type property on an approximation of a semicontinuous function and properties Op and Cl defined in terms of multifunctions.

Sulla proprietà di Scorza-Dragoni nel caso delle funzioni reali semicontinue rispetto alla seconda variabile

RIASSUNTO. — In questa Nota, per le funzioni reali $f(t, x)$ misurabili rispetto a t e semicontinue rispetto a x si dimostra l'equivalenza di quattro proprietà: della proprietà di Scorza-Dragoni; di quella di Baire, sull'approssimazione di una funzione semicontinua; nonchè delle qui chiamate proprietà Op e Cl, espresse in termini di multifunzioni.

1. - INTRODUCTION

As the classical theorem of Scorza-Dragoni has show ([SD]), a function $f\colon [0, 1]\times R \to R$ satisfying the so-called Caratheodory's conditions, i.e. f is measurable in t for each x and continuous in x for each t, has the following property: « for any $\varepsilon > 0$ there exists a closed set $T_\varepsilon \subset [0, 1]$ whose measure is greater then $1 - \varepsilon$ such that $f|_{T_\varepsilon \times R}$ is continuous with respect to both variables ». However if the continuity of function $f(t, \cdot)$ is substituted by semicontinuity, then the function f may not have the semicontinuous restriction of Scorza-Dragoni's type (cf. [B, Ex. 2,7]; for it is well know that the function f which is only semicontinuous with respect to every variable separately, may be very irregular, cf. [Sie, p. 65], [J-K, Sect. 4]). In the case when such

(*) Indirizzo dell'Autore: Instytut Matematyki, Uniwersytet Marii Curie-Sklodowskiej, pl. Marii Curie-Sklodowskiej 1, 20-031 Lublin (Polonia).
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restriction is possible we shall say that f has property SD_* (for lowersemicontinuity) or property SD^* (for uppersemicontinuity).

Now let A be a subset of $\mathbf{R}$ and let us consider the mapping F_A from $[0, 1]$ to the family of all subsets of $\mathbf{R}$ defined by $F_A(t) = \{x \in \mathbf{R}: f(t, x) \in A\}$, where f is described above. If f is Caratheodory's function, the mapping F_A is weakly measurable for each open or closed set A (see [H, Theorem 6.2 and 6.4]). For a function f which is only semicontinuous in x, F_A may not be weakly measurable; if F_A is weakly measurable we say that f has property Op (for A open) or property Cl (for A closed).

At last, the well known Baire's theorem asserts that every real semicontinuous function defined on metric space is a limit of a monotonic sequence of continuous functions. The question arises, if $f: [0, 1] \times \mathbf{R} \to \mathbf{R}$ measurable in t and semicontinuous in x is a limit of a monotonic sequence of Caratheodory's functions $f_n: [0, 1] \times \mathbf{R} \to \mathbf{R}$. If so, then we say that f has property B_* (in the case of a nondecreasing sequence) or property B^* (a nonincreasing sequence).

In this paper we show that the previously mentioned properties for function $f: T \times X \to \mathbf{R}$ are equivalent under suitable assumptions about T and X. Section 4 is concerned with the proof of this fact. In section 2 we give the necessary definitions and terminology and state one theorem important for our considerations. Section 3 contains a few lemmas which prepare the ground for the proof of our main thesis.

2. - Definitions and preliminaries

Let Z be a metric space. A function $f: Z \to \mathbf{R}$ is called lowersemicontinuous (lsc) (uppersemicontinuous—usc), if the set $\{x \in Z: a < f(x)\}$ ($\{x \in Z: f(x) < a\}$) is open for each $a \in \mathbf{R}$; it is equivalent to the condition that the set $\{x \in Z: f(x) < a\}$ ($\{x \in Z: a < f(x)\}$) is closed. The following theorem holds:

Theorem (Baire, [L], [Ho]): Every lsc (usc) function $f: Z \to \mathbf{R}$ is a limit of a nondecreasing (nonincreasing) sequence of continuous functions $f_n: Z \to \mathbf{R}$.

Let Y be a topological space. We denote by $\mathcal{P}(Y)$ the family of all subsets of Y including the empty set. Let S be an arbitrary set and μ a measure defined on σ-field Σ of subsets of S. By multifunction we mean a mapping from S into $\mathcal{P}(Y)$. The set $\{t \in S: F(t) \neq \emptyset\}$ is called the domain of $F: S \to \mathcal{P}(Y)$. A multifunction $F: S \to \mathcal{P}(Y)$ is weakly μ-measurable (in abbreviation—w. μ-meas) if the set $F^-(B) = \{t \in S: F(t) \cap B \neq \emptyset\} \in \Sigma$ for each open subset $B \subset Y$. It is easy to verify that

(i) $F: S \to \mathcal{P}(Y)$ is w. μ-meas iff (= if and only if) the multifunction $\overline{F}: S \to \mathcal{P}(Y)$, defined by $\overline{F}(t) = \overline{F(t)}$, is w. μ-meas,

(ii) if $F: S \to \mathcal{P}(Y)$ is w. μ-meas then domain F is a μ-measurable set.

A function $f\colon S\times X\to \mathbf{R}$, where X is a topological space, will be called C (resp. C_*, C^*) type, if $f(t,\cdot)$ is continuous (lsc, usc) for each $t\in S$ and $f(\cdot,x)$ is μ-measurable for each $x\in X$.

Now we assume that S is a compact Hausdorff topological space and a Borel, σ-finite, regular and complete measure. We say that $f\colon S\times X\to \mathbf{R}$ has

1) property $SD(SD_*, SD^*)$, if for every $\varepsilon>0$ there exists a closed subset S_ε of S, with $\mu(S\setminus S_\varepsilon)<\varepsilon$, such that $f|_{S_\varepsilon\times X}$ is continuous (lsc, usc),

2) property Op, if a multifunction $F_A\colon S\to \mathfrak{s}(X)$, defined by

$$F_A(t)=\{x\in X\colon f(t,x)\in A\},$$

is w. μ-meas for each open subset A of $\mathbf{R}$,

3) property Cl, if a multifunction $F_A\colon S\to \mathfrak{s}(X)$, defined as above, is w. μ-meas for each closed subset A of $\mathbf{R}$,

4) property B_* (B^*), if there exists a nondecreasing (nonincreasing) sequence of C-type functions $f_n\colon S\times X\to \mathbf{R}$, which converges to f.

Recently, in [R-V], the classical result of Scorza-Dragoni has been extended. We give it in the form appropriate for our considerations.

Theorem (Scorza-Dragoni): Let S and μ satisfy the same assumptions as before and let X be a separable metric space. Then every C-type function $f\colon S\times X\to \mathbf{R}$ has property SD.

Lemma 1: Let s be the homeomorphic transformation of $\mathbf{R}$ onto $I=(0,1)$ which preserves the order. Then

(i) $f\colon S\times X\to \mathbf{R}$ is C-, (C_*)-type function iff the composition $g=s\circ f\colon S\times X\to I$ is C, (C_*)-type function,

(ii) $f\colon S\times X\to \mathbf{R}$ has property Op iff the composition $g=s\circ f\colon S\times X\to I$ has property Op.

Proof: Follows from the equalities given below (see e.g. [Sik, p. 58]):

(i) $\{t\in S\colon f(t,x)>a\}=\{t\in S\colon s(f(t,x))>s(a)\}$,

$\{x\in X\colon f(t,x)>a\}=\{x\in X\colon s(f(t,x))>s(a)\}$,

$$\{t\in S\colon s(f(t,x))>a\}=\begin{cases}\emptyset, & \text{for } a>1,\\ \{t\in S\colon f(t,x)>s^{-1}(a)\}, & \text{for } 0<a<1,\\ S, & \text{for } a<0,\end{cases}$$

$$\{x\in X\colon s(f(t,x))>a\}=\begin{cases}\emptyset, & \text{for } a>1,\\ \{x\in X\colon f(t,x)>s^{-1}(a)\}, & \text{for } 0<a<1,\\ X, & \text{for } a<0,\end{cases}$$

(ii) $F_A(t) = \{x \in X: f(t,x) \in A\} = \{x \in X: s(f(t,x)) \in s(A)\} = G_{s(A)}(t)$,

$$G_A(t) = \{x \in X: s(f(t,x)) \in A\} = \begin{cases} \emptyset, & \text{for } A \cap I = \emptyset, \\ \{x \in X: f(t,x) \in s^{-1}(A \cap I)\} = \\ \qquad = F_{A \cap I}(t), & \text{for } A \cap I \neq \emptyset. \end{cases}$$

THEOREM 1: Let X be a Polish space (i.e. separable, complete, metric space) and let $f: S \times X \to \mathrm{R}$ be C_*-type function. If f has property B_*, then f has property Op also.

PROOF: We have

$$f_1(t,x) < \ldots < f_n(t,x) < f_{n+1}(t,x) < \ldots < f(t,x) = \lim_{n\to\infty} f_n(t,x),$$

where $f_n: X \times X \to \mathrm{R}$ is C-type function, $n = 1, 2, 3, \ldots$.

First, we prove that a multifunction $F_{(\alpha,\beta]}$ is w. μ-meas for every interval $(\alpha, \beta] \subset \mathrm{R}$. Let us fix $(\alpha, \beta]$ and consider multifunctions $F_n: S \to \mathfrak{F}(X)$, defined by

$$F_n(t) = \left\{x \in X: f_n(t,x) \in \left[\alpha + \frac{1}{n+j}, \beta\right]\right\},$$

where $j \in \mathrm{N}$ is such that $\alpha + 1/j < \beta$. By [H, Theorems 6.4 and 3.5] the multifunctions F_n are w. μ-meas. Since F_n has closed values, then in view of [H, Theorem 3.5] for every $n \in \mathrm{N}$ the multifunctions $\bigcap_{k=0}^{\infty} F_{n+k}: S \to \mathfrak{F}(X)$ are w. μ-meas and hence the multifunction $\bigcup_{n=1}^{\infty}\left(\bigcap_{k=0}^{\infty} F_{n+k}\right): S \to \mathfrak{F}(X)$ is w. μ-meas.

(The union and intersection of multifunctions are defined in the usual way.)

We show that

$$F_{(\alpha,\beta]} = \bigcup_{n=1}^{\infty}\left(\bigcap_{k=0}^{\infty} F_{n+k}\right).$$

Indeed, we have

$$x \in \bigcup_{n=1}^{\infty}\left(\bigcap_{k=0}^{\infty} F_{n+k}(t)\right) \Leftrightarrow x \in \bigcap_{k=0}^{\infty} F_{n_0+k}(t) \Leftrightarrow [x \in F_n(t) \text{ for } n > n_0] \Rightarrow$$

$$\Rightarrow \left[\alpha + \frac{1}{n} < \alpha + \frac{1}{n_0} < f_{n_0}(t,x) < f_n(t,x) < \beta \text{ for } n > n_0\right] \Rightarrow$$

$$\Rightarrow \left[\alpha < \alpha + \frac{1}{n_0} < f(t,x) < \beta\right] \Leftrightarrow x \in F_{(\alpha,\beta]}(t).$$

On the other hand, if $x \in F_{(\alpha,\beta]}(t)$, then $\alpha < f(t,x) = \alpha + \delta < \beta$, where $\delta > 0$. There exists n_0 such that $n_0 > 2/\delta$ and $f(t,x) - f_n(t,x) < \delta/2$ for $n > n_0$. Then, for $n > n_0$,

$$\alpha + \frac{1}{n} < \alpha + \frac{\delta}{2} = f(t,x) - \frac{\delta}{2} < f_n(t,x) < f(t,x) < \beta$$

and we have

$$x \in \bigcap_{k=0}^{\infty} F_{n_0+k}(t); \quad \text{hence} \quad x \in \bigcup_{n=1}^{\infty} \left(\bigcap_{k=0}^{\infty} F_{n+k}(t) \right).$$

Thus $F_{(\alpha,\beta]} = \bigcup_{n=1}^{\infty} \left(\bigcap_{k=0}^{\infty} F_{n+k} \right)$ and therefore $F_{(\alpha,\beta]}$ is w. μ-meas.

Since

$$(\alpha, \beta) = \bigcup_{n=1}^{\infty} \left(\alpha, \beta - \frac{1}{n+j} \right],$$

where $j \in \mathrm{N}$ is such that $\alpha < \beta - 1/j$, and since the multifunctions $F_{(\alpha,\beta-1/(n+j)]}$ are w. μ-meas, we conclude that the multifunctions $F_{(\alpha,\beta)} = \bigcup_{n=1}^{\infty} F_{(\alpha,\beta-1/(n+j)]}$ is w. μ-meas.

Now, if A is an open set of R, then $A = \bigcup_{n=1}^{\infty} (\alpha_n, \beta_n)$, where α_n, β_n are some rational numbers. Hence $F_A = \bigcup_{n=1}^{\infty} F_{(\alpha_n,\beta_n)}$ is the w. μ-meas multifunction which means that the function f has property *Op*.

3. - Some lemmas

Henceforth we assume that (T, m) is a measure space, where T is a compact Hausdorff metric space and m a Borel, σ-finite, regular and complete measure. The σ-field of m-measurable sets we denote by $\mathcal{M}$. We also assume that X is a Polish space.

If $S \subset T$ and $S \in \mathcal{M}$, then (S, m_s) is a measure space with the measure $m_S = m|_{\mathcal{M}_S}$, where $\mathcal{M}_S = \{M \cap S: M \in \mathcal{M}\}$. It is clear that $\mathcal{M}_S \subset \mathcal{M}$ and the measure m_s has the same properties as m.

Besides, if a multifunction $F: T \to \mathfrak{F}(X)$ is w. m-meas, then the restriction $F|_S: S \to \mathfrak{F}(X)$ is w. m_S-meas. On the other hand, if a multifunction $G: S \to \mathfrak{F}(X)$ is w. m_S-meas, then its « empty » extension $\tilde{G}: T \to \mathfrak{F}(X)$, defined by $\tilde{G}(t) = \emptyset$, if $t \notin S$, $\tilde{G}(t) = G(t)$, if $t \in S$, is w. m-meas.

Lemma 2: If a multifunction $F: T \to \mathfrak{F}(X)$ satisfies the following condition: « for every $\varepsilon > 0$ there exists a closed subset T_ε of T, with $m(T \setminus T_\varepsilon) < \varepsilon$, such that $F|_{T_\varepsilon}$ is w. m_{T_ε}-meas », then F is w. m-meas.

Proof: Putting $\varepsilon_n = 1/n$, $n \in \mathrm{N}$, we obtain a sequence of closed sets $T_n \subset T$ such that $m(T \setminus T_n) < \varepsilon_n$ and multifunctions $F_n = F|_{T_n}$ are w. m_{T_n}-meas. Thus their « empty » extensions $\tilde{F}_n: T \to \mathfrak{F}(X)$ are w. m-meas. Hence the multifunction $\tilde{F} = \bigcup_{n=1}^{\infty} \tilde{F}_n$ is w. m-meas. Since

$$\{t \in T: F(t) \neq \tilde{F}(t)\} \subset T \setminus \bigcup_{n=1}^{\infty} T_n \quad \text{and} \quad m\left(T \setminus \bigcup_{n=1}^{\infty} T_n\right) = 0,$$

then the multifunction F is w. m-meas.

LEMMA 3: Let $f\colon T\times X\to \mathbf{R}$ be C_*-type function and have property Op. Then for every multifunction $F_{(\alpha,\infty)}\colon T\to \mathfrak{P}(X)$ there exists a nondecreasing sequence of C-type functions $\varphi_n^*\colon T\times X\to [0,1]$ such that $\lim\limits_{n\to\infty}\varphi_n^*(t,x) = \chi(x;t,\alpha)$ where $\chi(\cdot\,;t,\alpha)$ denotes the characteristic function of the set $F_{(\alpha,\infty)}(t)$.

PROOF: Let $\alpha_n=\alpha+1/n$, $n\in \mathrm{N}$. We have

$$F_{(-\infty,\alpha]}(t)\subset \overline{F}_{(-\infty,\alpha_n)}(t)\subset F_{(-\infty,\alpha_n]}(t)\,.$$

Hence

$$F_{(-\infty,\alpha]}(t)\subset \bigcap_{n=1}^{\infty} F_{(-\infty,\alpha_n)}(t)\subset \bigcap_{n=1}^{\infty} F_{(-\infty,\alpha_n]}(t)=F_{(-\infty,\alpha]}(t)\,,$$

so

$$F_{(-\infty,\alpha]}(t)=\bigcap_{n=1}^{\infty} F_{(-\infty,\alpha_n)}(t)\,.$$

But the multifunctions $\overline{F}_{(-\infty,\alpha_n)}$ are w. m-meas (and have closed values), therefore by [H, Theorem 3.5] the multifunction

$$F_{(-\infty,\alpha]}=\bigcap_{n=1}^{\infty} \overline{F}_{(-\infty,\alpha_n)} \qquad \text{is w. } m\text{-meas}\,.$$

Now we define functions φ_n^* as follows:

$$\varphi_n^*(t,x)=n\cdot\min\left(\frac{1}{n},\, d(x, F_{(-\infty,\alpha]}(t))\right),$$

where $d(x,A)$ denotes the distance of x to A and $d(x,\emptyset)=+\infty$. For each fixed $t\in T$ the functions $\varphi_n^*(t,\cdot)$ are continuous, because the distance of a point from the nonempty set is a continuous function; if $F_{(-\infty,\alpha]}(t)=\emptyset$, then $\varphi_n^*(t,\cdot)=1=\text{const.}$ For each fixed $x\in X$ the functions $\varphi_n^*(\cdot,x)$ are m-meas by [H, Theorem 3.3 b], because the multifunctions $F_{(-\infty,\alpha]}$ are w. m-meas. Thus $\varphi_n^*\colon T\times X\to[0,1]$ are C-type functions.

Now if $x\in F_{(\alpha,\infty)}(t)$, then $x\notin F_{(-\infty,\alpha]}(t)$. Hence $d(x,F_{(-\infty,\alpha]}(t))=\eta>0$ and for $n>1/\eta$ we have $\varphi_n^*(t,x)=1$ that is $\lim\limits_{n\to\infty}\varphi_n^*(t,x)=1$.

If $x\notin F_{(\alpha,\infty)}(t)$, then $x\in F_{(-\infty,\alpha]}(t)$. Hence $\varphi_n^*(t,x)=0$ for each $n\in N$, that is $\lim\limits_{n\to\infty}\varphi_n^*(t,x)=0$.

Therefore $\lim\limits_{n\to\infty}\varphi_n^*(t,x)=\chi(x;t,\alpha)$.

LEMMA 4: Let $f\colon T\times X\to(0,1)$ be C_*-type function and have property Op.

Then there exists a nondecreasing sequence of C-type functions $f_n\colon T\times X\to[0,1]$ which converges to f.

Proof: Let $\alpha_i^n = i\cdot(\frac{1}{2})^n$, $i = 0, 1, 2, \ldots, 2^n$. Consider step-functions s_n: $T\times X \to [0, 1]$ defined by

$$s_n(t, x) = (\tfrac{1}{2})^n \cdot \sum_{i=1}^{2^n} \chi(x; t, \alpha_i^n).$$

It easily verified that $s_n \leq s_{n+1}$ and s_n converges (even uniformly) to f.

For a moment let us fix $n \in \mathrm{N}$. In view of Lemma 3 for each $i = 1, 2, 3, \ldots, 2^n$ there exists a nondecreasing (with respect to k) sequence $\varphi_{i,k}^n$: $T\times X \to [0, 1]$ of C-type functions such that

$$\lim_{k\to\infty} \varphi_{i,k}^n(t, x) = \chi(x; t, \alpha_i^n).$$

The functions φ_k^n: $T\times X \to [0, 1]$, defined by

$$\varphi_k^n(t, x) = (\tfrac{1}{2})^n \sum_{i=1}^{2^n} \varphi_{i,k}^n(t, x),$$

form a nondecreasing (with respect to k) sequence of C-type functions, which is converging to s_n. Indeed,

$$\varphi_k^n(t, x) = (\tfrac{1}{2})^n \sum_{i=1}^{2^n} \varphi_{i,k}^n(t, x) \leq (\tfrac{1}{2})^n \sum_{i=1}^{2^n} \varphi_{i,k}^n(t, x) = \varphi_{k+1}^n(t, x)$$

and

$$\lim_{k\to\infty} \varphi_k^n(t, x) = (\tfrac{1}{2})^n \sum_{i=1}^{2^n} \lim_{k\to\infty} \varphi_{i,k}^n(t, x) = (\tfrac{1}{2})^n \sum_{i=1}^{2^n} \chi(x; t, \alpha_i^n) = s_n(t, x).$$

Let now $f_n(t, x) = \sup_{i\leq n, j\leq n} \varphi_j^i(t, x)$.

Such defined functions f_n: $T\times X \to [0, 1]$ are of C-type and obviously form a nondecreasing sequence converging to f. In fact, we have

$$f(t, x) = \lim_{i\to\infty} s_i(t, x) = \sup_{i\in\mathrm{N}} s_i(t, x) = \sup_{i\in\mathrm{N}} \Big(\lim_{j\to\infty} \varphi_j^i(t, x)\Big) =$$
$$= \sup_{i\in\mathrm{N}} \Big(\sup_{j\in\mathrm{N}} \varphi_j^i(t, x)\Big) = \sup_{n\in\mathrm{N}} \Big(\sup_{i\leq n, j\leq n} \varphi_j^i(t, x)\Big) = \sup_{n\in\mathrm{N}} f_n(t, x) = \lim_{n\to\infty} f_n(t, x).$$

The proof of our lemma is completed.

Lemma 5: If f: $T\times X \to (0, 1)$ is C_*-type function and has property Op, then function $f+b$: $T\times X \to \mathrm{R}$, defined by $(f+b)(t, x) = f(t, x) + b(x)$, where b: $X \to \mathrm{R}$ is a continuous function, is of C_*-type and has property Op.

Proof: In virtue of Lemma 4 there exists C-type functions f_n: $T\times X \to [0, 1]$ such that $f_n \leq f_{n+1}$ and $\lim_{n\to\infty} f_n(t, x) = f(t, x)$.

Then the functions $f_n + b$ are of C-type and form a nondecreasing sequence coverging to $f + b$. Thus, $f + b$ has property B_* and therefore by Theorem 1 has property Op.

4. - The main results

Theorem 2: Let $f\colon T \times X \to \mathrm{R}$ be C_*-type function. Then the following statements are equivalent:

a) f has property SD_*,

b) f has property Op,

c) f has property Cl,

d) f has property B_*.

Proof: $a) \Rightarrow b)$. Let $\varepsilon > 0$. There exists a closed subset T_ε of T such that $m(T \setminus T_\varepsilon) < \varepsilon$ and $f|_{T_\varepsilon \times X}$ is lsc. Then by Baire's theorem there exists a sequence of continuous functions $f_n^\varepsilon\colon T_\varepsilon \times X \to \mathrm{R}$ such that $f_n^\varepsilon < f_{n+1}^\varepsilon$ and $\lim_{n\to\infty} f_n^\varepsilon(t, x) = f|_{T_\varepsilon \times X}(t, x)$. The functions f_n^ε are in particular, of C-type, thus by Theorem 1 a multifunctions $F_A^\varepsilon\colon T_\varepsilon \to \mathfrak{F}(X)$, defined by $F_A^\varepsilon(t) = \{x \in X\colon f|_{T_\varepsilon \times X}(t, x) \in A\}$ is w. m_{T_ε}-meas for each open set $A \subset \mathrm{R}$. But $F_A|_{T_\varepsilon} = F_A^\varepsilon$ and ε is arbitrary, therefore in view of Lemma 2 the multifunction $F_A\colon T \to \mathfrak{F}(X)$ is w. m-meas and so f has property Op.

$b) \Rightarrow c)$. Let $A \subset \mathrm{R}$ be a closed set. Then the sets $A_n = K(A, 1/n)$, where $K(A, r)$ denotes the open ball with a «center» A and radius r, are open. Hence the multifunctions F_{A_n}, by assumption, are w. m-meas and also $\overline{F}_{A_n}$. But $F_A(t) = \bigcap_{n=1}^{\infty} \overline{F}_{A_n}(t)$, because

$$F_A(t) \subset \overline{F}_{A_n}(t) \subset F_{\bar{A}_n}(t) \quad \text{and} \quad \bigcap_{n=1}^{\infty} F_{\bar{A}_n}(t) = F_A(t)$$

and consequently the multifunction F_A is by [H, Theorem 3.5] w. m-meas.

$c) \Rightarrow b)$. We have $F_{(\alpha,\beta)} = \bigcup_{n=1}^{\infty} F_{[\alpha_n, \beta_n]}$, where $\alpha_n = \alpha + 1/(n+j)$, $\beta_n = \beta - 1/(n+j)$ and $j \in \mathrm{N}$ is such that $\alpha + 1/j < \beta - 1/j$.

Further we have also

$$F_A = \bigcup_{n=1}^{\infty} F_{(a_n, b_n)} \quad \text{where} \quad A = \bigcup_{n=1}^{\infty} (a_n, b_n)$$

is an open subset of R. From these equalities the w. m-measurability of multifunction F_A follows.

$b) \Rightarrow d)$. In view of Lemma 1 it is sufficient to prove that a function $g = s \circ f$, where s is as in assumption of Lemma 1, is the limit of a nondecreasing sequence of C-type functions g_n: $T \times X \to (0, 1)$. Then the functions $f_n = s^{-1} \circ g_n$: $T \times X \to \mathbf{R}$ are of C-type and nondecreasingly converge to $f = s^{-1} \circ g$.

Consider the functions φ_n: $T \times X \times X \to \mathbf{R}$ and g_n: $T \times X \to \mathbf{R}$ defined by $\varphi_n(t, x, a) = g(t, a) + nd(a, x)$,

$$g_n(t, x) = \inf_{a \in X} \varphi_n(t, x, a), \qquad n = 1, 2, \ldots .$$

Obviously $g_n(t, x) < g_{n+1}(t, x)$. Furthermore for a fixed n we have: $\varphi_n(\cdot, x, a)$ is m-measurable for each $x, a \in X$, $\varphi_n(t, \cdot, a)$ is continuous for each $t \in T$, $a \in X$ (we observe that for fixed $t \in T$ the functions $\varphi_n(t, \cdot, a)$ are the family of equicontinuous functions with respect to the parameter $a \in X$), $\varphi_n(t, x, \cdot)$ is lowersemicontinuous for each $t \in T$, $x \in X$. By the same argument as in [L, p. 61] (cf. also [H, pp. 151-152]) we prove that

$$0 < g(t, x) < 1 \quad \text{and} \quad \lim_{n \to \infty} g_n(t, x) = g(t, x), \qquad (t, x) \in T \times X.$$

Now it remains to show that g_n is a C-type function. For each $t \in T$, $g_n(t, \cdot)$ is continuous as an infimum of the family of equicontinuous functions $\varphi_n(t, \cdot, a)$ (cf. [L, p. 57]). Further, we observe that if x is fixed, then $\varphi_n(\cdot, x, \cdot)$ is the sum of the function g, which is of C_*-type, and the continuous function $h = nd(\cdot, x)$. Since g has property Op, in view of Lemma 5 $g + h$ has property too. It means that multifunctions $\Phi^{(x)}_{n,A}$: $T \to \mathcal{S}(X)$, defined by $\Phi^{(x)}_{n,A}(t) = \{a \in X: \varphi_n(t, x, a) \in A\}$, are w. m-means for each open subset A of $\mathbf{R}$ and consequently their domains are m-measurable sets, that is $\{t \in T: \Phi^{(x)}_{n,A}(t) \neq \emptyset\} \in \mathcal{M}$ for each open $A \subset \mathbf{R}$.

Now consider a multifunction $\Psi^{(x)}_n$: $T \to \mathcal{S}(\mathbf{R})$ defined by

$$\Psi^{(x)}_n(t) = \{\varphi_n(t, x, a): a \in X\}.$$

We claim that $\Psi^{(x)}_n$ is w. m-measurable. Indeed, for each open set $A \subset \mathbf{R}$ we have

$$\{t \in T: \Psi^{(x)}_n(t) \cap A \neq \emptyset\} = \{t \in T: \varphi_n(t, x, a) \in A \text{ for some } a \in X\} =$$
$$= \{t \in T: \Phi^{(x)}_{n,A}(t) \neq \emptyset\} \in \mathcal{M}.$$

Thus by [H, Theorem 6.6] the function « $t \to \inf \Psi^{(x)}_n(t)$ » is m-measurable.

But $\inf \Psi^{(x)}_n(t) = \inf_{a \in X} \varphi_n(t, x, a) = g_n(t, x)$ which completes the proof of the m-measurability of function $g_n(\cdot, x)$. Therefore g_n is the C-type function.

a) ⇒ *d*). Let $f_n: T\times X \to \mathbf{R}$ be *C*-type, $f_n < f_{n+1}$, $n = 1, 2, \ldots$ and

$$\lim_{n\to\infty} f_n(t, x) = f(t, x)\,.$$

Let us choose an arbitrary $\varepsilon > 0$. Then in virtue of Scorza-Dragoni's theorem we can obtain of close sets T_n such that $T_n \subset T_{n-1}$, where $T_0 = T$, $m(T_{n-1}\setminus T_n) < (\frac{1}{2})^n \varepsilon$ and $f_n|_{T_n\times X}$ is continuous, $n=1, 2, \ldots$. Let $T_\varepsilon = \bigcap_{n=1}^{\infty} T_n \cdot T_\varepsilon$ is closed and $T_\varepsilon \subset T$. The functions $f_n|_{T_\varepsilon\times X}$ are continuous and form a non-decreasing sequence converging to the function $f|_{T_\varepsilon\times X}$ which is lowersemicontinuous with respect to both variables.

Furthermore

$$m(T\setminus T_\varepsilon) = m\Big(T\setminus \bigcap_{n=1}^{\infty} T_n\Big) = m\Big(\bigcup_{n=1}^{\infty}(T\setminus T_n)\Big) =$$

$$= m\Big(\bigcup_{n=1}^{\infty}(T_{n-1}\setminus T_n)\Big) = \sum_{n=1}^{\infty} m(T_{n-1}\setminus T_n) < \sum_{n=1}^{\infty}(\tfrac{1}{2})^n \varepsilon = \varepsilon\,.$$

Thus $f: T\times X \to \mathbf{R}$ has property SD_*.

The proof of Theorem 2 is completed.

It is easily observed that $f: T\times X\to\mathbf{R}$ is C_*-type function iff $(-f)$ is C^*-type function, f has property SD_* iff $(-f)$ has property SD^*, f has property *Op* (*Cl*) iff $(-f)$ has the same property and finally f has property B_* iff $(-f)$ has property B^*. Hence we have the following theorem

THEOREM 3: Let $f: T\times X\to\mathbf{R}$ be C^*-type function. Then the following statements are equivalent:

a) f has property SD^*,

b) f has property *Op*,

c) f has property *Cl*,

d) f has property B^*.

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