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# An Uniqueness Result in the Calculus of Variations (**)

## Un risultato di unicità nel Calcolo delle Variazioni

SUNTO. — Considerato il seguente problema in $G$ aperto convesso limitato di $R^N$, $N \geq 2$,

$$(\mathcal{F}) \qquad \operatorname{Inf}\left\{\int_G f(Dv)\,dx, \quad v \in C^{0,1}(G), \quad v = u_0 \quad \text{su } \partial G\right\}$$

con $f: R^N \to R$ convessa ma non strettamente convessa, si ottengono alcune condizioni sul dato al bordo $u_0$ affinchè $(\mathcal{F})$ abbia un'unica soluzione.

1. - Consider the following functional of Calculus of Variations

$$F(v) = \int_G f(Dv)\,dx$$

where $f: R^N \to R$ is a convex function and $G$ is a bounded convex subset of $R^N$ with boundary $\partial G$. Consider the problem:

$$(\mathcal{F}) \qquad \operatorname{Inf}\{F(v):\ v \in C^{0,1}(G),\ v = u_0 \text{ on } \partial G\}.$$

From a well known theorem of P. Hartman - G. Stampacchia (Th. 13.2 of [1]) problem $(\mathcal{F})$ has at least one solution if $u_0$ verifies the « bounded slope condition » with constant $L_0$ (see [1]), i.e. (B.S.C.) for every $x_0 \in \partial G$ there exist

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a pair of linear functions $\pi^+, \pi^-$

$$\pi^\pm(x) = \sum_{j=1}^{N} \alpha_j^\pm (x^j - x_0^j) + u_0(x_0),$$

satisfying for $x \in \partial G$

$$\pi^-(x) \leq u_0(x) \leq \pi^+(x)$$

and

$$|D\pi^\pm(x)|^2 = \sum_{j=1}^{N} |\alpha_j^\pm|^2 < L_0 .$$

We note explicitely that this result of existence does not require any assumptions of coerciveness on $f$. Moreover, if $f$ is strictly convex, problem ($\mathcal{P}$) has only one solution.

We are interested in finding conditions on problem ($\mathcal{P}$), when $f$ is convex but not strictly convex, in order to obtain a unique solution.

In [3] P. Marcellini proves a result of uniqueness for a problem of the same type as ($\mathcal{P}$) by supposing that it has a solution $u$ of class $C^1$ with $Du \neq 0$.

In this short note we prove that if $f$ is strictly convex in $\mathbb{R}^N - K$, where $K$ is a bounded subset of $\mathbb{R}^N$, and $u_0$ verifies a suitable condition, related to the convex hull of $K$, problem ($\mathcal{P}$) has only one solution.

The proof of uniqueness is not very involved, however this result shows the crucial role played by the boundary datum.

The problem of uniqueness for functionals which are convex but not strictly convex is deeply related to both existence and non existence of solutions of non convex problems, as P. Marcellini pointed out in [3] and [4].

Indeed, the decisive role acted by boundary data within the non-existence of minima of non convex problems, was already emphasized in [6].

2. – Let $C$ an open bounded convex subset of $\mathbb{R}^N$ and consider the problem in $G$, a convex bounded subset of $\mathbb{R}^N$ ($N > 2$):

$$\begin{cases} Dw(x) \in \bar{C} & \text{a.e. in } G \\ w = u_0 & \text{on } \partial G \end{cases} \tag{2.1}$$

where $u_0 \in C^{0,1}(G)$. Problem (2.1) is a boundary value problem for a special Hamilton-Jacobi equation, as it was just remarked in [5]. The Hamiltonian function $H$ is the following function:

$$H(p) = 1_{\bar{C}}(p) = \begin{cases} 0 & \text{if } p \in \bar{C} \\ +\infty & \text{if } p \notin \bar{C}. \end{cases}$$

The function $H$ is convex since $C$ is a convex subset of $\mathbb{R}^N$. Consider a boundary value problem for a general Hamilton-Jacobi equation:

$$\text{(HJ)} \qquad \begin{cases} H(Dv(x)) = 0 & \text{a.e. in } G \\ v = u_0 & \text{on } \partial G\,. \end{cases}$$

The following theorem holds (theorem 5.2 of [2]):

THEOREM 2.1: *Let $G$ be a bounded, smooth and connected domain of $\mathbb{R}^N$. Let $H$ be a continuous convex function in $\mathbb{R}^N$, such that $H(p) \to +\infty$ as $|p| \to +\infty$. Define for $(x, y) \in \bar{G} \times \bar{G}$* [1]:

$$L(x,y) = \inf\left\{\int_0^1 \sup_{H(p)=0}\left(-\frac{d\xi}{dt}(t), p\right) dt \quad \xi \in A(x,y)\right\}$$

*where*

$$A(x,y) = \Big\{\xi \in C^{0,1}([0,1], \mathbb{R}^N)\colon\ \xi(t) \in \bar{G} \quad \forall t \in [0,1]\,,$$
$$\frac{d\xi}{dt} \in L^\infty([0,1], \mathbb{R}^N),\ \xi(0) = x,\ \xi(1) = y\Big\}.$$

*The following condition*

$$u_0(x) - u_0(y) < L(x,y) \quad \forall x, y \in \partial G$$

*is a necessary and sufficient condition for the existence of function $v \in C^{0,1}(\bar{G})$ satisfying $H(Dv) < 0$ a.e. in $G$ and $v = u_0$ on $\partial G$. Moreover, define*

$$u(x) = \inf_{y \in \partial G} \{u_0(y) + L(x,y)\}\,,$$

*we have that $u \in C^{0,1}(\bar{G})$ and it is a solution of problem* (HJ).

Consider now problem (2.1), by proceding as in the proof of theorem 2.1, we obtain:

THEOREM 2.2: *Let*

$$\tilde{L}(x,y) = \inf\left\{\int_0^1 \sup_{p \in \tilde{C}}\left(-\frac{d\xi}{dt}, p\right) dt\,, \quad \xi \in \tilde{A}(x,y)\right\},$$

[1] If $p = (p_1, \ldots, p_N)$ and $q = (q_1, \ldots, q_N)$ we denote

$$(p, q) = \sum_{i=1}^{N} p_i q_i\,.$$

*with*

$$\tilde{A}(x,y)=\Big\{\xi\in C^{0,1}([0,1],\mathbf{R}^n),\ \xi(t)\in\bar{G}\ \text{and}$$
$$\left|\frac{d\xi}{dt}(t)\right|\le 1,\ \forall t\in[0,1],\ \xi(0)=x,\ \xi(1)=y\Big\}.$$

*The following condition*

(C) $$u_0(x)-u_0(y)\le\tilde{L}(x,y)$$

*is a necessary and sufficient condition for the existence of function* $v\in C^{0,1}(G)$ *such that* $Dv(x)\in\bar{C}$ *a.e. in* $G$ *and* $v=u_0$ *on* $\partial G$.

*Moreover, the function*

$$\bar{u}(x)=\inf_{y\in\partial G}\{u_0(y)+\tilde{L}(x,y)\}$$

*is a solution of* (2.1).

Moreover, by proceeding as in theorem 1.1 of [5], we obtain that the function $\bar{u}$ is such that $D\bar{u}(x)\in\partial C$ a.e. in $G$.

In particular, if

$$C=B(0,R)=\Big\{p\in\mathbf{R}^n:\ |p|=\Big(\sum_i p_i^2\Big)^{\frac{1}{2}}<R\Big\},\qquad R>0$$

problem (2.1) becomes

(2.2) $$\begin{cases}|Dv(x)|\le R & \text{a.e. in } G\\ v=u_0 & \text{on } \partial G\end{cases}$$

and the compatibility condition (C) is

(2.3) $$|u_0(x)-u_0(y)|\le R|x-y|,\qquad \forall x,y\in\partial G.$$

(see remark 5.3 of [2]).

It is easy to check that if $u_0$ verifies the (B.S.C.) with constant $L_0$, $u_0$ verifies the condition (C) with respect to $B(0,L_0)$.

Let now $f:\mathbf{R}^n\to\mathbf{R}$ be a convex function in $\mathbf{R}^n$. Consider the problem in $G$:

(2.4) $$\operatorname{Inf}\Big\{F(v)=\int_G f(Dv)\,dx,\ v\in C^{0,1}(G),\ v=u_0\ \text{on}\ \partial G\Big\},$$

with $u_0\in C^{0,1}$, verifying the B.S.C. on $\partial G$ with constant $L_0$.

From the theorem of Hartman-Stampacchia [1], there exists at least one solution of problem (2.4).

The following theorem of uniqueness holds:

THEOREM 2.3: *Consider problem* (2.4) *with* $u_0$ *verifying the B.S.C. with constant* $L_0$. *Suppose* $f$ *strictly convex in* $\mathbf{R}^\nu - K$, *with* $K$ *a bounded open of* $\mathbf{R}^\nu$. *If* $u_0$ *does not verify the condition* (C) *with respect to* $coK$, *the convex hull of* $K$, *problem* (2.4) *has a unique solution.*

PROOF: Let $u_1, u_2$ two solutions of (2.4). Since $F$ is convex, for all $\lambda \in [0,1]$, the function $\lambda u_1 + (1-\lambda)u_2$ is still a solution of (2.4).

Since $u_0$ does not verify (C) with respect to $coK$, there are no solutions of the following problem:

$$\begin{cases} Dv(x) \in \overline{coK} & \text{a.e. } x \in G, \\ v = u_0 & \text{on } \partial G, \end{cases}$$

then, for all functions $v \in C^{0,1}$ with $v = u_0$ on $\partial G$, there exists a subset $A$ of $G$ with positive measure such that

$$Dv(x) \in \mathbf{R}^\nu - \overline{coK} \quad \text{a.e. in } A .$$

Since $\mathbf{R}^\nu - \overline{coK} \subset \mathbf{R}^\nu - \bar{K}$, for $\lambda \in [0,1]$, there exists $A_\lambda \subset G$ with meas $A_\lambda > 0$ such that, for a.e. $x \in A_\lambda$

(2.5) $$D(\lambda u_1 + (1-\lambda)u_2)(x) \in \mathbf{R}^\nu - \bar{K} .$$

Now, since $f$ is strictly convex in $\mathbf{R}^\nu - \bar{K}$, from (2.5) we obtain:

$$\int_{A_\lambda} f(\lambda Du_1 + (1-\lambda)Du_2)dx < \lambda \int_{A_\lambda} f(Du_1)dx + (1-\lambda)\int_{A_\lambda} f(Du_2)dx .$$

Therefore, we get

$$F(\lambda u_1 + (1-\lambda)u_2) < \lambda F(u_1) + (1-\lambda)F(u_2) .$$

Then, (2.4) has only one solution.

We give now an example in which theorem 2.1 can be applied.

Let $g: \mathbf{R}_+ \to \mathbf{R}$ a convex function, such that $\lim_{t\to+\infty} g(t)t^{-1} = +\infty$.

Suppose that $g$ is strictly convex in $\mathbf{R}^\nu - ]r, R[$ with $r > 0$ and $R > 0$. For example, let

$$g(t) = \begin{cases} t^2 & 0 < t < r,\ t > R \\ r^2 + (R+r)(t-r) & r < t < R . \end{cases}$$

Consider the problem:

(2.6) $$\operatorname{Inf}\left\{\int_G g(|Dv|)dx,\ v \in C^{0,1}(G),\ v = u_0 \text{ on } \partial G\right\}.$$

In this case $K = \{p \in \mathbb{R}^N : r < |p| < R\}$ and then $coK$ is the ball of radius $R$ and center in $O$.

Assume that $u_0$ verifies the B.S.C. with constant $L_0 > R$ and $u_0$ does not verify (2.3), i.e. there exist $\bar{x}, \bar{y} \in \partial G$ such that

$$|u_0(\bar{x}) - u_0(\bar{y})| > R|\bar{x} - \bar{y}| .$$

In this hypothesis, by applying theorem 2.1, problem (2.6) has only one solution.

Theorem 2.1 gives a sufficient condition on $u_0$ to obtain the uniqueness for problem (2.4). However the condition is not necessary. In fact, consider problem (2.6) with

$$u_0(x) = \sum_{i=1}^{N} p_{0i} x_i + q \qquad x \in G ,$$

$$p_0 = (p_{01}, \ldots, p_{0N}) \in \mathbb{R}^N .$$

Suppose $|p_0| = t_0$ with $t_0 \in ]r, R[$. Then $u_0$ verifies the compatibility condition (2.3) with respect $coK$, but (2.6) has only one solution $u = u_0$ as pointed out in [3].

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