GIUSEPPE ZAMPIERI (*)

Local Existence at the Boundary of Analytic Solutions of P.D.E. with Analytic Coefficients (**)  

Esistenza locale di soluzioni al bordo per equazioni a derivate parziali con coefficienti analitici

Abstract

Let $M$ be a real analytic manifold, $X$ a complexification of $M$, $\Omega$ an open subset of $M$ with smooth boundary, $(\Omega$ on one side of $\partial \Omega)$, $A = A_\Omega$ the sheaf of analytic functions on $M$, $P = P(\Omega, D)$ a germ of differential operator at $\xi_0$, $\xi_0 \in \partial \Omega$, with analytic coefficients. We prove that $P_{\text{T}^{\partial}(A_\Omega)}_{x_0} = \Gamma_0(A_\Omega)_{x_0}$ when the conormal to $M \setminus \Omega$ is non-microcharacteristic (cf. [2]) for $P$ along $\Omega \times T^* \Omega \setminus X$ in $\pi^{-1}(\xi_0)$. In some case we even show that for the equation $Pu = 0$, $u \in \Gamma_0(A_\Omega)_{x_0}$, to have a solution $u \in \Gamma_0(A_\Omega)_{x_0}$, we only need non-microcharacteristicity in $\pi^{-1}(\xi_0) \cap SS_B(\partial)$, $SS_B$ being the microsupport at the boundary in the sense of [5]. To obtain this result we introduce, according to [8], a new sheaf $\mathcal{A}_2$: in local coordinates $\zeta = \xi + iy$ its stalk at $\xi_0$ is represented by holomorphic functions in the sets $U_\varepsilon = \{ |\zeta - \xi_0| < \varepsilon, \xi \in \Omega, |y| < \varepsilon \text{ dist}(\xi; \partial \Omega) \}$ for arbitrarily small $\varepsilon$. We then compare the cohomology of $P$ with values in the sheaves $\Gamma_0(A_\Omega)$ and $\mathcal{A}_2$ (Theorem 3.2), and then in $\mathcal{A}_0$ and $\mathcal{A}_2$ (Theorem 3.3). As for the first result, which holds in more general situation, see also [6] and [8].

(*) Dipartimento di Matematica - Università, Via Belzoni, 7, 35131 Padova, Italy.
(**) Memoria presentata il 1° giugno 1987 da Giuseppe Scozzi Dragoni, aso dei XL.
The frame of the paper is the theory developed in [2]; many tools used in the proofs are also related to [6], [8].

I like to express my gratitude to prof. P. Schapira for frequent and useful discussions.

1. - Review on Microlocalization

Let \( X \) be a real \( C^\infty \)-manifold, \( \pi: T^*X \to X \) the cotangent bundle, \( \omega_x \) the orientation sheaf. Let \( D(X) \) (resp \( D^+(X) \), resp \( D^b(X) \)) be the derived category of the category of complexes (resp lower bounded complexes, resp bounded complexes) of sheaves of abelian groups on \( X \). For \( A \subset X \) locally closed and for \( \mathcal{F} \in \text{Ob} \left(D^+(X)\right) \) a complex \( \mu_A(\mathcal{F}) \) is defined in [5]. We recall its main properties:

\[
\begin{align*}
\text{R}n_{\mu_A}(\mathcal{F}) &= \text{R}_\mathcal{F}(\mathcal{F}), \\
\text{supp} \mu_A(\mathcal{F}) &\subset \text{SS} \mathcal{F} \cap \text{SS} Z_A, \\
\text{SS} \mu_A(\mathcal{F}) &\subset C(\text{SS} \mathcal{F}, \text{SS} Z_A),
\end{align*}
\]

(cf [2] for the definition of microsupport \( \text{SS} \) and of normal cone \( C(\cdot, \cdot) \)).

Let \( M \subset X \) be a real \( C^2 \)-submanifold of codimension \( n \), \( T^*_M X \) the conormal bundle to \( M \), \( \omega_{M|\Omega} \) the relative orientation sheaf. For \( \mathcal{F} \in D^+(X) \) we set \( \mathcal{F}^* = R \text{com}_{M|\Omega}(\mathcal{F}, Z_M) \). We will often consider the case \( A = \Omega \) or \( A = \Omega \), where \( \Omega \) is an open subset of \( M \) such that:

\[
Z_{\Omega} \text{ is c.c. (cohomologically constructible) and } Z_{\Omega}^* = Z_{\Omega} \otimes \omega_{M|\Omega}[-n],
\]

(cf [2], Section 5, for the definition of c.c.). We recall that for \( \mathcal{F} \) c.c. in \( D^b(X) \), \( \mathcal{F}^* \) is also c.c. and \( \mathcal{F} = \mathcal{F}^{**} \). Thus (1.4) is equivalent to:

\[
Z_{\Omega} \text{ is c.c. and } Z_{\Omega}^* = Z_{\Omega} \otimes \omega_{M|\Omega}[-n].
\]

Easily to see all \( C^2 \)-convex \( \Omega \) and all open \( \Omega \) such that \( M \setminus \Omega \) is \( C^2 \)-convex and \( M \setminus \Omega = \text{Int} M \setminus \Omega \), satisfy (1.4). (\( C^2 \)-convex means convex at any point in some local \( C^2 \)-chart.)

Under assumption (1.4) we have, by Proposition 5.6.3 of [2] (and by (1.4), (1.5)):

\[
\text{R} \Gamma_{T^*_M X} \mu_{\Omega}(\mathcal{F}) = \mathcal{F}_{\Omega} \otimes \omega_{M|\Omega}[-n], \quad \text{R} \Gamma_{T^*_M X} \mu_{\Omega}(\mathcal{F}) = \mathcal{F}_{\Omega} \otimes \omega_{M|\Omega}[-n].
\]

We suppose now that \( \Omega \subset M \) is \( C^2 \)-convex. For a sheaf \( \mathcal{F} \) on \( X \) a new sheaf \( \Gamma_{\Omega}(\mathcal{F}) \) on \( M \) is defined in [8]. This is a subsheaf of \( \Gamma_{\Omega}(\mathcal{F}) \) which coincides with \( \mathcal{F} \) over \( \Omega \) and whose stalk at \( x_\Omega \), \( x_\Omega \in \Omega \), is, in local coordinates \((x, y)\)
at $x_0$ (with $M = \{ y = 0 \}$):

\begin{equation}
(1.7) \quad \tilde{\Gamma}_0(\mathcal{F})_{x_0} = \lim_{\mathcal{F}} \Gamma'(U, \mathcal{F}), \text{ for } U \supset (x_0, y) ; x \in \Omega, |y| < \varepsilon \text{ dist } (x, \delta \Omega) \cap W \text{ for some } \varepsilon > 0 \text{ and some neighborhood } W \text{ of } x_0 .
\end{equation}

Let $R\tilde{\Gamma}_0(\cdot)$ denote the derived functor; in [8] we prove that:

\begin{equation}
(1.8) \quad R\Gamma_{T^*_X}(\mu_0(\mathcal{F}))_{T^*_X} = R\tilde{\Gamma}_0(\mathcal{F}) \otimes \omega_{_{T^*_X}[−\pi]}.
\end{equation}

On the other hand, since $\pi(\text{supp } (\mu_0(\mathcal{F}))) \subset \pi(T^*_X) = M$, then:

\begin{equation}
(1.9) \quad R\pi_0((\mu_0(\mathcal{F}))_{T^*_X}) = R\Gamma_0(\mathcal{F}).
\end{equation}

Combining (1.8) and (1.9) (resp (1.1) and (1.6)), one obtains the fundamental Sato's triangles for $\Gamma_{T^*_X}(\mu_0(\mathcal{F}))$ (resp $\mu_0(\mathcal{F})$ or $\mu_0(\mathcal{F})$), which are the main tools of the present paper.

2. - Microlocalization of $\partial_X$

Let $M$ be a real analytic manifold of dimension $n$, $X$ a complexification of $M$, $\Omega \subset M$ an open subset with analytic boundary $N = \partial \Omega$ ($\Omega$ on one side of $\partial \Omega$), $Y$ a complexification of $N$. Let $\partial = \partial_X$, $A = A_X, \mathcal{B} = \mathcal{B}_X$ be the sheaves of holomorphic functions on $X$, analytic functions on $M$, hyperfunctions on $M$ respectively; let $\mathcal{A}_0 = \tilde{\mathcal{A}}_0(\partial_X)$. For $A \subset M$ locally closed, especially for $A = \Omega$ and $A = \partial$, we set:

\begin{equation}
(2.1) \quad \mathcal{C}_{A_X} = \mu_\ast(\mathcal{A}_X) \otimes \omega_{_{T^*_X}[\eta]}.
\end{equation}

We recall that for any closed cone $Z \subset T^*_X$ and for any $\mathcal{G} \in \text{Ob } (D^c_{\text{cone}}(T^*_X))$, we have a distinguished triangle:

\begin{equation}
(2.2) \quad R\Gamma_{T^*_X}(\mathcal{G}) \rightarrow R\pi_0 R\Gamma_0(\mathcal{G}) \rightarrow R\pi_0 R\Gamma_0(\mathcal{G}) \rightarrow 0.
\end{equation}

(where $\tilde{Z} = Z \setminus T^*_X$). In particular for $Z = T^*_X$ and $\mathcal{G} = \mathcal{C}_{A_X}$ (resp $\mathcal{G} = \mathcal{C}_{\partial_X}$), and by (1.1), (1.6), we obtain an exact sequence:

\begin{equation}
(2.3) \quad 0 \rightarrow A_{\Omega} \rightarrow \Gamma_0(\mathcal{B}_X) \rightarrow H^0(R\pi_0 \mathcal{C}_{A_X}) \rightarrow 0,
\end{equation}

(resp:

\begin{equation}
(2.4) \quad 0 \rightarrow A_{\Omega} \rightarrow \Gamma_0(\mathcal{B}_X) \rightarrow H^0(\pi_0 \mathcal{C}_{\partial_X}) \rightarrow 0.
\end{equation}

Along with (2.3) we will also make use of the exact sequence:

\begin{equation}
(2.5) \quad 0 \rightarrow A_{\Omega} \rightarrow \Gamma_0(A_X) \rightarrow H^0(R\pi_0 R\Gamma_0(\mathcal{C}_{\partial_X})) \rightarrow 0,
\end{equation}

which is a consequence of (2.2) for $Z = \pi^{-1}(M \setminus \Omega) \cup T^*_X$. 
We note now that $\partial \Omega$ being analytic, then (cf [5]):

\[(2.6) \quad H^p(\mathcal{C}_D \mathcal{B} \mathcal{L}) = (\mathcal{C}_D \mathcal{B} \mathcal{L})_{\mathcal{L}^X}.
\]

In the above assumption we recall two other statements whose proof can be found in [7].

**Lemma 2.1.** For any $x \in \partial \Omega$ and for a suitable neighborhood $X'$ of $x$, we have:

\[(2.7) \quad H^p(U, \mathcal{C}_D \mathcal{B} \mathcal{L}) = 0 \quad \forall U \subset T^{\ast}_X X', \forall j \geq 1.
\]

**Theorem 2.2.** $(\mathcal{C}_D \mathcal{B} \mathcal{L})_{\mathcal{L}^X}$ is conically flabby (i.e., its image in $T^{\ast}_X X[\mathcal{R}]$ is flabby).

If we let now $\Omega = (\mathcal{C}_D \mathcal{B} \mathcal{L})_{\mathcal{L}^X}$ and $Z = T^{\ast} X$ (resp $Z = T^{\ast} (M \backslash \Omega) \cup T^{\ast}_X X$) in (2.2), we obtain by the aid of (1.8), (1.9) (and also by Theorem 2.2), exact sequences:

\[(2.8) \quad 0 \rightarrow \mathcal{A}_0 \rightarrow \Gamma_0(\mathcal{B}_M) \rightarrow \pi_+ (\mathcal{C}_D \mathcal{B} \mathcal{L})_{\mathcal{L}^X} \rightarrow 0,
\]

(resp):

\[(2.9) \quad 0 \rightarrow \mathcal{A}_0 \rightarrow \Gamma_0(A_M) \rightarrow \pi_+ \Gamma_{\Omega \cup T^{\ast}_X X}(\mathcal{C}_D \mathcal{B} \mathcal{L})_{\mathcal{L}^X} \rightarrow 0.
\]

More generally we know that for any closed cone $Z \subset T^{\ast}_X X$, $R_{\pi,}$

\[\begin{align*}
\Gamma_x(\mathcal{C}_D \mathcal{B} \mathcal{L})_{\mathcal{L}^X} & \quad \text{is concentrated in degree 0 and that}: \\
H^p(R_{\pi,}) \Gamma_x & \quad \text{concentrated in degree 0,}\end{align*}
\]

where SS$\pi$ is the microsupport at the boundary defined in [5]. (When $Z \subset M \times T^{\ast}_X X$, then $f \in \Gamma_0(A_M)$ in the right of (2.10).)

This gives a large class of exact sequences extending (2.8), (2.9).

3. **Existence of Analytic Solutions at the Boundary**

Let $M$ be a real analytic manifold of dimension $n$ and $X$ a complexification of $M$. Let $\rho \in T^{\ast}_M X$ and $x = \pi(\rho)$; using the natural embeddings:

\[T^{\ast}_M X \times T^{\ast}_M M \rightarrow T^{\ast} X \times T^{\ast} X \rightarrow T^{\ast} T^{\ast} X,
\]

$T^{\ast}_M M$ is identified to a submanifold of $T^{\ast}_M T^{\ast} X$. $T^{\ast} T^{\ast} X$ is in turns identified to $TT^{\ast} X$ by means of $-H$, $H$ being the Hamiltonian isomorphism.

Let $\Omega$ be an open subset of $M$ with analytic boundary ($\Omega$ on one side of $\partial \Omega$), let $x \in \partial \Omega$, and let $\theta$ be the exterior conormal to $\Omega$ at $x$. Let $P = P(x, D)$ be a germ of (micro)differentiable operator at $p$, $p \in \pi^{-1}(x)$, with holomorphic coefficients, and let char $P$ denote its characteristic variety. In [6] (and [8])
the following result was stated:

**Lemma 3.1.** Assume that:

\[ -H(0) \notin C_\pi(\text{char } P, \bar{\Omega} \times T_M^*X). \]

Then \( P \) is an isomorphism of \( R^\pi_{\bar{\Omega} \times (M \setminus \Omega)} C_{\Omega \times X} \) in a neighborhood of \( p \).

Let \( Z \) be a closed cone of \( \bar{\Omega} \times T_M^*X \) with \( \bar{\Omega} \times \Omega \subset T_M^*X \). By (2.2) and by the results of § 2, we have an exact sequence:

\[ 0 \to \bar{\Lambda}_0 \to \pi_\star \Gamma_\pi((C_{\Omega \times X})_{T_M^*X}) \to \bar{\Lambda}_0 \star \Gamma_\pi((C_{\Omega \times X})_{T_M^*X}) \to 0. \]

We also have:

\[ \Gamma_\pi((C_{\Omega \times X})_{T_M^*X}) = \Gamma_\pi((T_{\pi^{-1}(\Omega \setminus \Omega)}(C_{\Omega \times X})_{T_M^*X}) = \Gamma_\pi((T_{\pi^{-1}(\Omega \setminus \Omega)}(C_{\Omega \times X})_{T_M^*X}), \]

due to \( \text{supp } C_{\Omega \times X} \subset \pi^{-1}(M \setminus \Omega) \cup T_M^*X \).

Thus if \( P \) is a differential operator at \( x \) which induces an isomorphism of \( \Gamma_\pi((M \setminus \Omega)(C_{\Omega \times X})_{T_M^*X}) \) in a neighborhood of \( \bar{Z} \cap \pi^{-1}(x) \), we deduce from (3.2) and (3.3) that \( P \) is an isomorphism of \( \{ \pi_\star \Gamma_\pi((C_{\Omega \times X})_{T_M^*X})/\bar{\Lambda}_0 \}_x \). On the other hand we note that:

\[ \pi_\star \Gamma_\pi((C_{\Omega \times X})_{T_M^*X})_x = \{ f \in \Gamma_\Omega(A_\Omega)_x : SS_\Omega(f) \subset Z \} \]

On account of (3.4) and Lemma 3.1 we have then established the following:

**Theorem 3.2.** Let \( \Omega \) be an open set of \( M \) with analytic boundary, \( x \) a point of \( \partial \Omega \), 0 the exterior conormal to \( \Omega \) at \( x \), \( P(x, D) \) a differential operator at \( x \), \( Z \) a closed cone of \( \bar{\Omega} \times T_M^*X \) verifying \( \Omega \times Z \subset T_M^*X \). Assume that (3.1) is fulfilled \( \forall p \in \bar{Z} \cap \pi^{-1}(x) \). Then \( P \) is an isomorphism of \( \{ f \in \Gamma_\Omega(A_\Omega)_x : SS_\Omega(f) \subset Z \}/(\bar{\Lambda}_0)_x \).

To get good existence criteria we need now:

**Theorem 3.3.** In the situation of Theorem 3.3, assume, instead of (3.1):

\[ \text{char } P \cap (SS Z_\Omega \setminus T_M^*X) \cap (U \times T_M^*X) \subset T_M^*X, \]

for a neighborhood \( U \) of \( x \).

Then:

\[ P((\bar{\Lambda}_0)_x) = (\bar{\Lambda}_0)_x. \]

**Proof.** Let \( N^*(\Omega) \) be the conormal cone to \( \Omega \); we recall that:

\[ \text{supp } C_{\Omega \times X} \subset SS Z_\Omega = T_M^*X \oplus N^*(\Omega)_x, \]
(where \(a\) denotes the antipodal). Let \(L = SS Z_{D^*} T_{\beta}^* X\) and consider the triangle:

\[(3.7) \quad (c_{D|x})_L \to (c_{D|x}) \to (c_{D|x})^{\beta}_L \to.\]

Applying the functor \(R\pi_*(\cdot)\) to (3.7), and using (1.1) and (1.9) we obtain:

\[(3.8) \quad R\pi_*(c_{D|x})_L = 0.\]

On account of the triangle:

\[(3.9) \quad R\pi_*(c_{D|x})_L \to R\pi_*(c_{D|x})_L \to R\pi_*(c_{D|x})_L [-1],\]

we then obtain from (3.8):

\[(3.10) \quad R\pi_*(c_{D|x})_L = R\pi_*(c_{D|x})_L [-1].\]

Finally applying \(R\pi_*(\cdot)\) to (3.7) and using (1.6), (1.8), (3.10), we obtain an exact sequence:

\[(3.11) \quad 0 \to A_0 \to A_0 \to H^0(R\pi_*(c_{D|x})_L) \to 0.\]

We take now full advantage of the language of derived categories. We denote by \(\mathcal{D}_x\) the sheaf of differential operators with holomorphic coefficients, set \(\mathcal{M} = \mathcal{D}_x / \mathcal{D}_x P\), and apply the functor \(R\text{hom}_{\mathcal{D}_x}(\mathcal{M}, \cdot)\) to (3.11).

We first note that \(Y\) (the complexification of \(N = \partial D\)) is non-characteristic for \(\mathcal{M}\), due to (3.5). It follows: \(\text{Ext}_{\mathcal{D}_x}(\mathcal{M}, A_0) = 0\). Therefore (3.6) will be a consequence of:

\[(3.12) \quad H^1(R\pi_*(R\text{hom}_{\mathcal{D}_x}(\mathcal{M}, c_{D|x})_L)) = 0.\]

To prove (3.12) we notice that because of (1.2) and (3.5) we have:

\[\text{supp } (R\text{hom}_{\mathcal{D}_x}(\mathcal{M}, c_{D|x})_L) \cap (U \times T^{\beta}_{\mathcal{M}} X) = \emptyset,\]

and therefore:

\[(3.13) \quad R\pi_*(\text{hom}_{\mathcal{D}_x}(\mathcal{M}, c_{D|x})_L) = R(\pi_*)_*(R\text{hom}_{\mathcal{D}_x}(\mathcal{M}, c_{D|x})_L).\]

We also notice that \(R(\pi_*)_*(c_{D|x})_L\) is concentrated in degree \(-1\) since

\[c_{D|x} L = c_{D|x} [1 + 1] \quad \text{and} \quad H^j(U \times L, c_{D|x}) = 0 \quad \forall j \geq 1 \quad \text{(by Theorem 2.2)}.\]

This gives (3.12) and completes the proof of the theorem.
Corollary 3.4. Let \( P \) verify (3.5), let \( f \in \Gamma_0(A_\omega)_x \), and assume (3.1) fulfilled \( \forall p \in \mathcal{Z} = \mathcal{S}_0(f) \cap \mathcal{X}^\omega(x) \). Then there is a solution \( u \in \Gamma_0(A_\omega)_x \), with \( \mathcal{S}_0(u) \subset \mathcal{S}_0(f) \), of the equation \( Pu = f \). Such \( u \) is unique modulo \((\mathcal{A}_0)_x\).

We note now that if (3.1) holds \( \forall p \in \mathcal{Z} = \mathcal{X}^\omega(x) \cap T^*_M X \), then (3.5) is automatically fulfilled. We have therefore:

**Corollary 3.5.** Let \( P \) verify (3.1) \( \forall p \in \mathcal{Z} = \mathcal{X}^\omega(x) \cap T^*_M X \). Then:

\[
P(\Gamma_0(A_\omega)_x) = \Gamma_0(A_\omega)_x.
\]

**Remarks 3.6.**

\( a \) The conclusion of Corollary 3.4 holds, without the additional hypothesis (3.5), when \( P \) has constant coefficients and \( \Omega \) is convex in \( \mathbb{R}^n \). In fact, owing to the existence theorem on convex regions by Ehrenpreis-Malgrange, (3.6) is automatically verified in such situation.

\( b \) When \( \mathcal{X}^\omega(x) \cap T^*_M X \cap \text{char } P = \emptyset \), then Corollary 3.5 is equivalent to the well-known theorem of existence of analytic solutions to elliptic equations.

\( c \) Let \( \mathcal{M} \) be a coherent \( D_x \)-module (i.e. a differential system with holomorphic coefficients) and assume \( \gamma' \) non-characteristic for \( \mathcal{M} \). Let \( k \) be the length of a free projective resolution of \( \mathcal{M} \). If we suppose that \( \mathcal{M} \) verifies (3.1) \( \forall p \in \mathcal{X}^\omega(x) \cap T^*_M X \), then we obtain, by adapting the proofs of Theorems 3.2 and 3.3:

\[
R \text{Kom}_{D_x} (\mathcal{M}, \mathcal{A}_0)_x \simeq R \text{Kom}_{D_x} (\mathcal{M}, \Gamma_0(A_\omega))_x ;
\]

\[
\mathcal{E}x^2_{\delta_x}(\mathcal{M}, \mathcal{A}_0)_x \longrightarrow \mathcal{E}x^2_{\delta_x}(\mathcal{M}, \mathcal{A}_0)_x \text{ surjective}.
\]

In particular we have \( \mathcal{E}x^2_{\delta_x}(\mathcal{M}, \mathcal{A}_0)_x = 0 \) when \( k = n \), due to the Cauchy-Kowalewsky-Kashiwara theorem. However, apart from the constant coefficients case, the cohomology of the system \( \mathcal{M} \) with values in \( \mathcal{A}_0 \) does not vanish, generally, in any degree \( j, 1 \leq j \leq n \).

**Example.** Let \( \mathcal{M} = (C^\omega)^R \simeq R^n \), let \( \mathcal{M} \) be the Cauchy-Riemann system \( \mathcal{O}_{C^n} \) on \( M \), and let \( \Omega = M \setminus \mathcal{O}' \) for \( \mathcal{O}' \subset M \) strictly pseudoconvex and with analytic boundary.

Thus the hypotheses of Corollary 3.5 are fulfilled (as \( \mathcal{M} \) is elliptic), but we do not have \( \mathcal{E}x^2_{\delta_x}(\mathcal{M}, \Gamma_0(A_\omega))_x = 0 \), \( \forall j \geq 1, x \in \mathcal{O} \).
4. - Applications

We choose local coordinates \((\zeta, \zeta') \in T^*X\), \(\zeta = x + i\eta\), \(\zeta = \xi + i\eta\), and assume:

\[ X = \mathbb{C} \times X', \quad M = \mathbb{R} \times M', \quad \Omega = \mathbb{R}^r \times M'. \]

We also write \(\zeta = (\zeta_1, \zeta')\), \(\zeta = (\xi, \zeta')\). We note that the exterior conormal to \(\Omega\) is \(\theta = -dx_1\), and that \(-H(-dx_1) = -\zeta\partial \zeta_1\). Let \(P = P(x, D)\) be a differential operator at \(x_0\), let \(P_m\) denote the principal part of \(P\), and let \(p = (x_0, i\eta_0) \in \partial \Omega \times T^*_x X\), \(|\eta_0| = 1\).

Then (3.1) is equivalent to:

\[ P_m \not\equiv 0 \quad \text{if} \quad \xi_1 < -\varepsilon \min \left\{ \left( \frac{1}{2}\left| Y(-\xi_1)\right| |\eta| \right) + |\mathcal{E}'| \right\}, \quad \min \left\{ \left( \frac{1}{2}\left| Y(-\xi_1)\right| |\eta| \right) + |\mathcal{E}'| \right\}, \quad (\zeta, \zeta') - p < \varepsilon, \]

for suitable \(\varepsilon > 0\), \(\varepsilon > 0\). (Here \(Y\) denotes the Heaviside function.)

In particular (4.1) holds \(\forall p \in \pi^{-1}(x_0) \cap T^*_x X\) iff for suitable \(\varepsilon, \varepsilon', \varepsilon\):

\[ P_m \not\equiv 0 \quad \text{for} \quad -\varepsilon' |\eta| < \xi_1 < -\varepsilon \min \left\{ \left( \frac{1}{2}\left| Y(-\xi_1)\right| |\eta| \right) + |\mathcal{E}'| \right\}, \quad |\zeta - x_0| < \varepsilon. \]

On the other hand (3.5) is equivalent to:

\[ P_m \not\equiv 0 \quad \text{for} \quad -\varepsilon' |\eta| < \xi_1 < 0, \quad x_1 = y = \xi' = 0, \quad |\zeta' - x_0| < \varepsilon, \]

for suitable \(\varepsilon', \varepsilon\).

By applying the Bochner's local tube theorem to \(1/P_m\) one immediately proves that (4.3) is equivalent to:

\[ P_m \not\equiv 0 \quad \text{for} \quad -\varepsilon' |\eta| < \xi_1 < -\varepsilon \min \left\{ \left| Y' \right| |\eta| + |\mathcal{E}'| \right\}, \quad x_1 = y = 0, \quad |\zeta - x_0| < \varepsilon, \]

with new \(\varepsilon, \varepsilon', \varepsilon\).

By similar argument one also obtains the following refinement of Proposition 5.3 of [6]:

**Proposition 4.1.** Let \(P_m(\xi_1, \xi' ; \zeta) = f(\xi_1, \xi' ; \zeta)\) \((k \geq 1\) integer\) for \(f\) holomorphic at \(p\) and verifying:

\[ f \not\equiv 0 \quad \text{for} \quad \xi_1 < 0, \quad y = \xi' = 0, \quad (\zeta, \zeta') - p < \varepsilon, \quad x_1 \geq 0. \]
Then for a new \( x \) and \( \varepsilon \):

\[
\begin{align*}
(4.6) \quad P_m \neq 0 \quad \text{for} \quad \varepsilon_1 < -\varepsilon \left( (|x_1|^{2k} + Y(x_1)) \inf (1, |x_1|^{15-13k}) |x_1| + \right. \\
+ Y(-x_1)|x_1|^{1/2} + |y'| |y| + |\varepsilon'| \left. \right), \quad \left( \varepsilon, \varepsilon |\xi| - x \right) \cdot \varepsilon < \varepsilon .
\end{align*}
\]

(Note that (4.6) obviously implies (4.1) when \( k \geq 2 \).)

**Example (cf [6]).** Let:

\[
M = \mathbb{R}^n \times (\mathbb{R}^n, \mathbb{R}^n), \quad \Omega = \mathbb{R}^n \times \mathbb{R}^n, \quad P_m = D_1^2 - (x_1^2 + x_1^200) D^2, \\
k \geq 2, \quad k' \geq 0 .
\]

By Proposition 4.1, (4.2) is fulfilled at \( x_0 = 0 \) and then by Corollary 3.5 we get:

\[
P(\Gamma_0(A_m))_{x_0} = \Gamma_0(A_m)_{x_0}.
\]

We give now an application of the Phragmén-Lindelöf-Hörmander method [1] to exhibit a large class of operators \( P \) for which microhyperbolicity (i.e. Condition (4.1)) is also necessary for existence of analytic solutions. Let \( w \in \mathbb{R}^n, \quad |w| = 1 \), let \( \Omega \) be the open half space of \( \mathbb{R}^n \) defined by \( x \cdot w < 0 \), let \( x_0 \in \partial \Omega \), and let \( P = P(D) \) be a differential operator with constant coefficients. Let \( K \) be the \((n - 1)\)-dimensional closed unit sphere orthogonal to \( w \), let \( K(t) = K \times \{ t : \varepsilon \in \mathbb{R}, |\varepsilon| \leq t \} \), and let \( H_K(\cdot) \) be the support function of \( K \). We also let \( V = \{ \xi \in \mathbb{C}^n : P_m(\xi) = 0 \} \) and denote by \( V_{\varepsilon, \varepsilon} \) the \( \varepsilon \)-neighborhood of \( i \eta \) on \( V \).

One can then prove the following variant of the results of [1] (cf [9]):

**Proposition 4.2.** In the above situation, assume:

\[
P(\Gamma_0(A_m))_{x_0} = \Gamma_0(A_m)_{x_0} .
\]

Then \( \forall \eta \in \tilde{V} \cap T_{\eta} K \) and \( \forall t, 0 < t \leq 1 \), there exist a compact convex subset \( K' \subseteq \Omega \) and constants \( d, r_0 \) and \( \varepsilon \) such that \( \forall \eta \leq r_0 \) the following implication holds on the class of weakly plurisubharmonic functions \( \psi \) on \( V' \):

\[
\begin{align*}
(4.8) \quad &\psi(\zeta) \leq H_{K}(\text{Re} \zeta) + \delta \tau \quad \forall \zeta \in V_{t_0, t}, \\
&\psi(\zeta) \leq H_{K(\varepsilon)}(\text{Re} \zeta) \quad \forall \zeta \in V_{t_0, t},
\end{align*}
\]

implies:

\[
(4.9) \quad \psi(\zeta) \leq H_{-a + x'}(\text{Re} \zeta) \quad \forall \zeta \in V_{t_0, t} .
\]

We say that \( P \) is microhyperbolic at \( i \eta \) to \( \pm w \) \( d \varepsilon \) iff \( \pm \overline{w(\partial / \partial \overline{\varepsilon})} \notin C_{v_0}(V, i\mathbb{R}^n) \); we say that it is non-microcharacteristic iff \( \pm \overline{w(\partial / \partial \overline{\varepsilon})} \notin C_{v_0}(V, \{ i \eta \}) \).
THEOREM 4.3. Let \( \Omega = \{ x \cdot w < 0 \} \), let \( x_0 \in \Omega \), and assume \( P \) non-micro-characteristic to \( \pm w \) at a characteristic in of multiplicity \( \leq 2 \). If (4.7) holds then \( P \) is microhyperbolic to \( \pm w \).

PROOF. If (4.7) is fulfilled then we know that \( \forall t, 0 < t < 1 \), and for a suitable \( K' = K'_t \), the implication (4.8) \( \Rightarrow \) (4.9) is satisfied. Then reasoning as in [9] we can see that for a suitably small \( t \neq 0 \) we have \( 0 < H_{K'_t}(-w) \) which is a contradiction due to \( K'_t \subset C \Omega \).

EXAMPLE. Let

\[
M = \mathbb{R}^s, \quad \Omega = \{ x_1 < 0 \}, \quad x_0 = 0, \quad w = (1, 0, \ldots), \quad \eta = (0, \ldots, 0, 1), \quad \xi = (\xi_1, \xi, \zeta, \zeta_0) \in C \times C' \times C' \times C = C^s, \quad r \neq 0, \quad t \neq 0.
\]

Then for any one of the polynomials \( P(D) \):

\[
D_1^2 + D_2^2, \quad D_1^2 - D_2^2 + D_2^2, \quad (D_1^2 - D_2^2)D_2^2 + D_4^4,
\]

we know, on account of Theorem 4.3, that (4.7) is not fulfilled.

REFERENCES