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New Classes of Variational Inequalities and Applications to Equilibrium Problems (**) (***)

SUMMARY. — We prove an existence theorem for the traffic equilibrium problem in the continuous case and we study a variational inequality over a not usual convex set depending on the divergence.

Nuove classi di disuguaglianze variazionali. Applicazioni a problemi di equilibrio

SOMMARIO. — Si dimostra un teorema di esistenza per il problema dell'equilibrio del traffico nel caso continuo e si studia una disequazione variazionale su un convesso, non usuale, che dipende dalla divergenza.

1. - Introduction

We are concerned with the traffic equilibrium problem in the continuous case (for the discrete case see e.g. [2]). To describe this problem let us consider a subset Ω of the plane $\mathbb{R}^2$ of generic point $x = (x_1, x_2)$ and, following the ideas of [1], let us suppose that at each point x the traffic flows in the direction of the increasing axes x_1 and x_2.

The flux in each point $x \in \Omega$ is described by a vectorial field $u(x)$, whose components $u_1(x)$, $u_2(x)$ represent the traffic density along the directions x_1 and x_2 respectively; it will be

$$u_1(x) \geq 0 , \qquad u_2(x) \geq 0$$

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and $u_1(x)$, $u_2(x)$ have non negative fixed traces $\varphi_1(x)$ and $\varphi_2(x)$ on $\partial\Omega$ (or on a part of $\partial\Omega$) respectively; to this end we suppose that Ω has a conveniently smooth boundary $\partial\Omega$ and that $u(x) \in H^1(\Omega, \mathbf{R}^2)$; in such a way the function $\varphi(x) = (\varphi_1(x), \varphi_2(x))$ belongs to $H^{\frac{1}{2}}(\partial\Omega, \mathbf{R}^2)$.

If we associate to each point $x \in \Omega$ a scalar field $t(x) \in L^2(\Omega)$, that represents the density of the flow originating or terminating at x, we can write the flow conservation law in the form

$$\iint_{\Omega'} t\,dx + \oint_{\partial\Omega'} u \cdot n\,ds = 0 \tag{1.1}$$

where n is the normal at the point $x \in \partial\Omega'$ and Ω' is any subdomain of Ω with smooth boundary $\partial\Omega'$. Let us observe that, if we write (1) for $\Omega' = \Omega$, we obtain the relation

$$\iint_{\Omega} t\,dx_1\,dx_2 + \oint_{\partial\Omega} [\varphi_1(x)n_1 + \varphi_2(x)n_2]\,ds = 0 \tag{1.2}$$

which expresses the connexion between $t(x)$ and the traces $\varphi_1(x)$, $\varphi_2(x)$. Moreover from (1.1), since $u \in H^1(\Omega, \mathbf{R}^2)$, we infer

$$\operatorname{Div} u + t(x) = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + t(x) = 0\,. \tag{1.3}$$

Now we can consider the « personal cost » $\vec{c}$ that will be a vector whose components $c_1(x, u(x))$, $c_2(x, u(x))$ represent the travel cost along the axes x_1 and x_2 respectively; whereas, for the case where we are not interested in the user's personal travel cost but rather in an overall travel cost spent in the network we can consider the total cost, which can be expressed in the form

$$c(x, u(x)) = c_1(x, u(x))\,u_1(x) + c_2(x, u(x))\,u_2(x)\,,$$

and the global total cost, whose expression is

$$F(u) = \int_{\Omega} [c_1(x, u(x))\,u_1(x) + c_2(x, u(x))\,u_2(x)]\,dx\,. \tag{1.4}$$

It is well known that the transportation network operates under one of the following decision rules:

I) the first produces a « user-optimizing » flow pattern with the equilibrium property that no user has any incentive to change unilaterally his decision;

II) the second rule deals with the « systems optimizing » flow pattern, i.e. the pattern that minimize the travel cost over the entire network.

In this paper we are concerned with the second rule of decision and we look for a distribution flow $u^*(x)$ minimizing the global total cost. We explain since now that we can reach some existence theorems minimizing $F(x)$ not over the « natural » conves K given by

$$K = \{u(x) = (u_1(x), u_2(x)) \in H^1(\Omega, \mathbf{R}^2) | u_1(x), u_2(x) \geq 0, \\ u_1(x)|_{\partial\Omega} = \varphi_1(x), u_2(x)|_{\partial\Omega} = \varphi_2(x), \operatorname{div} u + t(x) = 0\} \tag{1.5}$$

but over the closure of K in $L^2(\Omega, \mathbf{R}^2)$, $\bar{K}^{L^2(\Omega)}$; the reason of this fact is that the « coerciveness » of $F(u)$ over $\bar{K}^{L^2(\Omega)}$ is guaranteed.

2. - Existence theorems

We achieve a first theorem assuming that $c_1(x, u(x))$ and $c_2(x, u(x))$ are linear; that is

$$\begin{cases} c_1(x, u(x)) = a_{11}(x)u_1(x) + a_{12}(x)u_2(x) + b_1(x) & a_{ij}, b_i \geq 0 \,, \\ c_2(x, u(x)) = a_{21}(x)u_1(x) + a_{22}(x)u_2(x) + b_2(x) & i, j = 1, 2 \,, \end{cases} \tag{2.1}$$

and that the following hypotheses are fulfilled:

a) $a_{ij}(x) \in L^\infty(\Omega)$, $i, j = 1, 2$; $b_1(x), b_2(x) \in L^2(\Omega)$;

b) there exists a constant $\nu > 0$ such that

$$a_{11}u_1^2 + (a_{12} + a_{21})u_1 u_2 + a_{22}u_2^2 \geq \nu(u_1^2 + u_2^2) \,.$$

Then we have the following

Theorem 2.1: *Under the hypotheses* (2.1) *and a*), *b*), *there exists a point* $u_0 \in \bar{K}^{L^2(\Omega)}$ *such that:*

$$F(u_0) = \min_{u \in \bar{K}^{L^2(\Omega)}} \int_\Omega c(x, u(x)) u(x)\, dx \,. \tag{2.2}$$

Let us start by observing that $F(u)$ is a convex functional. In fact if we denote by A the matrix whose elements are the coefficients a_{ij} and by B the vector whose components are b_i, we can rewrite the functional $F(u)$ in the form

$$F(u) = \int_\Omega (Au|u) + (B|u)\, dx \tag{2.3}$$

where the simbol (|) denotes the inner product; then, since it results

$$(Au - Av, u - v) \geq 0 \qquad \forall u, v \in \bar{K}^{L^2(\Omega)} \tag{2.4}$$

the convexity is achieved.

Moreover we have

$$(2.5)\qquad \int_{\Omega}[(Au|u)+(B|u)]\,dx \geq \nu \int_{\Omega}|u|^2\,dx$$

and the coerciveness over $\bar{K}^{L^2(\Omega)}$ is insured: taking into account a well known theorem, the existence of the minimum is so acquired and also the uniqueness.

If we assume that $a_{12}=a_{21}$ the well known theory of variational inequalities allows us to say that it results

$$(2.6)\qquad 2\int_{\Omega}(Au^*|u-u^*)+(B|u-u^*)\,dx \geq 0 \qquad \forall u \in \bar{K}^{L^2(\Omega)}$$

and one can try to apply the algorithms for the computation of solutions of variational inequalities to compute the solutions of (2.6).

A very special result is espressed in the following theorem

THEOREM 2.2: *If c does not depend on $u(x)$ and is conservative, the functional $F(u)$ is constant.*

In fact, because $c(x)$ is conservative, there exists a function $f(x)$ such that $c_1(x)=\partial f(x)/\partial x_1$, $c_2(x)=\partial f(x)/\partial x_2$; then if $u \in K$ we have

$$F(u)=\int_{\Omega}[c_1(x)u_1+c_2(x)u_2]\,dx=$$

$$=\int_{\partial\Omega} f(x)[\varphi_1(x)n_1+\varphi_2(x)n_2]\,d\sigma-\int_{\Omega} f(x)\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\right)dx$$

and considering for $u \in \bar{K}^{L^2(\Omega)}$ an approaching succession $u^n(x)$ of elements of K we get

$$F(u_n)=\int_{\partial\Omega} f(x)(\varphi_1(x)n_1+\varphi_2(x)n_2)\,d\sigma+\int_{\Omega} f(x)t(x)\,dx=C$$

and also $F(u)=C$ for every $u \in \bar{K}^{L^2(\Omega)}$.

3. - THE CASE $t(x)=0$ IN A RECTANGULAR GRID

In this section we consider the case in which $t(x)=0$ and Ω is a rectangular grid $]0,a[\times]0,b[$ whose vertexes are $A=(0,0)$, $B=(a,0)$, $C=(a,b)$, $D=(0,b)$. Let us assume that the traces are fixed in the following way:

$$(3.1)\qquad \begin{cases} u_1|_{AD}=\varphi_1(x_2) & x_2 \in\,]0,b[\,, \\ u_1|_{BC}=\psi_1(x_2) & x_2 \in\,]0,b[\,, \end{cases}$$

$$(3.2)\qquad \begin{cases} u_2|_{AB}=\varphi_2(x_1) & x_1 \in\,]0,a[\,, \\ u_2|_{CD}=\psi_2(x_1) & x_1 \in\,]0,a[\,, \end{cases}$$

and let us suppose that the following hypothesis holds:

i) $a_{ij} \in C^1(\bar{\Omega})$, $a_{ij} = a_{ji}$, $i, j = 1, 2$, $b_i(x) \in C^1(\bar{\Omega})$, $i = 1, 2$.

Now let us observe that, because $t(x) = 0$, if $u \in K$ there exists a function $U(x)$ such that:

$$\frac{\partial U}{\partial x_2} = u_1 \, , \qquad \frac{\partial U}{\partial x_1} = -u_2 \quad (^1) \, . \tag{3.3}$$

A simple calculation shows that the function U, which is defined apart from an arbitrary constant C, has the following traces

$$U(0, x_2) = \int_0^{x_2} \varphi_1(t)\, dt + C = \Phi_1(x_2) \qquad x_2 \in [0, b] \, ,$$

$$U(a, x_2) = \int_0^{x_2} \psi_1(t)\, dt - \int_0^{a} \varphi_2(t)\, dt + C = \Psi_1(x_2) \qquad x_2 \in [a, b] \, ,$$

$$U(x_1, 0) = -\int_0^{x_1} \varphi_2(t)\, dt + C = \Phi_2(x_1) \qquad x_1 \in [a, b] \, ,$$

$$U(x_1, b) = -\int_0^{x_1} \psi_2(t)\, dt - \int_0^{b} \varphi_1(t)\, dt + C = \Psi_2(x_1) \qquad x_1 \in [a, b] \, ,$$

and we can say that for every $u \in R^{L^2(\Omega)}$ there exists a function U such that (3.3) holds and, if we consider another function $v \in R^{L^2(\Omega)}$ and denote by V the function for which (3.3) holds, the function $U - V$ belongs to $H_0^1(\bar{\Omega}, \mathbf{R}^2)$.

Let us consider the Variational Inequality (2.6) and let us observe that it becomes

$$\int_\Omega \Big[2\Big(a_{11} \frac{\partial U^0}{\partial x_2} - a_{12} \frac{\partial U^0}{\partial x_1} \Big) \frac{\partial (U - U^0)}{\partial x_2} + 2\Big(a_{22} \frac{\partial U^0}{\partial x_1} - a_{21} \frac{\partial U^0}{\partial x_2} \Big) \frac{\partial (U - U^0)}{\partial x_1} + \tag{3.5}$$

$$+ b_1 \frac{\partial (U - U^0)}{\partial x_2} - b_2 \frac{\partial (U - U^0)}{\partial x_1} \Big] dx \geq 0 \qquad \forall U \text{ s.t. } \Big(\frac{\partial U}{\partial x_2}, -\frac{\partial U}{\partial x_1} \Big) \in R^{L^2(\Omega)} \, .$$

(¹) To achieve this result, for all $u \in K$, we can consider a succession $u_n \in C^1(\bar{\Omega}, \mathbf{R}^2)$ such that u_1^n, u_2^n, $\partial u_1^n / \partial x_1$, $\partial u_2^n / \partial x_2$ converge to u_1, u_2, $\partial u_1 / \partial x_1$, $\partial u_2 / \partial x_2$ in $L^2(\Omega)$ and $u_1^n|_{AD} = \varphi_1^n(x_2)$, $u_1^n|_{BC} = \psi_1^n(x_2)$, $u_2^n|_{AB} = \varphi_2^n(x_1)$, $u_2^n|_{CD} = \psi_2^n(x_1)$ converge to φ_1, ψ_1, φ_2, ψ_2 in L^2 respectively; moreover we can construct this succession so that $\operatorname{Div} u_n = 0$ (see (4.1)). Then there exists a function U_n such that $\partial U_n / \partial x_2 = u_1^n$, $\partial U_n / \partial x_1 = -u_2^n$ and, taking into account the estimate

$$\int_\Omega |U_n - U_m|^2 dx < 2a^2 b \int_0^a |\varphi_2^n - \varphi_2^m|^2 dx_1 + 2b^2 \int_\Omega |u_1^n - u_1^m|^2 dx \, , \tag{3.4}$$

we obtain the thesis.

Then, if we denote by U^0 the solution of the Dirichlet problem

$$\begin{cases} a_{22}\dfrac{\partial^2 U^0}{\partial x_1^2}+a_{11}\dfrac{\partial^2 U^0}{\partial x_2^2}-(a_{12}+a_{21})\dfrac{\partial^2 U^0}{\partial x_1\partial x_2}- \\ \qquad -\left(\dfrac{\partial a_{12}}{\partial x_2}-\dfrac{\partial a_{22}}{\partial x_1}\right)\dfrac{\partial U^0}{\partial x_1}-\left(\dfrac{\partial a_{21}}{\partial x_1}-\dfrac{\partial a_{11}}{\partial x_2}\right)\dfrac{\partial U^0}{\partial x_2}+\dfrac{1}{2}\left(\dfrac{\partial b_1}{\partial x_2}-\dfrac{\partial b_2}{\partial x_1}\right)=0\,, \\ U^0(0,x_2)=\Phi_1(x_2)\,, \qquad U^0(a,x_2)=\Psi_1(x_2)\,, \qquad x_2\in\,]0,b[\,, \\ U^0(x_1,0)=\Phi_2(x_1)\,, \qquad U^0(x_1,b)=\Psi_2(x_1)\,, \qquad x_1\in\,]0,a[\,, \end{cases}$$

and if we suppose that

$$\frac{\partial U^0}{\partial x_2}>0\,, \qquad \frac{\partial U^0}{\partial x_1}<0 \quad \text{in } \Omega\,, \tag{3.6}$$

the vector $(\partial U^0/\partial x_2, -\partial U^0/\partial x_1)$ is the unique solution of variational inequality (2.6). In fact it results

$$\int_\Omega\left[2\left(a_{11}\frac{\partial U^0}{\partial x_2}-a_{12}\frac{\partial U^0}{\partial x_1}\right)\frac{\partial(U-U^0)}{\partial x_2}+\left(a_{22}\frac{\partial U^0}{\partial x_1}-a_{21}\frac{\partial U^0}{\partial x_2}\right)\frac{\partial(U-U^0)}{\partial x_1}+\right.$$
$$\left.+b_1\frac{\partial(U-U^0)}{\partial x_2}-b_2\frac{\partial(U-U^0)}{\partial x_1}\right]dx=\int_{\partial\Omega}\left[2\left(a_{11}\frac{\partial U^0}{\partial x_2}-a_{12}\frac{\partial U^0}{\partial x_1}\right)(U-U^0)n_2+\right.$$
$$\left.+2\left(a_{22}\frac{\partial U^0}{\partial x_1}-a_{21}\frac{\partial U^0}{\partial x_2}\right)(U-U^0)n_1+b_1(U-U^0)n_2-b_2(U-U^0)n_1\right]d\sigma-$$
$$-2\int_\Omega\left[\frac{\partial}{\partial x_2}\left(a_{11}\frac{\partial U^0}{\partial x_2}-a_{12}\frac{\partial U^0}{\partial x_1}\right)+\frac{\partial}{\partial x_1}\left(a_{22}\frac{\partial U^0}{\partial x_1}-a_{21}\frac{\partial U^0}{\partial x_2}\right)+\right.$$
$$\left.+\frac{1}{2}\left(\frac{\partial b_1}{\partial x_2}-\frac{\partial b_2}{\partial x_1}\right)\right](U-U^0\,dx=0)$$

for every U s.t. $(\partial U/\partial x_2, -\partial U/\partial x_1)\in K^{L^2(\Omega)}$.

One can compare this case with the example of [1], pp. 239-301.

If hypothesis i) does not hold, we can consider the solution of Variational Inequality (3.5) as a « weak solution » of the traffic equilibrium problem.

4. - The case $t(x)\neq 0$ in a rectangular grid

Let us start by observing that we can rewrite the condition $\operatorname{Div} u+t(x)=0$ in the following way

$$\operatorname{Div} u+t(x)=\frac{\partial}{\partial x_1}\left(u_1+\frac{1}{2}\int_0^{x_1}t(\tau_1,x_2)\,d\tau_1\right)+ \tag{4.1}$$
$$+\frac{\partial}{\partial x_2}\left(u_2+\frac{1}{2}\int_0^{x_2}t(x_1,\tau_2)\,d\tau_2\right)=0\,.$$

Then if we assume $t(x) \in H^1(\Omega)$ and if we set

$$(4.2) \qquad \left|\begin{array}{l} v_1 = u_1 + \frac{1}{2}\int\limits_0^{x_1} t(\tau_1, x_2)\, d\tau_1\,, \\[2ex] v_2 = u_2 + \frac{1}{2}\int\limits_0^{x_2} t(x_1, \tau_2)\, d\tau_2\,, \\[2ex] v = (v_1, v_2)\,, \qquad t_1(x) = \frac{1}{2}\int\limits_0^{x_1} t(\tau_1, x_2)\, d\tau_1\,, \\[2ex] t_2(x) = \frac{1}{2}\int\limits_0^{x_2} t(x_1, \tau_2)\, d\tau_2\,, \qquad \theta(x) = (t_1, t_2)\,, \end{array}\right.$$

the functional $F(u)$ becomes, taking into account (2.4) and assuming that $a_{12} = a_{21}$,

$$(4.3) \qquad F^*(v) = F(v - \theta) = \int\limits_\Omega A(v-\theta)|(v-\theta) + (B|v-\theta)\, dx =$$

$$= \int\limits_\Omega \{(Av|v) - [(Av|\theta) + (A\theta|v)] + (A\theta|\theta) + (B|v) - (B|\theta)\}\, dx =$$

$$= \int\limits_\Omega \left[(Av|v) - 2\left(\left(A\theta - \frac{B}{2}\right)\middle| v\right) + (A\theta - B|\theta)\right] dx$$

and the convex K becomes

$$K^* = \Big\{v \in H^1(\Omega, R^2) | v_1(x) > t_1(x), v_2(x) > t_2(x),$$

$$v_1(0, x_2) = \varphi_1(x_2), v_1(a, x_2) = \psi_1(x_2) + \frac{1}{2}\int\limits_0^a t(\tau_1, x_2)\, d\tau_1,$$

$$v_2(x_1, 0) = \varphi_2(x_1), v_2(x_1, b) = \psi_2(x_1) + \frac{1}{2}\int\limits_0^b t(x_1, \tau_2)\, d\tau_2, \operatorname{Div} v = 0\Big\}$$

and the solution $v^0 = u^0 + \theta(x)$ of the minimizing problem verifies

$$\int\limits_\Omega (Av^0|v - v^0) - \left(A\theta - \frac{B}{2}\middle| v - v^0\right) dx > 0 \qquad \forall v \in R^{L^2(\Omega)}.$$

Taking into account the results of section 3, we can say that there exists a function V^0 such that $\partial V^0/\partial x_2 = v_1^0$, $-(\partial V^0/\partial x_1) = v_2^0$ and such that

$$\int\limits_\Omega \left[\left(a_{11}\frac{\partial V^0}{\partial x_2} - a_{12}\frac{\partial V^0}{\partial x_1}\right)\frac{\partial(V - V^0)}{\partial x_2} + \left(a_{22}\frac{\partial V^0}{\partial x_1} - a_{21}\frac{\partial V^0}{\partial x_2}\right)\frac{\partial(V - V^0)}{\partial x_1} - \right.$$

$$\left. - \left(a_{11}t_1 + a_{12}t_2 - \frac{b_1}{2}\right)\frac{\partial(V - V^0)}{\partial x_2} + \left(a_{21}t_1 + a_{22}t_2 - \frac{b_2}{2}\right)\frac{\partial(V - V^0)}{\partial x_1}\right] dx > 0$$

$$\forall V \text{ s.t. } \left(\frac{\partial V}{\partial x_2}, -\frac{\partial V}{\partial x_1}\right) \in R^{*L^2(\Omega)}.$$

Then, assuming that the hypothesis i) holds, if we denote by V^0 the solution of the Dirichlet problem

$$\begin{cases} a_{22}\dfrac{\partial^2 V^0}{\partial x_1^2}+a_{11}\dfrac{\partial^2 V^0}{\partial x_2^2}-(a_{12}+a_{21})\dfrac{\partial^2 V^0}{\partial x_1\,\partial x_2}-\left(\dfrac{\partial a_{12}}{\partial x_2}-\dfrac{\partial a_{22}}{\partial x_1}\right)\dfrac{\partial V^0}{\partial x_1}- \\ \qquad -\left(\dfrac{\partial a_{21}}{\partial x_1}-\dfrac{\partial a_{11}}{\partial x_2}\right)\dfrac{\partial V^0}{\partial x_1}-\left[\dfrac{\partial(a_{11}t_1+a_{12}t_2-b_1/2)}{\partial x_2}-\dfrac{\partial(a_{21}t_1+a_{22}t_2-b_2/2)}{\partial x_1}\right]=0\,, \\ V^0(0,x_2)=\Phi_1(x_2)\,, \qquad V^0(a,x_2)=\Psi_1(x_2)+\dfrac{1}{2}\displaystyle\int_0^{x_1}\!\!\int_0^{a} r(\tau_1,\tau_2)\,d\tau_1\,d\tau_2\,, \\ V^0(x_1,0)=\Phi_2(x_1)\,, \qquad V^0(x_1,b)=\Psi_2(x_1)+\dfrac{1}{2}\displaystyle\int_0^{b}\!\!\int_0^{x_2} r(\tau_1,\tau_2)\,d\tau_1\,d\tau_2\,, \end{cases}$$

and if we suppose that

$$\frac{\partial V^0}{\partial x_2}>t_1(x)\,,\qquad \frac{\partial V^0}{\partial x_1}<t_2(x)\,, \tag{4.3}$$

the vector $(\partial V^0/\partial x_2, -\partial V^0/\partial x_1)$, as in the case $r(x)=0$, is the unique solution of our problem.

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