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On a Mathematical Model Describing Interaction Between Two Classical Gravitational Fields (**) (***)

SUMMARY. — A mathematical model is studied which describes the interaction between a classical gravitational field generated by a distributed mass (for instance a gas) and one due to concentrated masses (for instance solid bodies moving through the gas). A global existence theorem is proved for the solution of a system of variational equations and inequalities which can be associated in a natural way to such a model.

Su un modello matematico che descrive l'interazione tra due campi gravitazionali classici

SUNTO. — Si studia un modello matematico che descrive l'interazione di un campo gravitazionale classico prodotto da masse distribuite (ad esempio un gas) con uno prodotto da masse concentrate (ad esempio corpi solidi in moto attraverso il gas). Si dimostra un teorema di esistenza di una soluzione in grande per un sistema di equazioni e disequazioni variazionali che può associarsi in modo naturale a tale modello.

1. - INTRODUCTION

In the present paper we consider a problem connected with the interaction of two classical gravitational fields, produced respectively by concentrated and distributed masses. An example of a problem of this kind can be found in astrophysics, when studying the motion of stars in an interstellar gas assuming that the temperature variations and the influence of the magnetic field are negligible (see, for instance, [1]).

Consider a material body of mass m represented by a sphere $s(\mathbf{y})$ of radius δ and boundary $\sigma(\mathbf{y})$, centered at the point $\mathbf{y}(t) = \{y_1(t), y_2(t), y_3(t)\} \in R^3$, im-

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mersed in a gas which occupies a fixed region $\Omega \subset R^3$ with boundary Γ; we shall assume that the gas is ideal and that its motion is isentropic, with constant specific heat, i.e. that the pressure p depends on the density through the formula $p = N\varrho^\gamma$ ($N, \gamma > 1$ constants).

Denoting by $\boldsymbol{u}$ the velocity of the gas and by Φ and Ψ respectively the potentials generated by the mass m and the gas, the unknown functions of our problem $\boldsymbol{u}$, ϱ, $\boldsymbol{y}$, Φ, Ψ must satisfy the following constitutive equations:

$$(1.1)\qquad \varrho\frac{\partial \boldsymbol{u}(x,t)}{\partial t} + \varrho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \mu\Delta\boldsymbol{u} - \xi\nabla(\nabla\cdot\boldsymbol{u}) + N\nabla\varrho^\gamma = -\varrho\nabla(\Phi+\Psi),$$

$$(1.2)\qquad \frac{\partial\varrho(x,t)}{\partial t} + \nabla(\varrho\boldsymbol{u}) = 0 \qquad (x \in \Omega - s(\boldsymbol{y}(t));\ t \in [0,T]),$$

$$(1.3)\qquad \frac{d^2\boldsymbol{y}(t)}{dt^2} = -\nabla\Psi(\boldsymbol{y}(t),t) \qquad (t\in[0,T]),$$

$$(1.4)\qquad \Psi(x,t) = -\frac{1}{4\pi G}\int_\Omega \frac{\varrho(\xi,t)}{|x-\xi|}\,d\xi \qquad (x\in R^3;\ t\in[0,T]),$$

$$(1.5)\qquad \Phi(x,t) = -\frac{3m}{4\pi G\delta^3}\int_{s(\boldsymbol{y}(t))}\frac{d\xi}{|x-\xi|} \qquad (x\in R^3 - s(\boldsymbol{y}(t));\ t\in[0,T]).$$

Observe that (1.1) represents the classical Navier-Stokes equations, in which μ and ξ represent the shear and dilatational viscosity coefficients (which we assume to be constant) (¹); together with (1.2), which represents the principle of conservation of mass, it describes the motion of a viscous, compressible fluid subject to the potentials Φ and Ψ. Equation (1.3) expresses the law of motion of the material body, while (1.4), (1.5) define the gravitational potentials generated respectively in R^3 by the gas and in $R^3 - s$ by the mass m (G being the universal gravitational constant).

To equations (1.1)-(1.5) must be added the obvious initial conditions

$$(1.6)\qquad \boldsymbol{u}(x,0) = \boldsymbol{u}_0(x) \qquad \big(x\in\Omega - s(\boldsymbol{y}(0))\big),$$

$$(1.7)\qquad \varrho(x,0) = \varrho_0(x) \qquad \big(x\in\Omega - s(\boldsymbol{y}(0))\big),$$

$$(1.8)\qquad \boldsymbol{y}(0) = \boldsymbol{y}_0, \qquad \boldsymbol{y}'(0) = \boldsymbol{y}'_0$$

and the boundary conditions

$$(1.9)\qquad u_r(x,t) = \frac{\partial u_\tau(x,t)}{\partial\nu} = 0 \qquad (x\in\Gamma,\ t\in[0,T]),$$

$$(1.10)\qquad \boldsymbol{u}(x,t) = \boldsymbol{y}'(t) \qquad \big(x\in\sigma(\boldsymbol{y}(t)),\ t\in[0,T]\big),$$

(¹) In the cases considered in [1], μ and ξ are assumed to be so small that the corresponding terms can be neglected.

where ν is the exterior normal to Γ, u_τ, u_ν the tangential and normal components of the velocity.

Relations (1.9) interpret the condition that no gas passes through Γ and that, since no gas is present outside Ω, no drag is exercised from outside on particles moving along Γ. (1.10) finally represents the no-slip condition on σ. The problem considered above is an initial-boundary value problem in which a portion of the boundary (i.e. σ) depends on an unknown function (i.e. $\boldsymbol{y}$).

We shall in what follows consider a simplified problem, assuming that Ω is a sphere, $\boldsymbol{u}$ and ϱ can be defined in the whole of Ω and, consequently, that (1.5) holds in the whole of R^3 and (1.1), (1.2) hold in Ω; hence condition (1.10) is eliminated. This simplification appears justified provided we assume that m is a «nearly pointwise» mass. Hence, the system of equations we shall consider is constituted by (1.1)-(1.5) ((1.1), (1.2) in $\Omega\times[0, T]$, (1.4), (1.5) in $R^3\times[0, T]$) with the initial conditions (1.6), (1.7), (1.8) and the boundary conditions (1.9).

Equations (1.1)-(1.5) will always be intended in the sense of the theory of distributions, assuming that

$$(1.11)\qquad \begin{cases} \boldsymbol{u}(t) = \{\boldsymbol{u}(x, t), x\in\Omega\} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega)) \cap \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cap H^1(0, T; L^2) \\ \varrho(t) = \{\varrho(x, t), x\in\Omega\} \in L^2(0, T; H^3) \cap H^1(0, T; L^2)\,, \end{cases}$$

while $\boldsymbol{y}(t)$, $\Psi(x, t)$, $\Phi(x, t)$ belong to appropriate functional spaces determined by (1.3), (1.4), (1.5).

Equations (1.1)-(1.5) are not however the only relationships that are related to the problem in question; the deduction of (1.1), (1.2) from the general principles of conservation of mass and momentum is in fact rigourous only if certain assumptions on the physical quantities $\boldsymbol{u}$ and ϱ are fulfilled, i.e. only when $\boldsymbol{u}$ and ϱ satisfy certain consistency conditions. These can, in the present case, be expressed as follows

i) The velocity $|\boldsymbol{u}|$ must be non-relativistic:

$$(1.12)\qquad |\boldsymbol{u}| < M_1\,.$$

ii) The density must not be either «too large» or «too small», since (1.2) does not hold for solids or for extremely rarified gases:

$$(1.13)\qquad 0 < \alpha_1 < \varrho < \alpha_2\,.$$

iii) The «shock zones» have, in the propagation processes of viscous fluids, a width of the same order of magnitude as the mean free path of the molecules (see, for instance, [2]); hence, being ϱ bounded

$$(1.14)\qquad |\nabla p| < M_2 \Rightarrow |\nabla\varrho| < M_3\,.$$

It is obvious that the eventual solutions of (1.1)-(1.9) will have a physical meaning (i.e. be physically consistent) in $Q_{\bar{t}} = \Omega \times (0, \bar{t})$ provided the consistency conditions (1.12), (1.13), (1.14) are also satisfied in $Q_{\bar{t}}$; if, in particular, (1.12), (1.13), (1.14) were not satisfied for any $\bar{t} > 0$, we would have to conclude that our model, at least for the data considered, is physically inconsistent.

We shall therefore be interested in the study of (1.1)-(1.9) only where its solutions are physically consistent. Assume now that $\boldsymbol{u}$, ϱ are physically consistent in $Q_{\bar{t}}$; it follows then from (1.2), (1.12), (1.13), (1.14)

$$\|\varrho(t)\|_{H^1(0,\bar{t};L^2(\Omega))} \leq c\,. \tag{1.15}$$

Hence, setting

$$\bar{\varrho}(t) = \begin{cases} \varrho(t) & \text{when } 0 < t < t', \\ \varrho(t') & \text{when } t' < t < T, \end{cases} \tag{1.16}$$

with

$$t' = \max\{t : \|\varrho\|_{H^1(0,t;L^2(\Omega))} \leq c\} \qquad (\Rightarrow \bar{\varrho}(t) = \varrho_0 \text{ if } t' = 0)\,,$$

we may, for what has been said above, substitute equations (1.1), (1.2), (1.4) with

$$\bar{\varrho}\frac{\partial \boldsymbol{u}(x,t)}{\partial t} + \bar{\varrho}(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \mu\Delta\boldsymbol{u} - \xi\nabla(\nabla\cdot\boldsymbol{u}) + N\nabla\bar{\varrho}^{\gamma} = -\bar{\varrho}\nabla(\Phi + \Psi)\,, \tag{1.1'}$$

$$\frac{\partial\varrho(x,t)}{\partial t} + \nabla(\bar{\varrho}\boldsymbol{u}) = 0\,, \tag{1.2'}$$

$$\Psi(x,t) = -\frac{1}{4\pi G}\int_{\Omega}\frac{\bar{\varrho}(\xi,t)}{|x-\xi|}\,d\xi\,. \tag{1.4'}$$

Let us now introduce the following differential inequalities, associated to (1.1'), (1.2')

$$\int_0^t d\eta \int_{\Omega}\Big(\frac{\partial\boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \frac{\mu}{\bar{\varrho}}\Delta\boldsymbol{u} - \frac{\xi}{\bar{\varrho}}\nabla(\nabla\cdot\boldsymbol{u}) + \frac{N\nabla\bar{\varrho}^{\gamma}}{\bar{\varrho}} + \nabla(\Phi+\Psi)\Big)(\boldsymbol{u}-\boldsymbol{h})\,d\Omega \leq 0\,, \tag{1.1''}$$

$$\int_0^t d\eta \int_{\Omega}\Big(\frac{\partial\varrho}{\partial t} + \nabla\cdot(\bar{\varrho}\boldsymbol{u})\Big)(\varrho - l)\,d\Omega \leq 0\,, \qquad 0 < t < T\,, \tag{1.2''}$$

where the unknown functions $\boldsymbol{u}$, ϱ and the test functions $\boldsymbol{h}$, l belong to the functional spaces defined by (1.11) and

$$\boldsymbol{u}, \boldsymbol{h} \in K_1 = \{\boldsymbol{v} \in L^2(Q_T);\ |\boldsymbol{v}| \leq M_1 \text{ a.e.}\}\,,$$

$$\varrho, l \in K_2 = \{\zeta \in L^2(Q_T);\ 0 < \alpha_1 \leq \zeta \leq \alpha_2,\ |\nabla\zeta| \leq M_2 \text{ a.e.}\}\,.$$

The following fundamental proposition can be proved by standard procedures (see, for example, [3], [4]).

Assume that $\boldsymbol{u}$, ϱ satisfy (1.1''), (1.2'') and the consistency conditions (1.12), (1.13), (1.14) a.e. in $Q_{\bar{t}}$; then $\boldsymbol{u}$, ϱ satisfy also a.e. in $Q_{\bar{t}}$ equations (1.1'), (1.2') (and also, consequently, (1.1), (1.2)) (2).

Assume therefore that a global existence theorem holds in Q_T for the solution of (1.1''), (1.2''), (1.3), (1.4'), (1.5), (1.6), (1.7), (1.8), (1.9) and that there exists $\bar{t} > 0$ such that (1.12), (1.13), (1.14) hold in $Q_{\bar{t}}$; then there exists in $Q_{\bar{t}}$ a solution of (1.1)-(1.9).

Hence, from a global existence theorem of the system with inequalities we can deduce either

i) a global existence theorem for the original system, if $\bar{t} = T$;

ii) a local existence theorem for the original system if $0 < \bar{t} < T$;

iii) the conclusion that the model is not physically consistent if $\bar{t} = 0$, i.e. the solutions of (1.1)-(1.9), if they exist, have no physical meaning.

It should be noted that the case ii) differs significantly from the usual local existence theorems which may eventually be proved for (1.1)-(1.9). The normal procedure consists in fact in proving local existence theorems through a priori estimates which hold when $0 < t < t'$, where t' depends also on quantities, such as embedding constants, etc. which do not have a physical interpretation.

In our case, on the other hand, $\bar{t}$ has a precise physical meaning since, when $t > \bar{t}$, the solutions, even if they exist, are no longer physically consistent.

In the present paper we shall precisely give an existence theorem for the global solutions of (1.1''), (1.2''), (1.3), (1.4'), (1.5), (1.6), (1.7), (1.8), (1.9), from which one of the three propositions indicated above can be deduced.

The scheme of the work is as follows. In § 2 we shall give some basic notations and definitions; in § 3 some preliminary existence and uniqueness theorems will be proved, while in § 4 the global existence theorem mentioned earlier in this paragraph will be given.

2. - Basic notations and definitions

Let Ω be a sphere of R^3, with boundary Γ. Denoting by v_1, v_2, v_3 the components of a vector $\boldsymbol{v}$ defined in Ω, we shall introduce the following notations:

$$\mathcal{V} = \left\{\boldsymbol{v} : v_i \in C^\infty(\bar{\Omega});\ v_i = \frac{\partial v_i}{\partial \nu} = 0 \quad \text{in a neighbourhood of } \Gamma\ (i = 1, 2, 3)\right\};$$

$$V^{s,p} = \text{closure of } \mathcal{V} \text{ in } W^{s,p}(\Omega)\,; \qquad V^{s,2} = V^s \qquad (s, p > 0)\,;$$

(2) In fact, it can be deduced from (1.2'), analogously to (1.15) that, if (1.12), (1.13), (1.14) hold in $[0, \bar{t}]$, then $\|\dot{\varrho}(t)\|_{H^1(0,\bar{t},\,L^2(\Omega))} \leqslant \varepsilon$.

$$H^s = W^{s,2}(\Omega)\,; \qquad H^0 = L^2(\Omega)\,;$$

$$(V^{s,2})' = \text{dual of } V^{s,2}, \text{ with } (H^s)' = H^{-s}\,;\ (\mathbf{u}, \mathbf{v})_{V^s} = (\mathbf{u}, \mathbf{v})_{H^s}\,;$$

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega \sum_{i,j=1}^{3} \frac{\partial u_j}{\partial x_i}\frac{\partial v_j}{\partial x_i}\, d\Omega \qquad a(\mathbf{u}, \mathbf{u}) = \|\mathbf{u}\|^2_{V^1}\,.$$

Observe that $\mathbf{v} \in \mathcal{V} \Rightarrow \partial\mathbf{v}(P)/\partial\mathbf{v} = 0$, $\forall P$ in a neighbourhood of Γ; in fact, we have

$$\frac{\partial\mathbf{v}(P)}{\partial\mathbf{v}} = \frac{\partial\mathbf{v}_r(P)}{\partial\nu} + \frac{\partial\mathbf{v}_t(P)}{\partial\mathbf{v}} = \frac{\partial\mathbf{v}_r(P)}{\partial\mathbf{v}} + \lim_{\Delta\mathbf{v}\to 0} \frac{\mathbf{v}(P+\Delta\mathbf{v}) - \mathbf{v}(P)}{\Delta\mathbf{v}} = 0\,.$$

We shall say that the functions $\mathbf{u}(x,t)$, $\varrho(x,t)$, $\Phi(x,t)$, $\Psi(x,t)$, $\mathbf{y}(t)$ (the first two defined in Q_T, the third and fourth in $R^3 \times (0,T)$, the last in $(0,T)$) *represent a solution of the system* (1.1″), (1.2″), (1.3), (1.4′), (1.5) *satisfying the initial and boundary conditions* (1.6), (1.7), (1.8), (1.9) *if:*

i) $\mathbf{u}(t) \in L^\infty(0,T; V^1 \cap K_1) \cap L^2(0,T; V^2) \cap H^{1/2-\varepsilon}(0,T; (V^\alpha)')$

$$(\varepsilon > 0,\ \alpha > \tfrac{3}{2})\,;$$

$\varrho(t) \in L^\infty(0,T; K_2)\,; \qquad \Psi(t) \in L^\infty(0,T; W^{1,\infty}(R^3))\,;$

$\Phi(t) \in C^2(0,T; C^1(R^3))\,; \quad \mathbf{y}(t) \in H^2(0,T)\,,\ \mathbf{y}(0) = \mathbf{y}_0,\ \mathbf{y}'(0) = \mathbf{y}'_0\,;$

ii) $\mathbf{u}, \varrho, \Phi, \Psi, \mathbf{y}$ *satisfy, a.e. on* $(0,T)$, *the relations*

$$\text{(2.1)} \quad \frac{1}{2}\|\mathbf{u}(t) - \mathbf{h}(t)\|^2_{V^0} + \int_0^t \Big(\mathbf{h}' + \nabla(\Phi + \Psi) + N\frac{\nabla\bar{\varrho}^\gamma}{\bar{\varrho}} - \frac{\mu}{\bar{\varrho}}\Delta\mathbf{u} - \frac{\xi}{\bar{\varrho}}\nabla(\nabla\cdot\mathbf{u}) + (\mathbf{u}\cdot\nabla)\mathbf{u}, \mathbf{u} - \mathbf{h}\Big)_{V^0} d\eta \le \frac{1}{2}\|\mathbf{u}_0 - \mathbf{h}(0)\|^2_{V^0}$$

$$\forall \mathbf{h}(t) \in H^1(0,T; V^0) \cap L^\infty(0,T; K_1)\,;$$

$$\text{(2.2)} \quad \frac{1}{2}\|\varrho(t) - l(t)\|^2_{L^2} + \int_0^t (l' + \nabla\cdot(\bar{\varrho}\mathbf{u}), \varrho - l)_{L^2}\, d\eta \le \frac{1}{2}\|\varrho_0 - l(0)\|^2_{L^2}$$

$$\forall l(t) \in H^1(0,T; L^2) \cap L^\infty(0,T; K_2)\,;$$

$$\text{(2.3)} \quad \frac{d^2\mathbf{y}(t)}{dt^2} = -\nabla\Psi(\mathbf{y}(t), t)\,;$$

$$\text{(2.4)} \quad \Psi(x,t) = -\frac{1}{4\pi G}\int_\Omega \frac{\bar{\varrho}(\xi,t)}{|x-\xi|}\, d\xi\,;$$

$$\text{(2.5)} \quad \Phi(x,t) = -\frac{3m}{4\pi G\delta^3}\int_{s(y(t))} \frac{d\xi}{|x-\xi|}\,.$$

It should be noted that (2.1), (2.2) correspond to the classical weak formulation of inequalities (1.1'), (1.2') and of the corresponding initial and boundary conditions; moreover, as already pointed out in § 1, the functional classes for Ψ, Φ, y are deduced directly from those for $\boldsymbol{u}$, ϱ by means of equations (2.3), (2.4), (2.5).

3. - Some auxiliary theorems

We now prove some auxiliary existence and uniqueness theorems which will be utilized in the proof of the final existence theorem.

Theorem 3.1: *Assume that* $\boldsymbol{u}_0 \in V^1 \cap K_1$, $f(t) \in L^2(0, T; L^2)$, $\beta(t) \in \in L^\infty(0, T; K_2)$. *There exists then an unique function* $\boldsymbol{u}(t)$ *such that*:

i$_1$) $\boldsymbol{u}(t) \in L^\infty(0, T; V^1 \cap K_1) \cap L^2(0, T; V^2) \cap H^{1/2-\varepsilon}(0, T; (V^\alpha)')$

$(\varepsilon > 0, \alpha > \frac{3}{2})$;

ii$_1$) $\boldsymbol{u}(t)$ *satisfies* a.e. *in* $(0, T)$ *the inequality*

$$(3.1)\qquad \frac{1}{2}\|\boldsymbol{u}(t)-\boldsymbol{h}(t)\|^2_{V^0}+\int_0^t\Big(\boldsymbol{h}'-\boldsymbol{f}+N\frac{\nabla\beta^\gamma}{\beta}-\frac{\mu}{\beta}\Delta\boldsymbol{u}-\frac{\xi}{\beta}\nabla(\nabla\cdot\boldsymbol{u})+$$

$$+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u},\boldsymbol{u}-\boldsymbol{h}\Big)_{V^0}d\eta\le\frac{1}{2}\|\boldsymbol{u}_0-\boldsymbol{h}(0)\|^2_{V^0},\quad \forall\boldsymbol{h}(t)\in H^1(0,T;V^\alpha)\cap L^\infty(0,T;K_1).$$

Let $\{\boldsymbol{g}_j\}$ be a basis in $\mathcal{V}$; under the assumptions made we can assume that $\{\boldsymbol{g}_j\}$ is the orthonormal sequence of the eigenfunctions in V^1 of the operator $-\Delta$: $\Delta\boldsymbol{g}_j+\lambda_j\boldsymbol{g}_j=0$.

Setting $\boldsymbol{u}_n(t)=\sum_{j=1}^n \alpha_{jn}(t)\boldsymbol{g}_j$, consider the Faedo-Galerkin system associated to (3.1), containing the penalization term $B(\boldsymbol{u}_n)$,

$$(3.2)\qquad \Big(\boldsymbol{u}_n'(t)-\frac{\mu}{\beta(t)}\Delta\boldsymbol{u}_n(t)-\frac{\xi}{\beta(t)}\nabla(\nabla\cdot\boldsymbol{u}_n(t))+(P\boldsymbol{u}_n(t)\cdot\nabla)\boldsymbol{u}_n(t)+$$

$$+N\frac{\nabla\beta^\gamma(t)}{\beta(t)}-f(t)+nB(\boldsymbol{u}_n(t)),\boldsymbol{g}_j\Big)_{V^0}=0\qquad(j=1,\dots,n),$$

$$(3.3)\qquad \boldsymbol{u}_n(0)=\boldsymbol{u}_{0,n},$$

where P is the projection operator from V^1 to K_1, $\boldsymbol{u}_{0,n}$ is the projection of $\boldsymbol{u}_0$ on the subspace spanned by $\boldsymbol{g}_1,\dots,\boldsymbol{g}_n$ and we have set

$$B(\boldsymbol{v})=\begin{cases}0 & \text{if } |\boldsymbol{v}|\le M_1,\\ \boldsymbol{v}\left(1-\dfrac{M_1}{|\boldsymbol{v}|}\right) & \text{if } |\boldsymbol{v}|>M_1.\end{cases}$$

The differential system (3.2) obviously admits an unique local solution, $\forall$ fixed n, satisfying (3.3). Let us now, following a classical procedure, prove some a priori estimates on $(0, T)$ for $\boldsymbol{u}_n$, which will enable us to obtain a global existence and uniqueness theorem for (3.2), (3.3) and then pass to the limit when $n \to \infty$.

Multiplying (3.2) by $\alpha_{jn}(t)$ and adding we obtain

$$(3.4)\quad \left(\boldsymbol{u}_n' - \frac{\mu}{\beta}\Delta\boldsymbol{u}_n - \frac{\xi}{\beta}\nabla(\nabla\cdot\boldsymbol{u}_n) + (P\boldsymbol{u}_n\cdot\nabla)\boldsymbol{u}_n + N\frac{\nabla\beta^\gamma}{\beta} - \boldsymbol{f} + nB(\boldsymbol{u}_n), \boldsymbol{u}_n\right)_{V^0} = 0 .$$

On the other hand we obtain, bearing in mind that, by the definition of V^1, $\boldsymbol{u}_n\cdot\partial\boldsymbol{u}_n/\partial\nu = 0$ on Γ,

$$(3.5)\quad \left(\frac{\Delta\boldsymbol{u}_n}{\beta}, \boldsymbol{u}_n\right)_{V^0} = -a\left(\boldsymbol{u}_n, \frac{\boldsymbol{u}_n}{\beta}\right) + \int_\Gamma \frac{1}{\beta}\frac{\partial\boldsymbol{u}_n}{\partial\nu}\cdot\boldsymbol{u}_n\, d\Gamma =$$

$$= -\left\|\frac{1}{\sqrt{\beta}}\boldsymbol{u}_n\right\|_{V^1}^2 + \left(\left(\frac{\nabla\beta}{\beta^2}\cdot\nabla\right)\boldsymbol{u}_n, \boldsymbol{u}_n\right)_{V^0} < -\frac{1}{\alpha_1}\|\boldsymbol{u}_n\|_{V^1}^2 + \left(\left(\frac{\nabla\beta}{\beta^2}\cdot\nabla\right)\boldsymbol{u}_n, \boldsymbol{u}_n\right)_{V^0} .$$

Analogously,

$$(3.6)\quad \left(\frac{1}{\beta}\nabla(\nabla\cdot\boldsymbol{u}_n), \boldsymbol{u}_n\right)_{V^0} = -\left(\nabla\cdot\boldsymbol{u}_n, \nabla\cdot\frac{\boldsymbol{u}_n}{\beta}\right)_{V^0} + \int_\Gamma \frac{1}{\beta}\boldsymbol{u}_{n\nu}\nabla\cdot\boldsymbol{u}_n\, d\Gamma =$$

$$= -\left\|\frac{\nabla\cdot\boldsymbol{u}_n}{\sqrt{\beta}}\right\|_{V^0}^2 + \left(\frac{\nabla\beta}{\beta^2}\nabla\cdot\boldsymbol{u}_n, \boldsymbol{u}_n\right)_{V^0} < -\frac{1}{\alpha_1}\|\nabla\cdot\boldsymbol{u}_n\|_{V^0}^2 + \left(\frac{\nabla\beta}{\beta^2}\nabla\cdot\boldsymbol{u}_n, \boldsymbol{u}_n\right)_{V^0} .$$

Since $(B\boldsymbol{u}_n, \boldsymbol{u}_n)_{V^0} > 0$, we obtain from (3.4), (3.5), (3.6)

$$(3.7)\quad \frac{d}{dt}\|\boldsymbol{u}_n(t)\|_{V^0}^2 + \frac{\mu}{\alpha_2}\|\boldsymbol{u}_n(t)\|_{V^1}^2 + \frac{\xi}{\alpha_2}\|\nabla\cdot\boldsymbol{u}_n(t)\|_{V^0}^2 <$$

$$< C_1(\|\nabla\beta(t)\|_{L^\infty}\|\boldsymbol{u}_n(t)\|_{V^1}\|\boldsymbol{u}_n(t)\|_{V^0} + \|\boldsymbol{u}_n(t)\|_{V^1}\|\boldsymbol{u}_n(t)\|_{V^0} +$$

$$+ \|\nabla\beta^\gamma(t)\|_{L^2}\|\boldsymbol{u}_n\|_{V^0} + \|\boldsymbol{f}(t)\|_{L^2}\|\boldsymbol{u}_n(t)\|_{V^0}) .$$

Hence, by Gronwall's lemma and the assumptions made on β and $\boldsymbol{f}$

$$(3.8)\quad \|\boldsymbol{u}_n(t)\|_{L^\infty(0,T;V^0)\cap L^2(0,T;V^1)} < M_4 ,$$

where M_4 does not depend on n.

From (3.8) follows that the local solution $\boldsymbol{u}_n(t)$ of (3.2), (3.3) can be extended to the whole of $(0, T)$.

In order to obtain further a priori estimates, we now multiply (3.2) by $\lambda_j\alpha_{jn}(t)$ and add, obtaining

$$(3.9)\quad \left(\boldsymbol{u}_n' - \frac{\mu}{\beta}\Delta\boldsymbol{u}_n - \frac{\xi}{\beta}\nabla(\nabla\cdot\boldsymbol{u}_n) + (P\boldsymbol{u}_n\cdot\nabla)\boldsymbol{u}_n + N\frac{\nabla\beta^\gamma}{\beta} - \boldsymbol{f} + \right.$$

$$\left. + nB(\boldsymbol{u}_n), \Delta\boldsymbol{u}_n\right)_{V^0} = 0 .$$

We have, on the other hand,

$$(3.10)\qquad \left(\frac{1}{\beta}\nabla(\nabla\cdot\boldsymbol{u}_n), \Delta\boldsymbol{u}_n\right)_{\Gamma^*} = -\left(\frac{\nabla\cdot\boldsymbol{u}_n}{\beta}, \Delta(\nabla\cdot\boldsymbol{u}_n)\right)_{\Gamma^*} + $$
$$+\left(\frac{\nabla\cdot\boldsymbol{u}_n}{\beta^2}, \Delta\boldsymbol{u}_n\cdot\nabla\beta\right)_{\Gamma^*} + \int_\Gamma \frac{\nabla\cdot\boldsymbol{u}_n}{\beta}(\Delta\boldsymbol{u}_n\cdot\boldsymbol{\nu})\,d\Gamma =$$
$$= \left(\frac{1}{\beta}\nabla(\nabla\cdot\boldsymbol{u}_n), \nabla(\nabla\cdot\boldsymbol{u}_n)\right)_{\Gamma^*} - \left(\frac{\nabla\cdot\boldsymbol{u}_n}{\beta^2}\nabla\beta, \nabla(\nabla\cdot\boldsymbol{u}_n)\right)_{\Gamma^*} -$$
$$-\int_\Gamma \frac{\boldsymbol{u}_n}{\beta}\frac{\partial}{\partial\nu}(\nabla\cdot\boldsymbol{u}_n)\,d\Gamma + \left(\frac{\nabla\cdot\boldsymbol{u}_n}{\beta^2}, \Delta\boldsymbol{u}_n\cdot\nabla\beta\right)_{\Gamma^*} + \int_\Gamma \frac{\nabla\cdot\boldsymbol{u}_n}{\beta}(\Delta\boldsymbol{u}_n\cdot\boldsymbol{\nu})\,d\Gamma.$$

Let us show that in (3.10) all the boundary terms vanish. Observe, in fact, first of all, that since, by the definition of the $\boldsymbol{g}_j$'s, $\boldsymbol{g}_j\cdot\boldsymbol{\nu}|_\Gamma = 0$, it is

$$(3.11)\qquad \Delta\boldsymbol{u}_n\cdot\boldsymbol{\nu}|_\Gamma = \sum_{j=1}^{n}\alpha_{jn}\Delta\boldsymbol{g}_j\cdot\boldsymbol{\nu}|_\Gamma = \sum_{j=1}^{n}\alpha_{jn}\lambda_j\boldsymbol{g}\cdot\boldsymbol{\nu}|_\Gamma = 0\,.$$

On the other hand, defining in an obvious way in a neighbourhood of Γ the direction $\boldsymbol{\nu}$, we have

$$(3.12)\qquad \frac{\partial}{\partial\nu}(\nabla\cdot\boldsymbol{u}_n) = \nabla\cdot\frac{\partial\boldsymbol{u}_n}{\partial\nu} = \nabla\cdot\left(\sum_{j=1}^{n}\alpha_{jn}\frac{\partial\boldsymbol{g}_j}{\partial\nu}\right).$$

Since $\boldsymbol{g}_j \in \mathfrak{V}$ it is, as already pointed out,

$$(3.13)\qquad \frac{\partial\boldsymbol{g}_j}{\partial\nu} = 0$$

in a neighbourhood of Γ. Hence, by (3.12), (3.13),

$$(3.14)\qquad \frac{\partial}{\partial\nu}(\nabla\cdot\boldsymbol{u}_n)|_\Gamma = 0\,.$$

It follows then from (3.10) that

$$(3.15)\qquad \left(\frac{1}{\beta}\nabla(\nabla\cdot\boldsymbol{u}_n), \Delta\boldsymbol{u}_n\right)_{\Gamma^*} = \left(\frac{1}{\beta}\nabla(\nabla\cdot\boldsymbol{u}_n), \nabla(\nabla\cdot\boldsymbol{u}_n)\right)_{\Gamma^*} -$$
$$-\left(\frac{\nabla\cdot\boldsymbol{u}_n}{\beta^2}\nabla\beta, \nabla(\nabla\cdot\boldsymbol{u}_n)\right)_{\Gamma^*} + \left(\frac{\nabla\cdot\boldsymbol{u}_n}{\beta^2}, \Delta\boldsymbol{u}_n\cdot\nabla\beta\right)_{\Gamma^*}.$$

Bearing in mind that (since $B' \geq 0$)

$$(3.16)\qquad -(B(\boldsymbol{u}_n), \Delta\boldsymbol{u}_n)_{\Gamma^*} = a(B(\boldsymbol{u}_n), \boldsymbol{u}_n) \geq 0$$

and that, by the assumptions made, $v \in V^1$, $\Delta v \in L^2 \to v \in V^2$, we have then, by (3.9), (3.15), (3.16),

$$(3.17) \qquad \frac{1}{2}\frac{d}{dt}\|u_n(t)\|^2_{V^0} + \frac{\mu}{\alpha_2}\|u_n(t)\|^2_{V^1} + \frac{\xi}{\alpha_2}\|\nabla(\nabla\cdot u_n(t))\|^2_{V^0} < $$
$$< C_2(\|\nabla\beta(t)\|_{L^\infty}\|u_n(t)\|_{V^1}\|u_n(t)\|_{V^0} + \|u_n(t)\|_{V^1}\|u_n(t)\|_{V^0} + $$
$$+ \|\nabla\beta'(t)\|_{L^2}\|u_n(t)\|^2_{V^1} + \|f(t)\|_{L^2}\|u_n(t)\|_{V^0}),$$

with C_2 independent of n.

Hence, by Gronwall's lemma and the assumptions made on β and f,

$$(3.18) \qquad \|u_n(t)\|_{L^\infty(0,T;V^1)\cap L^2(0,T;V^2)} < M_3,$$

where M_3 does not depend on n.

It is then possible to select from $\{u_n\}$ a subsequence (again denoted by $\{u_n\}$) such that

$$(3.19) \qquad \lim_{n\to\infty} u_n(t) = u(t)$$

in the weak topology of $L^2(0, T; V^2)$ and the weak* topology of $L^\infty(0, T; V^1)$.

Moreover, by (3.4), (3.18),

$$(3.20) \qquad \int_0^T (B(u_n(t)), u_n(t))_{V^0}\, dt < \frac{C_3}{n}$$

and, consequently,

$$(3.21) \qquad \lim_{n\to\infty} \int_0^T (B(u_n(t)), u_n(t))_{V^0}\, dt = 0 .$$

From (3.20) it follows immediately, by the definition of B, that $u(t) \in L^2(0, T; K_1)$.

Let now ψ be an arbitrary function of $L^2(0, T; \mathcal{V})$ and denote by ψ_n its projection on the subspace $\mathcal{V}_n$ ($\mathcal{V}_n$ linear manifold spanned by $g_1, \dots, g_n$); by (3.2), (3.18) we have

$$(3.22) \qquad \left|\int_0^T (u_n(t), \Psi(t))_{V^0}\, dt\right| = \left|\int_0^T (u_n(t), \Psi_n(t))_{V^0}\, dt\right| <$$
$$< C_4\|\Psi_n(t)\|_{L^2(0,T;V^0)} + \left|n\int_0^T \big(B(u_n(t)), \Psi_n(t)\big)_{V^0}\, dt\right| .$$

On the other hand, by the definition of B and (3.20)

$$\text{(3.23)} \qquad \frac{C_3}{n} > \int_{Q_T} B(\boldsymbol{u}_n)\cdot \boldsymbol{u}_n \, dQ = \int_{Q_T} |B(\boldsymbol{u}_n)|\, |\boldsymbol{u}_n|\, dQ > M_1 \int_{Q_T} |B(\boldsymbol{u}_n)|\, dQ$$

and, consequently,

$$\text{(3.24)} \qquad \left| n\int_0^T \big(B(\boldsymbol{u}_n(t)), \boldsymbol{\Psi}_n(t)\big)_{V^*} dt \right| < n \|\boldsymbol{\Psi}_n(t)\|_{L^\infty(0,T;L^\infty)} \int_{Q_T} |B(\boldsymbol{u}_n)|\, dQ <$$
$$< C_4 \|\boldsymbol{\Psi}_n(t)\|_{L^\infty(0,T;L^\infty)}.$$

Substituting (3.24) into (3.22), we obtain then, by well known embedding theorems,

$$\text{(3.25)} \qquad \left| \int_0^T (\boldsymbol{u}_n'(t), \boldsymbol{\Psi}(t))_{V^*} dt \right| < C_4 \|\boldsymbol{\Psi}_n(t)\|_{L^2(0,T;V^\alpha)} +$$
$$+ C_5 \|\boldsymbol{\Psi}_n(t)\|_{L^\infty(0,T;L^\infty)} < C_6 \|\boldsymbol{\Psi}(t)\|_{H^{1/2+\varepsilon}(0,T;V^\alpha)}$$

where ε is an arbitrary positive number and $\alpha > \frac{3}{2}$.

Hence, by (3.25),

$$\text{(3.26)} \qquad \|\boldsymbol{u}_n'(t)\|_{H^{-1/2-\varepsilon}(0,T;(V^\alpha)')} < C_6\,,$$

which implies that $\boldsymbol{u}(t) \in H^{1/2-\varepsilon}(0, T; (V^\alpha)')$. Hence $\boldsymbol{u}(t)$ satisfies condition i_3).

It can be noted that from (3.19), (3.26) follows, by well known interpolation theorems, that

$$\text{(3.27)} \qquad \lim_{n\to\infty} \boldsymbol{u}_n = \boldsymbol{u} \qquad \text{a.e. in } Q_T\,.$$

Following now a classical procedure (see, for instance [5], ch. 3 and [6]) it can be shown, passing to the limit in (3.2), that $\boldsymbol{u}(t)$ satisfies, $\forall \boldsymbol{h}(t) \in H^1(0, T; V^\alpha) \cap L^\infty(0, T; K_1)$, the inequality

$$\text{(3.28)} \qquad \frac{1}{2} \|\boldsymbol{u}(t) - \boldsymbol{h}(t)\|_{V^*}^2 + \int_0^t \Big(\boldsymbol{h}' - \boldsymbol{f} + N\frac{\nabla \beta^\gamma}{\beta} - \frac{\mu}{\beta}\Delta \boldsymbol{u} - \frac{\xi}{\beta}\nabla(\nabla\cdot\boldsymbol{u}) +$$
$$+ (P\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{h}\Big)_{V^*} d\eta < \frac{1}{2} \|\boldsymbol{u}_0 - \boldsymbol{h}(0)\|_{V^*}^2.$$

Since, obviously, $P\boldsymbol{u} = \boldsymbol{u}$, (3.28) coincides with (3.1) and the existence of a solution is proved.

The proof of the uniqueness of such a solution is straightforward (see, for instance, again [5] ch. 3 and [7]) and we shall not repeat it here.

THEOREM 3.2: *Assume that $\varrho_0 \in K_2$, $\mathbf{g}(t) \in L^2(0, T; H^1)$. There exists then an unique function $\varrho(t)$ such that*

i$_2$) $\varrho(t) \in L^\infty(0, T; K_2)$;

ii$_2$) $\varrho(t)$ *satisfies a.e. in* $(0, T)$ *the inequality*

$$\tag{3.29} \tfrac{1}{2}\|\varrho(t)-l(t)\|_{L^2}^2+\int_0^t\{(l', \varrho-l)_{L^2}+(\nabla\cdot\mathbf{g}, \varrho-l)_{L^2}\}\,d\eta\leq\tfrac{1}{2}\|\varrho_0-l(0)\|_{L^2}^2, \quad \forall l(t)\in H^1(0, T; L^2)\cap L^\infty(0, T; K_2).$$

Observe, in fact, that, as is well known (see, for instance, [5], ch. 3) there exists, $\forall\varepsilon>0$, an unique solution $\varrho_\varepsilon(t)$ satisfying i$_2$) and the inequality

$$\tag{3.30} \tfrac{1}{2}\|\varrho_\varepsilon(t)-l(t)\|_{L^2}^2+\int_0^t\{(l', \varrho_\varepsilon-l)_{L^2}+\varepsilon a(\varrho_\varepsilon, \varrho_\varepsilon-l)+(\nabla\cdot\mathbf{g}, \varrho_\varepsilon-l)_{L^2}\}\,d\eta\leq \leq\tfrac{1}{2}\|\varrho_0-l(0)\|_{L^2}^2 .$$

Since $\varrho_\varepsilon(t)\in L^2(0, T; K_2)$,

$$\tag{3.31} \lim_{\varepsilon\to 0}\varrho_\varepsilon(t)=\varrho(t)\in L^2(0, T; K_2)$$

in the weak topology of $L^2(0, T; H^1)$; letting $\varepsilon\to 0$ in (3.30) it follows then immediately, by (3.31), that $\varrho(t)$ satisfies (3.29).

The uniqueness of such a solution can be proved directly in a classical way, (see, for instance, [5], ch. 3).

REMARK 1: We recall that the potential $\Psi(x, t)$ given by (1.4') is the solution in R^3 of the Poisson equation

$$\tag{3.32} \Delta\Psi(x, t)=-4\pi G\tilde{\varrho}(x, t)$$

with homogeneous asymptotic boundary conditions on the variables x. Bearing in mind the definition of $\tilde{\varrho}$ and Theorem 3.2, it follows that

$$\Delta\Psi(t)\in L^\infty(0, T; W^{1,\infty}(R^3))\cap H^1(0, T; H^1(R^3))\subset$$
$$\subset[L^2(0, T; W^{1,\infty}(R^3)), H^1(0, T; H^1(R^3))]_{4/7}=H^{4/7}(0, T; W^{1,4/7}(R^3))\subset C^0(R^3).$$

Hence, since $\tilde{\varrho}(t)\in K_2$ a.e. in $(0, T)$, we may conclude that equation (1.3) admits an unique solution, in the classical sense, $\mathbf{y}(t)\in C^2(0, T)$.

REMARK 2: Setting, in (3.1), $\mathbf{f}=\nabla(\Phi+\Psi)+N(\nabla\tilde{\varrho}^\gamma/\tilde{\varrho})$ (Φ, Ψ, $\tilde{\varrho}$ defined resp. by (2.4), (2.5), (1.16)) it is obvious that $\mathbf{f}(t)\in L^2(0, T; L^2)$.

Consequently, we can apply Theorem 3.1 and obtain by (3.18), (3.26),

$$(3.33)\qquad \|u(t)\|_{L^\infty(0,T;V^0)\cap L^2(0,T;V^1)\cap H^{1-\eta}(0,T;(V^1)')} \le c_1$$

where c_1 depends only on u_0, α_1, α_2, μ, ξ, m, δ, M_1, M_2, ε (these last three defined by (1.12), (1.14) and (1.15) respectively), i.e. on the data of the problem. From (3.33) il follows also that

$$(3.34)\qquad \|u(t)\|_{L^4(0,T;V^{1/2})} \le c_2$$

and, bearing in mind (3.26),

$$(3.35)\qquad \|u(t)\|_{H^{1/2-\eta}(0,T;(V^0)')} \le c_3 .$$

Analogously, if in (3.29) we set $\boldsymbol{g} = \varrho \boldsymbol{u}$, we can apply Theorem 3.2 and obtain, in particular, that

$$(3.36)\qquad \|\varrho(t)\|_{L^4(0,T;W^{1,4})} \le c_4 ,$$

c_2, c_3, c_4, depending obviously only on u_0, ϱ_0, α_1, α_2, η, ξ, m, δ, M_1, M_2, ε.

4. - We now prove the existence of $\boldsymbol{u}$, ϱ, Φ, Ψ, $\boldsymbol{y}$ satisfying conditions i), ii). This will be done by means of the Schauder-Tychonov fixed point theorem.

Let $\sigma(t)$, $\boldsymbol{v}(t)$ be a couple of functions, with

$$\sigma(t) \in L^4(0,T;W^{1,4}(\Omega)) , \qquad \boldsymbol{v}(t) \in L^4(0,T,V^{1,4})$$

and let $\bar\sigma(t)$ be defined in accordance to (1.16).

Consider the following system

$$(4.1)\qquad \frac{1}{2}\|\boldsymbol{u}(t)-\boldsymbol{h}(t)\|^2_{V^0} + \int_0^t \Big(\boldsymbol{h}' + \nabla(\Phi+\Psi) + N\frac{\nabla\bar\sigma^r}{\bar\sigma} - \frac{\mu}{\bar\sigma}\Delta\boldsymbol{u} - \frac{\xi}{\bar\sigma}\nabla(\nabla\cdot\boldsymbol{u}) + \\ + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, \boldsymbol{u}-\boldsymbol{h}\Big)_{V^0} d\eta \le \frac{1}{2}\|\boldsymbol{u}_0-\boldsymbol{h}(0)\|^2_{V^0} ,$$

$$(4.2)\qquad \tfrac{1}{2}\|\varrho(t)-l(t)\|^2_{L^2(\Omega)} + \int_0^t (l' - \nabla\cdot(\bar\sigma\boldsymbol{v}), \varrho - l)_{L^2(\Omega)} d\eta \le \tfrac{1}{2}\|\varrho_0 - l(0)\|^2_{L^2(\Omega)} ,$$

$$(4.3)\qquad \boldsymbol{y}'(t) = -\nabla\Psi(\boldsymbol{y}(t), t) ,$$

$$(4.4)\qquad \Psi(x,t) = -\frac{1}{4\pi G}\int_\Omega \frac{\bar\sigma(\xi,t)}{|x-\xi|}\, d\xi ,$$

$$(4.5)\qquad \Phi(x,t) = -\frac{3m}{4\pi G\delta^3}\int_{s(y(t))} \frac{d\xi}{|x-\xi|} ,$$

where, $\boldsymbol{h}$, l are arbitrary test functions belonging respectively to $H^1(0, T; V^a) \cap$ $\cap L^\infty(0, T; K_1)$ and to $H^1(0, T; L^2) \cap L^\infty(0, T; K_2)$ and $\boldsymbol{u}_0 \in V^1 \cap K_1$, $\varrho_0 \in K_2$ with: $\boldsymbol{y}(0) = \boldsymbol{y}_0$, $\boldsymbol{y}'(0) = \boldsymbol{y}'_0$.

Observe that, as already pointed out in the preceding paragraph, given $\sigma(t)$, (4.3), (4.4), (4.5) enable us to calculate $\boldsymbol{y}(t)$, $\Phi(x, t)$, $\Psi(x, t)$; moreover $\nabla(\Psi(t) + \Phi(t)) \in L^2(Q)$, with $\|\nabla(\Psi(t) + \Phi(t))\|_{L^2(Q)}$ independent of $(\sigma, \boldsymbol{v})$.

Bearing in mind the definition of ϑ, we can conclude that, since the assumptions of Theorems 3.1 and 3.2 are satisfied, there exists an unique couple of functions $(\varrho, \boldsymbol{u})$ which satisfy (4.1), (4.2) and (3.1), (3.29). System (4.1)-(4.5) therefore defines a one-to-one transformation S: $(\varrho, \boldsymbol{u}) = S(\sigma, \boldsymbol{v})$ from $L^4(0, T; W^{1,4}) \times L^4(0, T; V^{1,3})$ to itself.

It is obvious that the proof of the existence of a solution of the problem considered is equivalent to the proof that the transformation S has a fixed point. We therefore now prove the following preliminary theorem.

THEOREM 4.1: *The transformation S defined above has the following properties:*

a) There exists in $L^4(0, T; W^{1,4}) \times L^4(0, T, V^{1,3})$ a set $E = E_1 \times E_2$ which is closed, convex, weakly compact and such that $SE \subseteq E$.

b) S is weakly continuous.

Let us begin by proving *a*). We set, for this,

$$E_1 = \{\sigma(t) \in L^4(0, T; W^{1,4}) \cap L^\infty(0, T; K_2)\},$$

$$E_2 = \{\boldsymbol{v}(t) \in L^4(0, T, V^{1,3}) \cap L^\infty(0, T; K_1), \|\boldsymbol{v}\|_{H^{1,\infty}(0,T;(V^a)')} < c_3;\ \|\boldsymbol{v}\|_{L^4(0,T;V^{1,3})} < c_2\},$$

where c_3 and c_2 are the constants introduced in § 3 (see (3.34), (3.35)).

It is then obvious that the set E defined in this way is closed, convex, weakly compact and (bearing in mind the results obtained in § 3) that $SE \subseteq E$.

We now prove point *b*).

Let $\{\sigma_n, \boldsymbol{v}_n\}$ be a sequence such that:

$$\{\sigma_n, \boldsymbol{v}_n\} \rightharpoonup \{\sigma, \boldsymbol{v}\} \tag{4.6}$$

in the weak topology of $L^4(0, T; W^{1,4}) \times L^4(0, T; V^{1,3})$ and set

$$\{\varrho_n, \boldsymbol{u}_n\} = S\{\sigma_n, \boldsymbol{v}_n\}. \tag{4.7}$$

We must prove that

$$\{\varrho_n, \boldsymbol{u}_n\} \rightharpoonup \{\varrho, \boldsymbol{u}\} \tag{4.8}$$

in the weak topology of $L^4(0, T; W^{1,4}) \times L^4(0, T; V^{1,3})$ with $\{\varrho, \boldsymbol{u}\} = S\{\sigma, \boldsymbol{v}\}$.

Observe, first of all, that, since $\{\varrho_n, \boldsymbol{u}_n\} \in E$, there exists a subsequence $\{\varrho_{n'}, \boldsymbol{u}_{n'}\}$ for which (4.7) holds. Moreover, by the definition of E and of ϑ

and by (3.34)-(3.36)

(4.9) $\bar{\sigma}_{n'} \to \bar{\sigma}$ a.e. in Q and in the weak topology of $L^2(0, T; H^1(\Omega))$,

(4.10) $\boldsymbol{v}_{n'} \to \boldsymbol{v}$ a.e. in Q and in the weak topology of $L^2(0, T; V^1)$,

(4.11) $\boldsymbol{u}_{n'} \to \boldsymbol{u}$ a.e. in Q, in the strong topology of $L^2(0, T; V^1)$ and the weak topology of $L^2(0, T; V^2)$.

Let now n' diverge in (4.7) (in which we have substituted n' to n); bearing in mind (4.9), (4.10), (4.11) and the definition of S we have $\{\varrho, \boldsymbol{u}\} = S\{\sigma, \boldsymbol{v}\}$.

By the uniqueness theorems proved in § 3, we may conclude that the whole sequence $\{\varrho_n, \boldsymbol{u}_n\}$ converges to $\{\varrho, \boldsymbol{u}\}$.

The weak continuity of S is therefore proved.

We can now prove the final existence theorem.

THEOREM 4.2: *Under the assumptions made on the data in the Theorems* 3.1, 3.2, *the transformation* S *has a fixed point*; *therefore there exists at least one solution of system* (1.1'), (1.2'), (1.3), (1.4'), (1.5) *satisfying the initial and boundary conditions* (1.6)-(1.9).

The proof follows directly from Theorem 4.1 and the Schauder-Tychonov fixed point theorem.

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