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Local Existence at the Boundary of Analytic Solutions of P.D.E. with Analytic Coefficients (**)

Esistenza locale di soluzioni al bordo per equazioni a derivate parziali con coefficienti analitici

SUNTO. — Si usa la teoria della microlocalizzazione al bordo di [5], nei suoi ulteriori sviluppi di [7] e [8], per stabilire criteri di esistenza di soluzioni di equazioni a derivate parziali con coefficienti analitici.

ABSTRACT

Let M be a real analytic manifold, X a complexification of M, Ω an open subset of M with smooth boundary, (Ω on one side of $\partial\Omega$), $\mathcal{A} = \mathcal{A}_M$ the sheaf of analytic functions on M, $P = P(x, D)$ a germ of differential operator at x_0, $x_0 \in \partial\Omega$, with analytic coefficients. We prove that $P\Gamma_\Omega(\mathcal{A}_M)_{x_0} = \Gamma_\Omega(\mathcal{A}_M)_{x_0}$ when the conormal to $M \setminus \Omega$ is non-microcharacteristic (cf [2]) for P along $\bar{\Omega} \times_M T^*_M X$ in $\pi^{-1}(x_0)$. In some case we even show that for the equation $Pu = f$, $f \in \Gamma_\Omega(\mathcal{A}_M)_{x_0}$, to have a solution $u \in \Gamma_\Omega(\mathcal{A}_M)_{x_0}$, we only need non-microcharacteristicity in $\pi^{-1}(x_0) \cap SS_\Omega(f)$, SS_Ω being the microsupport at the boundary in the sense of [5]. To obtain this result we introduce, according to [8], a new sheaf $\tilde{\mathcal{A}}_\Omega$: in local coordinates $z = x + iy$ its stalk at x_0 is represented by holomorphic functions in the sets $U_\varepsilon = \{|z - x_0| < \varepsilon, x \in \Omega, |y| < \varepsilon \operatorname{dist}(x, \partial\Omega)\}$ for arbitrarily small ε. We then compare the cohomology of P with values in the sheaves $\Gamma_\Omega(\mathcal{A}_M)$ and $\tilde{\mathcal{A}}_\Omega$ (Theorem 3.2), and then in $\tilde{\mathcal{A}}_\Omega$ and $\mathcal{A}_{\bar{\Omega}}$ (Theorem 3.3). As for the first result, which holds in more general situation, see also [6] and [8].

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The frame of the paper is the theory developped in [2]; many tools used in the proofs are also related to [6], [8].

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1. - Review on microlocalization

Let X be a real C^2-manifold, $\pi: T^*X \to X$ the cotangent bundle, ω_X the orientation sheaf. Let $D(X)$ (resp $D^+(X)$, resp $D^b(X)$) be the derived category of the category of complexes (resp lower bounded complexes, resp bounded complexes) of sheaves of abelian groups on X. For $A \subset X$ locally closed and for $\mathcal{F} \in \mathrm{Ob}(D^+(X))$ a complex $\mu_A(\mathcal{F})$ is defined in [5]. We recall its main properties:

$$R\pi_* \mu_A(\mathcal{F}) = R\Gamma_A(\mathcal{F}), \tag{1.1}$$

$$\mathrm{supp}\, \mu_A(\mathcal{F}) \subset \mathrm{SS}\, \mathcal{F} \cap \mathrm{SS}\, \boldsymbol{Z}_A, \tag{1.2}$$

$$\mathrm{SS}\, \mu_A(\mathcal{F}) \subset C(\mathrm{SS}\, \mathcal{F}, \mathrm{SS}\, \boldsymbol{Z}_A), \tag{1.3}$$

(cf [2] for the definition of microsupport SS and of normal cone $C(\cdot,\cdot)$).

Let $M \subset X$ be a real C^2-submanifold of codimension n, $T^*_M X$ the conormal boundle to M, $\omega_{M/X}$ the relative orientation sheaf. For $\mathcal{F} \in D^+(X)$ we set $\mathcal{F}^* = \boldsymbol{R}\,\mathcal{H}om_{\boldsymbol{Z}_X}(\mathcal{F}, \boldsymbol{Z}_X)$. We will often consider the case $A = \Omega$ or $A = \bar{\Omega}$, where Ω is an open subset of M such that:

(1.4) $\boldsymbol{Z}_{\bar\Omega}$ is c.c. (cohomologically constructible) and $\boldsymbol{Z}^*_{\bar\Omega} = \boldsymbol{Z}_{\Omega} \otimes \omega_{M/X}[-n]$,

(cf [2], Section 5, for the definition of c.c.). We recall that for $\mathcal{F}$ c.c. in $D^b(X)$, $\mathcal{F}^*$ is also c.c. and $\mathcal{F} = \mathcal{F}^{**}$. Thus (1.4) is equivalent to:

(1.5) $\boldsymbol{Z}_{\Omega}$ is c.c. and $\boldsymbol{Z}^*_{\Omega} = \boldsymbol{Z}_{\bar\Omega} \otimes \omega_{M/X}[-n]$.

Easily to see all C^2-convex Ω and all open Ω such that $M \setminus \Omega$ is C^2-convex and $M \setminus \Omega = \overline{\mathrm{Int}\, M \setminus \Omega}$, satisfy (1.4). ($C^2$-convex means convex at any point in some local C^2-chart.)

Under assumption (1.4) we have, by Proposition 5.6.3 of [2] (and by (1.4), (1.5)):

$$R\Gamma_{T^*_M X}\mu_{\Omega}(\mathcal{F}) = \mathcal{F}_{\bar\Omega} \otimes \omega_{M/X}[-n]; \quad R\Gamma_{T^*_M X}\mu_{\bar\Omega}(\mathcal{F}) = \mathcal{F}_{\Omega} \otimes \omega_{M/X}[-n]. \tag{1.6}$$

We suppose now that $\Omega \subset M$ is C^2-convex. For a sheaf $\mathcal{F}$ on X a new sheaf $\tilde{\Gamma}_{\Omega}(\mathcal{F})$ on M is defined in [8]. This is a subsheaf of $\Gamma_{\Omega}(\mathcal{F})$ which coincides with $\mathcal{F}$ over Ω and whose stalk at x_0, $x_0 \in \partial\Omega$ is, in local coordinates (x, y)

at x_0 (with $M = \{y = 0\}$):

$$(1.7)\qquad \tilde{\Gamma}_\Omega(\mathcal{F})_{x_0} = \varinjlim_U \Gamma(U, \mathcal{F}), \text{ for } U \supset \{(x, y) : x \in \Omega,\ |y| < \varepsilon \operatorname{dist}(x, \partial\Omega)\} \cap W$$
for some $\varepsilon < 0$ and some neighborhood W of x_0.

Let $R\tilde{\Gamma}_\Omega(\cdot)$ denote the derived functor; in [8] we prove that:

$$(1.8)\qquad R\Gamma_{T^*_MX}(\mu_\Omega(\mathcal{F}))_{T^*_MX} = R\tilde{\Gamma}_\Omega(\mathcal{F}) \otimes \omega_{M/X}[-n].$$

On the other hand, since $\pi(\operatorname{supp}(\mu_\Omega(\mathcal{F}))) \subset \pi(T^*_MX) = M$, then:

$$(1.9)\qquad R\pi_*\big((\mu_\Omega(\mathcal{F}))_{T^*_MX}\big) = R\Gamma_\Omega(\mathcal{F}).$$

Combining (1.8) and (1.9) (resp (1.1) and (1.6)), one obtains the fundamental Sato's triangles for $(\mu_\Omega(\mathcal{F}))_{T^*_MX}$ (resp $\mu_\Omega(\mathcal{F})$ or $\mu_{\bar{\Omega}}(\mathcal{F})$), which are the main tools of the present paper.

2. - Microlocalization of $\mathcal{O}_X$

Let M be a real analytic manifold of dimension n, X a complexification of M, $\Omega \subset M$ an open subset with analytic boundary $N = \partial\Omega$ (Ω on one side of $\partial\Omega$), Y a complexification of N. Let $\mathcal{O} = \mathcal{O}_X$, $\mathcal{A} = \mathcal{A}_M$, $\mathcal{B} = \mathcal{B}_M$ be the sheaves of holomorphic functions on X, analytic functions on M, hyperfunctions on M respectively; let $\tilde{\mathcal{A}}_\Omega = \tilde{\Gamma}_\Omega(\mathcal{O}_X)$. For $A \subset M$ locally closed, especially for $A = \Omega$ and $A = \bar{\Omega}$, we set:

$$(2.1)\qquad \mathcal{C}_{A|X} = \mu_A(\mathcal{O}_X) \otimes \omega_{M/X}[n].$$

We recall that for any closed cone $Z \subset T^*X$ and for any $\mathcal{G} \in \operatorname{Ob}(D^+_{\text{conic}}(T^*X))$, we have a distinguished triangle:

$$(2.2)\qquad R\Gamma_{T^*_XX}(\mathcal{G}) \to R\pi_* R\Gamma_Z(\mathcal{G}) \to R\dot{\pi}_* R\Gamma_{\dot{Z}}(\mathcal{G}) \xrightarrow{+1},$$

(where $\dot{Z} = Z \setminus T^*_XX$). In particular for $Z = T^*X$ and $\mathcal{G} = \mathcal{C}_{\Omega|X}$ (resp $\mathcal{G} = \mathcal{C}_{\bar{\Omega}|X}$), and by (1.1), (1.6), we obtain an exact sequence:

$$(2.3)\qquad 0 \to \mathcal{A}_{\bar{\Omega}} \to \Gamma_\Omega(\mathcal{B}_M) \to H^0(R\dot{\pi}_* \mathcal{C}_{\Omega|X}) \to 0,$$

(resp:

$$(2.4)\qquad 0 \to \mathcal{A}_\Omega \to \Gamma_{\bar{\Omega}}(\mathcal{B}_M) \to \dot{\pi}_* \mathcal{C}_{\bar{\Omega}|X} \to 0.)$$

Along with (2.3) we will also make use of the exact sequence:

$$(2.5)\qquad 0 \to \mathcal{A}_{\bar{\Omega}} \to \Gamma_\Omega(\mathcal{A}_M) \to H^0(R\dot{\pi}_* R\Gamma_{\dot{\pi}^{-1}(M \setminus \Omega)}(\mathcal{C}_{\Omega|X})) \to 0,$$

which is a consequence of (2.2) for $Z = \dot{\pi}^{-1}(M \setminus \Omega) \cup T^*_XX$.

We note now that $\partial\Omega$ being analytic, then (cf [5]):

$$H^0(\mathcal{C}_{\Omega|X}) = (\mathcal{C}_{\Omega|X})_{T^*_M X}\,. \tag{2.6}$$

In the above assumption we recall two other statements whose proof can be found in [7].

LEMMA 2.1. *For any $x \in \partial\Omega$ and for a suitable neighborhood X' of x, we have:*

$$H^j(U, \mathcal{C}_{X'|X}) = 0 \qquad \forall U \subset T^*_X X'\,,\ \forall j \geq 1\,. \tag{2.7}$$

THEOREM 2.2. *$(\mathcal{C}_{\Omega|X})_{T^*_M X}$ is conically flabby (i.e. its image in $T^*_M X/R^+$ is flabby).*

If we let now $\mathcal{S} = (\mathcal{C}_{\Omega|X})_{T^*_M X}$ and $Z = T^*X$ (resp $Z = \tilde{\pi}^{-1}(M \setminus \Omega) \cup T^*_X X$) in (2.2), we obtain by the aid of (1.8), (1.9) (and also by Theorem 2.2), exact sequences:

$$0 \to \tilde{\mathcal{A}}_\Omega \to \Gamma_\Omega(\mathcal{B}_M) \to \tilde{\pi}_*(\mathcal{C}_{\Omega|X})_{T^*_M X} \to 0\,, \tag{2.8}$$

(resp:

$$0 \to \tilde{\mathcal{A}}_\Omega \to \Gamma_\Omega(\mathcal{A}_M) \to \tilde{\pi}_*\Gamma_{\tilde{\pi}^{-1}(M\setminus\Omega)}((\mathcal{C}_{\Omega|X})_{T^*_M X}) \to 0\,. \tag{2.9}$$

More generally we know that for any closed cone $Z \subset T^*_M X$, $R\pi_* \cdot R\Gamma_Z(\mathcal{C}_{\Omega|X})_{T^*_M X}$ is concentrated in degree 0 and that:

$$H^0(R\pi_* R\Gamma_Z\,(\mathcal{C}_{\Omega|X})_{T^*_M X})_x = \{f \in \Gamma_\Omega(\mathcal{B}_M)_x\ \ SS_\Omega(f) \subset Z\}\,,\quad x \in M\,, \tag{2.10}$$

where SS_Ω is the microsupport at the boundary defined in [5]. (When $\Omega \underset{M}{\times} Z \subset T^*_X X$, then $f \in \Gamma_\Omega(\mathcal{A}_M)_x$ in the right of (2.10).)

This gives a large class of exact sequences extending (2.8), (2.9).

3. - EXISTENCE OF ANALYTIC SOLUTIONS AT THE BOUNDARY

Let M be a real analytic manifold of dimension n and X a complexification of M. Let $p \in T^*_M X$ and $x = \pi(p)$; using the natural embeddings:

$$T^*_M X \underset{M}{\times} T^*M \hookrightarrow T^*X \underset{X}{\times} T^*X \hookrightarrow T^*T^*X,$$

$T^*_p M$ is identified to a submanifold of $T^*_p T^*X$. T^*T^*X is in turns identified to TT^*X by means of $-H$, H being the Hamiltonian isomorphism.

Let Ω be an open subset of M with analytic boundary (Ω on one side of $\partial\Omega$), let $x \in \partial\Omega$, and let θ be the exterior conormal to Ω at x. Let $P = P(x, D)$ be a germ of (micro)differential operator at p, $p \in \pi^{-1}(x)$, with holomorphic coefficients, and let char P denote its characteristic variety. In [6] (and [8]) the following result was stated:

LEMMA 3.1. *Assume that:*

$$(3.1)\qquad -H(\theta) \notin C_p(\operatorname{char} P, \bar{\Omega} \times_M T^*_M X).$$

Then P is an isomorphism of $\mathbf{R}\Gamma_{\pi^{-1}(M \setminus \Omega)} \mathcal{C}_{\Omega|X}$ in a neighborhood of p.

Let Z be a closed cone of $\bar{\Omega} \times_M T^*_M X$ with $Z \times_M \Omega \subset T^*_X X$. By (2.2) and by the results of § 2, we have an exact sequence:

$$(3.2)\qquad 0 \to \tilde{\mathcal{A}}_\Omega \to \pi_* \Gamma_Z((\mathcal{C}_{\Omega|X})_{T^*_M X}) \to \dot{\pi}_* \Gamma_{\dot{Z}}((\mathcal{C}_{\Omega|X})_{T^*_M X}) \to 0 .$$

We also have:

$$(3.3)\qquad \Gamma_{\dot{Z}}((\mathcal{C}_{\Omega|X})_{T^*_M X}) = \Gamma_{\dot{Z}}(\Gamma_{\pi^{-1}(M \setminus \Omega)}(\mathcal{C}_{\Omega|X})_{T^*_M X}) = \Gamma_{\dot{Z}}((\Gamma_{\pi^{-1}(M \setminus \Omega)} \mathcal{C}_{\Omega|X})_{T^*_M X}),$$

due to $\operatorname{supp} \mathcal{C}_{\Omega|X} \subset \pi^{-1}(M \setminus \Omega) \cup T^*_M X$.

Thus if P is a differential operator at x which induces an isomorphism of $\Gamma_{\pi^{-1}(M \setminus \Omega)} \mathcal{C}_{\Omega|X}$ in a neighborhood of $\dot{Z} \cap \dot{\pi}^{-1}(x)$, we deduce from (3.2) and (3.3) that P is an isomorphism of $\left(\pi_* \Gamma_Z((\mathcal{C}_{\Omega|X})_{T^*_M X}) / \tilde{\mathcal{A}}_\Omega\right)_x$. On the other hand we note that:

$$(3.4)\qquad \pi_* \Gamma_Z((\mathcal{C}_{\Omega|X})_{T^*_M X})_x = \{f \in \Gamma_\Omega(\mathcal{A}_M)_x : SS_\Omega(f) \subset Z\} .$$

On account of (3.4) and Lemma 3.1 we have then established the following:

THEOREM 3.2. *Let Ω be an open set of M with analytic boundary, x a point of $\partial\Omega$, θ the exterior conormal to Ω at x, $P(x, D)$ a differential operator at x, Z a closed cone of $\bar{\Omega} \times_M T^*_M X$* verifying $\Omega \times_M Z \subset T^*_X X$. *Assume that* (3.1) *is fulfilled* $\forall p \in \dot{Z} \cap \dot{\pi}^{-1}(x)$. *Then P is an isomorphism of* $\{f \in \Gamma_\Omega(\mathcal{A}_M)_x : SS_\Omega(f) \subset Z\} / (\tilde{\mathcal{A}}_\Omega)_x$.

To get good existence criteria we need now:

THEOREM 3.3. *In the situation of Theorem* 3.3, *assume, instead of* (3.1):

$$(3.5)\qquad \overline{\operatorname{char} P \cap (SS\, \mathbf{Z}_\Omega \setminus T^*_M X)} \cap (U \times_M T^*_M X) \subset T^*_X X ,$$

for a neighborhood U of x.

Then:

$$(3.6)\qquad P((\tilde{\mathcal{A}}_\Omega)_x) = (\tilde{\mathcal{A}}_\Omega)_x .$$

PROOF. Let $N^*(\Omega)$ be the conormal cone to Ω; we recall that:

$$\operatorname{supp} \mathcal{C}_{\Omega|X} \subset SS\, \mathbf{Z}_\Omega = T^*_M X \oplus_M N^*(\Omega)^a ,$$

(where «a» denotes the antipodal). Let $L = \mathrm{SS}\, \boldsymbol{Z}_\Omega \setminus T^*_M X$ and consider the triangle:

$$(3.7) \qquad (\mathcal{C}_{\Omega|X})_L \to \mathcal{C}_{\Omega|X} \to (\mathcal{C}_{\Omega|X})_{T^*_M X} \xrightarrow{+1} .$$

Applying the functor $\boldsymbol{R}\pi_*(\cdot)$ to (3.7), and using (1.1) and (1.9) we obtain:

$$(3.8) \qquad \boldsymbol{R}\pi_*(\mathcal{C}_{\Omega|X})_L = 0 .$$

On account of the triangle:

$$(3.9) \qquad \boldsymbol{R}\Gamma_{T^*_Y X}(\mathcal{C}_{\Omega|X})_L \to \boldsymbol{R}\pi_*(\mathcal{C}_{\Omega|X})_L \to \boldsymbol{R}\dot{\pi}_*(\mathcal{C}_{\Omega|X})_L \xrightarrow{+1} ,$$

we then obtain from (3.8):

$$(3.10) \qquad \boldsymbol{R}\Gamma_{T^*_Y X}(\mathcal{C}_{\Omega|X})_L = \boldsymbol{R}\dot{\pi}_*(\mathcal{C}_{\Omega|X})_L[-1] .$$

Finally applying $\boldsymbol{R}\Gamma_{T^*_Y X}(\cdot)$ to (3.7) and using (1.6), (1.8), (3.10), we obtain an exact sequence:

$$(3.11) \qquad 0 \to \mathcal{A}_\Omega \to \tilde{\mathcal{A}}_\Omega \to H^0(\boldsymbol{R}\dot{\pi}_*(\mathcal{C}_{\Omega|X})_L) \to 0 .$$

We take now full advantage of the language of derived categories. We denote by $\mathcal{D}_X$ the sheaf of differential operators with holomorphic coefficients, set $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X P$, and apply the functor $\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \cdot)$ to (3.11).

We first note that Y (the complexification of $N = \partial\Omega$) is non-characteristic for $\mathcal{M}$, due to (3.5). It follows: $\mathcal{E}xt^1_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_\Omega) = 0$. Therefore (3.6) will be a consequence of:

$$(3.12) \qquad H^1\big(\boldsymbol{R}\dot{\pi}_*(\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{\Omega|X}))_L\big) = 0 .$$

To prove (3.12) we notice that because of (1.2) and (3.5) we have:

$$\mathrm{supp}\,\big((\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{\Omega|X}))_L\big) \cap (U \underset{M}{\times} T^*_M X) = \emptyset ,$$

and therefore:

$$(3.13) \quad \boldsymbol{R}\dot{\pi}_*\big((\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{\Omega|X}))_L\big)_x = \boldsymbol{R}(\dot{\pi}|_L)_*(\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{\Omega|X})|_L)_x .$$

We also notice that $\boldsymbol{R}(\dot{\pi}|_L)_*(\mathcal{C}_{\Omega|X}|_L)$ is concentrated in degree -1 since

$$\mathcal{C}_{\Omega|X}|_L = \mathcal{C}_{N|X}[+1] \quad \text{and} \quad H^j(U \underset{M}{\times} L, \mathcal{C}_{N|X}) = 0 \quad \forall j \geq 1 \text{ (by Theorem 2.2)} .$$

This gives (3.12) and completes the proof of the theorem.

Corollary 3.4. *Let P verify (3.5), let $f \in \Gamma_\Omega(\mathcal{A}_M)_x$, and assume (3.1) fulfilled $\forall p \in \dot{Z} = SS_\Omega(f) \cap \dot{\pi}^{-1}(x)$. Then there is a solution $u \in \Gamma_\Omega(\mathcal{A}_M)_x$, with $SS_\Omega(u) \subset SS_\Omega(f)$, of the equation $Pu = f$. Such u is unique modulo $(\tilde{\mathcal{A}}_\Omega)_x$.*

We note now that if (3.1) holds $\forall p \in \dot{Z} = \dot{\pi}^{-1}(x) \cap T^*_M X$, then (3.5) is automatically fulfilled. We have therefore:

Corollary 3.5. *Let P verify (3.1) $\forall p \in \dot{Z} = \dot{\pi}^{-1}(x) \cap T^*_M X$. Then:*

$$P(\Gamma_\Omega(\mathcal{A}_M)_x) = \Gamma_\Omega(\mathcal{A}_M)_x . \tag{3.14}$$

Remarks 3.6. *a*) The conclusion of Corollary 3.4 holds, without the additional hypothesis (3.5), when P has constant coefficients and Ω is convex in $\boldsymbol{R}^n$. In fact owing to the existence theorem on convex regions by Ehrenpreis-Malgrange, (3.6) is automatically verified in such situation.

b) When $\dot{\pi}^{-1}(x) \cap T^*_M X \cap \operatorname{char} P = \emptyset$, then Corollary 3.5 is equivalent to the well-known theorem of existence of analytic solutions to elliptic equations.

c) Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module (i.e. a differential system with holomorphic coefficients) and assume Y non-characteristic for $\mathcal{M}$. Let k be the length of a free projective resolution of $\mathcal{M}$. If we suppose that $\mathcal{M}$ verifies (3.1) $\forall p \in \dot{\pi}^{-1}(x) \cap T^*_M X$, then we obtain, by adapting the proofs of Theorems 3.2 and 3.3:

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{A}}_\Omega)_x \simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Gamma_\Omega(\mathcal{A}_M))_x ; \tag{3.15}$$

$$\mathcal{E}xt^k_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_{\bar{\Omega}})_x \to \mathcal{E}xt^k_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{A}}_\Omega)_x \text{ surjective} .$$

In particular we have $\mathcal{E}xt^k_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{A}}_\Omega)_x = 0$ when $k = n$, due to the Cauchy-Kowalewsky-Kashiwara theorem. However, apart from the constant coefficients case, the cohomology of the system $\mathcal{M}$ with values in $\tilde{\mathcal{A}}_\Omega$ does not vanish, generally, in any degree j, $1 \leq j \leq n$.

Example. Let $M = (\boldsymbol{C}^n)^R \simeq \boldsymbol{R}^{2n}$, let $\mathcal{M}$ be the Cauchy-Riemann system $\mathcal{O}_{\bar{\boldsymbol{C}}^n}$ on M, and let $\Omega = M \setminus \Omega'$ for $\Omega' \subset M$ strictly pseudoconvex and with analytic boundary.

Thus the hypotheses of Corollary 3.5 are fulfilled (as $\mathcal{M}$ is elliptic), but we do not have $\mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}, \Gamma_\Omega(\mathcal{A}_M))_x = 0$, $\forall j \geq 1$, $x \in \partial\Omega$.

4. - Applications

We choose local coordinates $(z, \zeta) \in T^*X$, $z = x + iy$, $\zeta = \xi + i\eta$, and assume:

$$X = \boldsymbol{C} \times X', \qquad M = \boldsymbol{R} \times M', \qquad \Omega = \boldsymbol{R}^+ \times M'.$$

We also write $z = (z_1, z')$, $\zeta = (\zeta_1, \zeta')$. We note that the exterior conormal to Ω is $\theta = -dx_1$ and that $-H(-dx_1) = -\partial/\partial\xi_1$. Let $P = P(x, D)$ be a differential operator at x_0, let P_m denote the principal part of P, and let $p = (x_0, i\eta_0) \in \partial\Omega \underset{M}{\times} T^*_M X$, $|\eta_0| = 1$.

Then (3.1) is equivalent to:

$$(4.1) \qquad P_m \neq 0 \text{ if } \xi_1 < -c[(|y| + Y(-x_1)|x_1|)|\eta| + |\xi'|],\ |(z, \zeta/|\zeta|) - p| < \varepsilon,$$

for suitable $c > 0$, $\varepsilon > 0$. (Here Y denotes the Heaviside function.)

In particular (4.1) holds $\forall p \in \pi^{-1}(x_0) \cap T^*_M X$ iff for suitable c, c', ε:

$$(4.2) \qquad P_m \neq 0 \text{ for } -c'|\eta| < \xi_1 < -c[(|y| + Y(-x_1)|x_1|)|\eta| + |\xi'|],\ |z - x_0| < \varepsilon.$$

On the other hand (3.5) is equivalent to:

$$(4.3) \qquad P_m \neq 0 \text{ for } -c'|\eta| < \xi_1 < 0, \qquad x_1 = y = \xi' = 0,\ |x' - x_0| < \varepsilon,$$

for suitable c', ε.

By applying the Bochner's local tube theorem to $1/P_m$ one immediately proves that (4.3) is equivalent to:

$$(4.4) \quad P_m \neq 0 \text{ for } -c'|\eta| < \xi_1 < -c[|y'||\eta| + |\xi'|],\quad x_1 = y_1 = 0,\ |z - x_0| < \varepsilon,$$

with new c, c', ε.

By similar argument one also obtains the following refinement of Proposition 5.3 of [6]:

Proposition 4.1. *Let $P_m(z_1, z'; \zeta) = f(z_1^k, z'; \zeta)$ ($k \geq 1$ integer) for f holomorphic at p and verifying:*

$$(4.5) \qquad f \neq 0 \qquad \text{for } \xi_1 < 0,\ y = \xi' = 0,\ |(z, \zeta) - p| < \varepsilon,\ x_1 \geq 0.$$

Then for a new ε and c:

$$(4.6)\qquad P_m \neq 0 \quad \text{for } \xi_1 < -c\big[(|y_1|^{k/2} + Y(x_1)\inf(1, |x_1|^{(k-2k')/2})|y_1| + Y(-x_1)|x_1|^{k/2} + |y'|)|\eta| + |\xi'|\big], \qquad |(\xi, \zeta/|\zeta|) - p| < \varepsilon.$$

(Note that (4.6) obviously implies (4.1) when $k \geq 2$.)

Example (cf [6]). Let:

$$M = \mathbf{R}^n \ni (x_1, x'), \quad \Omega = \mathbf{R}^+ \times \mathbf{R}^{n-1}, \quad P_m = D_1^2 - (x_1^2 + x'^{2k'}) D'^2, \qquad k \geq 2, \ k' \geq 0.$$

By Proposition 4.1, (4.2) is fulfilled at $x_0 = 0$ and then by Corollary 3.5 we get:

$$P(\Gamma_\Omega(\mathcal{A}_M))_{x_0} = \Gamma_\Omega(\mathcal{A}_M)_{x_0}.$$

We give now an application of the Phragmén-Lindelöf-Hörmander method [1] to exhibit a large class of operators P for which microhyperbolicity (i.e. Condition (4.1)) is also necessary for existence of analytic solutions. Let $w \in \dot{\mathbf{R}}^n$, $|w| = 1$, let Ω be the open half space of $\mathbf{R}^n$ defined by $x \cdot w < 0$, let $x_0 \in \partial\Omega$, and let $P = P(D)$ be a differential operator with constant coefficients. Let K be the $(n-1)$-dimensional closed unit sphere orthogonal to w, let $K(t) = K \times \{sw : s \in \mathbf{R}, |s| \leq t\}$, and let $H_K(\cdot)$ be the support function of K. We also let $V = \{\zeta \in \mathbf{C}^n : P_m(\zeta) = 0\}$ and denote by $V_{i\eta,\varepsilon}$ the ε-neighborhood of $i\eta$ on V.

One can then prove the following variant of the results of [1] (cf [9]):

Proposition 4.2. *In the above situation, assume:*

$$(4.7)\qquad P(\Gamma_\Omega(\mathcal{A}_M))_{x_0} = \Gamma_\Omega(\mathcal{A}_M)_{x_0}.$$

Then $\forall i\eta \in V \cap \dot{T}^*_M X$ and $\forall t$, $0 < t \leq 1$, there exist a compact convex subset $K' \subset\subset \Omega$ and constants δ, r_0 and c such that $\forall r \leq r_0$ the following implication holds on the class of weakly plurisubharmonic functions φ on V:

$$(4.8)\qquad \begin{cases} \varphi(\zeta) \leq H_K(\operatorname{Re}\zeta) + \delta r & \forall \zeta \in V_{i\eta, r}, \\ \varphi(\zeta) \leq H_{K(t)}(\operatorname{Re}\zeta) & \forall \zeta \in V_{i\eta, r}, \end{cases}$$

implies:

$$(4.9)\qquad \varphi(\zeta) \leq H_{-tw+K'}(\operatorname{Re}\zeta) \qquad \forall \zeta \in V_{i\eta, cr}.$$

We say that P is *microhyperbolic* at $i\eta$ to $\pm w\,dx$ iff $\pm w(\partial/\partial\xi) \notin C_{i\eta}(V, i\mathbf{R}^n)$; we say that it is *non-microcharacteristic* iff $\pm w(\partial/\partial\xi) \notin C_{i\eta}(V, \{i\eta\})$.

THEOREM 4.3. *Let $\Omega = \{x \cdot w < 0\}$, let $x_0 \in \partial\Omega$, and assume P non-microcharacteristic to $\pm w\, dx$ at a characteristic $i\eta$ of multiplicity ≤ 2. If (4.7) holds then P is microhyperbolic to $\pm w\, dx$.*

PROOF. If (4.7) is fulfilled then we know that $\forall t$, $0 < t < 1$, and for a suitable $K' = K'_t$ the implication (4.8) $\Rightarrow$ (4.9) is satisfied. Then reasoning as in [9] we can see that for a suitably small $t \neq 0$ we have $0 < H_{K'_t}(-w)$ which is a contradiction due to $K'_t \subset\subset \Omega$.

EXAMPLE. Let

$$M = \mathbf{R}^n, \quad \Omega = \{x_1 < 0\}, \quad x_0 = 0, \quad w = (1, 0, \dots), \quad \eta = (0, \dots, 0, 1),$$

$$\zeta = (\zeta_1, \zeta', \zeta'', \zeta_n) \in \mathbf{C} \times \mathbf{C}^r \times \mathbf{C}^s \times \mathbf{C} = \mathbf{C}^n, \qquad r \neq 0, \ s \neq 0.$$

Then for any one of the polynomials $P(D)$:

$$D_1^2 + D'^2, \qquad D_1^2 - D'^2 + D''^2, \qquad (D_1^2 - D'^2) D_n^2 + D''^4,$$

we know, on account of Theorem 4.3, that (4.7) is not fulfilled.

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