Convergence in Homological Algebra

Convergenza in algebra omologica

SOMMARIO. — Si studiano proprietà di convergenza di subquotienti in categorie esatte e la loro compatibilità con i morfismi indotti. Le applicazioni riguardano la convergenza della successione spetrale di un complesso filtrato; saranno estese in un lavoro successivo a teorie omologiche più generali, come il sistema di categorie relative.

0. - INTRODUCTION

0.1. This paper aims to prepare a background for studying convergence properties in homological theories, mainly in the theories which produce spectral sequences (with non-trivial convergence), as the filtered complex, the exact couple [Ms], the system of relative homologies [Ei, Dc].

Generally (e.g. see [EM]) these properties have been treated in the context of abelian categories with countable products and sums, using the additive structure to transform properties of mappings into properties of objects (the "mapping cylinder" construction) and projective and inductive limits to formulate the hypotheses of convergence.

We study here the "convergence" of subquotients in the more general context of exact categories (in the sense of Puppe-Mitchell [?]): this allows to use the universal model of the above (exact) theories, developed by the
author in [G8, G9] and formalizing the Zeeman diagram [Ze, HW] in the case of the filtered complex. The lack of additivity prevents the use of such constructions as the mapping cylinder; on the other hand, it seems to lead to the essential core of the question.

The appropriate notion of convergence appears here to consist of unions and intersections of subquotients, with suitable conditions of regularity: e.g. the convergence theorem for the spectral sequence of a filtered complex $A_*$ (5.4) proves that, under suitable hypotheses on the filtration:

1. $E_{\alpha \beta}^m = \bigcap E_{\alpha \beta}^n$
   («regular», decreasing intersection of subquotients of $A_*$),

2. $H_{\alpha}(A_*) = \bigcup E_{\alpha \beta}^m$
   («regular», telescopic union of subquotients of $A_*$);

these conditions of regularity allow in particular to obtain «mapping theorems», i.e. to transfer the isomorphism property for the «induced morphisms», from a family of subquotients to its union or intersection.

0.2. Now, let us make precise the notion of subquotient and their order. Let $A$ be an object of the exact category $E$ and $\text{Sub}_E(A)$ its modular lattice of subobjects.

A subquotient $H/K$ of $A$ is determined by two subobjects $H$, $K$ with $H > K$, or—diagrammatically—by a bicartesian square:

\[
\begin{array}{ccc}
H/K & \xleftarrow{\pi} & H \\
\downarrow & & \downarrow \\
A/K & \xleftarrow{\pi} & A.
\end{array}
\]

(1)

Thus a subquotient of $A$ may be more effectively defined as a subobject $s$: $H/K \to A$ in the involutive category $A = \text{Rel } E$ of relations over $E$ ($s = \pi \pi - \pi s$, where $\sim$ denotes the involution of $A$). Accordingly, the subquotients of $A$ are provided with a canonical order $s < s'$, characterized in the following terms: $H/K < H'/K'$ iff $H < H'$ and $K > K'$ in the lattice of subobjects of $A$ in $E$. The unions (or intersections) we are speaking of concern this order, and are generally calculated by unions of numerators and intersections of denominators (or dually).

Equivalently, but avoiding any problem of choice of representants, a subquotient of $A$ is a projection $e: A \to A$, i.e. a symmetrical idempotent endo-relation of $A$ ($e = \hat{e} = e\hat{e}$):

\[
e = \hat{e} = (A \leftrightarrow H \leftrightarrow H/K \leftrightarrow H \leftrightarrow A);
\]

(2)
the corresponding canonical order $e < e'$ on the set $\text{Prj}(A)$ of projections of $A$ is now characterized as: $e = ee'$ (or equivalently: $e = e'e$).

We shall use both these descriptions of subquotients, the latter (projections in $A$) being more suitable for theoretical considerations, the former (subobjects in $A$) being more adapted for applications in homological algebra.

0.3. Thus we introduce in ch. 1 unions and intersections of projections in general involutive categories, together with three conditions of regularity: $P$-regularity, i.e., consistency with the transfer mappings of projections; $D$-regularity, i.e., consistency with the relation of domination $e \subseteq e'$ (1.3); $PD$-regularity, combining the above and being (probably) the "good" notion.

Chapter 2 recalls the definition of $RE$-category (slightly extending the categories of relations over exact categories) and characterizes in this case the previous notions, in particular for decreasing filtering intersections (2.7), increasing filtering unions (2.8) and telescopic unions (2.9).

In ch. 3 we consider relations induced on projections and the corresponding mapping theorems for "regular" unions and intersections of projections: in particular for decreasing filtering intersections (3.6), increasing filtering unions (3.5) and telescopic unions or differences (3.7). Chapter 4 investigates a condition of regularity for induction, typically occurring in homological theories, which makes it to agree with composition.

Chapter 5 applies these results, yielding the convergence theorem for filtered complexes described above and the corresponding mapping theorem (5.5). Analogous results for the system of relative homologies will be given in [G9].

Finally the appendix (ch. 6) concerns distributive $RE$-categories (e.g.: the classifying $RE$-categories of the above mentioned theories), clarifying some aspects of the regularity conditions in this case.

1. - UNIONS AND INTERSECTIONS OF PROJECTIONS IN $RI$-CATEGORIES

In this chapter $A$ will always be a category with regular involution ($RI$-category for short): this means a category provided with an involution $\sim: A \to A$ (a contravariant endofunctor, identical on the objects and involutory) which is regular: $a = a\alpha(a)$ for each morphism $a$). For example, all categories of relations over exact categories are so [C1, C2; G6].

$A$ is selfdual. The morphism $a$ is monic iff $\alpha(a) = 1$, iff it has a left inverse; dually for epis; it is an isomorphism iff $\alpha(a) = 1$ and $\alpha(a) = 1$.

The considerations of 0.2 should be kept in mind, to substantiate the following arguments.

1.1 Projections: For every object $A$ we write $\text{Pr}(A)$ or $\text{Pr}j(A)$ the set of projections of $A$, i.e. symmetrical idempotent endomorphisms $e: A \to A$
(e = \varepsilon = ee), equipped with the canonical (partial) order <:

(1) \quad e < f \; \text{iff} \; e = ef \; \text{iff} \; e = fef',

whose greatest element is \(1_A\).

It is well known that the product of two projections \(e, f\) is idempotent, generally not symmetrical. We write \(\triangleright\) the commutativity relation in \(\text{Prj}(A)\):

(2) \quad e \triangleright f \; \text{iff} \; ef = fe \; \text{(iff ef is a projection)},

notice that if \(e \triangleright f\) then \(ef = fe\) is the intersection of \(e\) and \(f\) in \(\text{Prj}(A)\).

Moreover we consider in \(\text{Prj}(A)\) the reflexive relation of domination \(e \unlhd f\) and the associated symmetrical relation \(e \Phi f\):

(3) \quad e \unlhd f \; \text{if} \; e = efe, \quad e \Phi f \; \text{if} \; e = efe \text{ and } f = fef' (\dagger),

which generally are not transitive, together with the idempotent binary operation \&:

(4) \quad e \& f = efe (\dagger),

which generally is neither associative nor commutative. Clearly:

(5) \quad e < f \; \text{iff} \; e = f \& e,

(6) \quad e \unlhd f \; \text{iff} \; e = e \& f,

(7) \quad e \triangleright f \; \text{iff} \; e \& f < f, \; \text{iff} \; f \& e < e, \; \text{iff} \; e \& f < f \& e, \; \text{iff} \; e \& f = f \& e,

(8) \quad e < f \; \text{implies} \; e \unlhd f,

(9) \quad \text{if} \; e < f \unlhd e \; \text{then} \; e = f,

(10) \quad e \& f \Phi f \& e.

1.2. Transfer mappings: For every morphism \(a : A \to A'\) in \(A\) we have the associated transfer mappings of projections:

(1) \quad a: \text{Prj}(A) \to \text{Prj}(A'), \quad a(e) = aea,

(2) \quad a^*: \text{Prj}(A') \to \text{Prj}(A), \quad a^*(e') = \alpha e' \alpha,

(\dagger) {\text{If } A = \text{Rel } E, \text{ (H/K)A (H'K') means that the canonical correspondence from H/K to H'K' (induced by } 1_A\text{) is monic, while } \Phi \text{ means that the latter is iso (3.2.7.3.2.8). These relations, as well as the operation } \&, \text{ are described by meets and joins in } \text{Sub} (A) \text{ in 2.3. They are transitive (for every object } A\text{) iff the operation } \& \text{ is always associative, iff the category } E \text{ is distributive (see 6.4).

}
and two associated projections:

\[ e(a) = a^\nu(1_a) = \bar{a} \in \text{Prj}(A), \]
\[ i(a) = a_\nu(1_a) = \alpha \bar{a} \in \text{Prj}(A'). \]

respectively simulating the coinage and the image of \( a \). Clearly \( a^\nu = \bar{a} \), and \( i(a) = e(\bar{a}) \).

The procedure \( A \mapsto \text{Prj}(A) \), \( a \mapsto (a_\nu, a^\nu) \) can be formalized as a functor from \( A \) into a suitable RI-category ([G5], §7.4).

Remark that, for \( e, f \in \text{Prj}(A) \):

\[ e \& f = e_\nu(f) = e^\nu(f). \]

1.3. Null morphisms: The morphism \( a \in A(A', A') \) is said to be null if, for every \( a' \in A(A', A') \), \( a a' = a \). Null morphisms from an ideal of \( A \).

A null projection \( e \in \text{Prj}(A) \) is dominated by each parallel projection, while a projection which is dominated by a null one, is null. Transfer mappings preserve null projections.

Two projections \( e, f \in \text{Prj}(A) \) are said to be disjoint whenever \( e \& f \) (or equivalently \( f \& e \)) is null.

1.4. Lemma: Consider, in the RI-category \( A \), a morphism \( a: A \rightarrow A' \) and projections \( e, f, g \in \text{Prj}(A) \), \( e' \in \text{Prj}(A') \). Then:

a) if \( e \triangleleft f \triangleleft g \) then \( e \triangleleft g \),
b) if \( e \triangleleft f \triangleleft g \) then \( e \triangleleft g \),
c) if \( e \triangleleft f \) then: \( a_\nu(e) \triangleleft a_\nu(f) \), \( g \& e \triangleleft g \& f \), \( e \& g \triangleleft f \& g \),
d) if \( e \triangleleft a_\nu(e) \) and \( e \triangleleft \bar{a}a \) then \( e \triangleleft a^\nu(e') \),
e) \( e \triangleleft a^\nu(e') \) iff \( (a_\nu(e) \triangleleft e' \) and \( e \triangleleft \bar{a}a \)),
f) if \( a_\nu(e) \triangleleft e' \) then \( a^\nu a_\nu(e) \triangleleft a^\nu(e') \).

Proof:

a) \( ege = (ef)e = e(ge) = ef = efge = ege = e \).
b) \( ege = (ef)g = e(ge) = ege = e \).

c) First: \( a(e\bar{a}) = a(ef)\bar{a} = a(e(\bar{a}) = a(ef)\bar{a} = a \bar{a}; \)

the second property follows from the first (by 1.2.5); as to the third:

\[ (ege)(ef)(ege) = egege = ege. \]

d) \( e = (\bar{a}a)\bar{a}(\bar{a}a) = a^\nu a_\nu(e) \triangleleft a^\nu(e'), \) by \( e \).
e) If $e \mathbin{Cl} a^e(\epsilon)$, then $e \mathbin{Cl} a^e(1) = \partial a$, by (a) and (e); moreover $a_\epsilon(\epsilon) \mathbin{Cl} \epsilon$
since:

$$a_\epsilon(\epsilon) \cdot a^e(\epsilon) = \partial (a e^e(d e^e a)) = a \epsilon (d e^e a) \epsilon = a \epsilon (a^e(\epsilon) \cdot \epsilon) = a \epsilon \partial a = a_\epsilon(\epsilon).$$

Conversely, when the right-hand property of (e) holds:

$$\epsilon \cdot a^e(\epsilon) \cdot \epsilon = e \epsilon (d e^e a) \epsilon = (e \cdot (d e^e a) (d e^e a) (d e^e \epsilon) \epsilon) = e \epsilon ((a \epsilon (d e^e a) \epsilon) \epsilon) = e \epsilon = (e \cdot d e^e a) \epsilon = \epsilon \cdot d e^e a = e\epsilon = \epsilon.$$

$$f) (a^e a^e(\epsilon)) (a^e(\epsilon)) (a^e a^e(\epsilon)) = (d e^e (d e^e a) (d e^e a) (d e^e \epsilon) \epsilon) =$$

$$= \epsilon (d e^e a) \epsilon = \epsilon (d e^e a) \epsilon = a^e a^e(\epsilon).$$

1.5. Definition: Let $\epsilon = \bigcup_i \epsilon_i \ (i \in I)$ in the (canonically) ordered set

$\Prj \ (A)$. We say that this union is:

a) $P$-regular if, for each morphism $a$ with domain $A$, $a_\epsilon(\epsilon) = \bigcup_i a_\epsilon(\epsilon_i)$
in $\Prj \ (\text{Cod} \ a)$;

b) $D$-regular if, for each $f \in \Prj \ (A)$, the condition $\epsilon \cdot f \mathbin{Cl} f (\forall i \in I)$ implies

$\epsilon \cdot f$;

c) PD-regular if, for each morphism $a$ with domain $A$, $a_\epsilon(\epsilon) = \bigcup_i a_\epsilon(\epsilon_i)$
is a $D$-regular union in $\Prj \ (\text{Cod} \ a)$.

Analogously we define $P_\epsilon, D_\epsilon, PD$-regular intersections $\epsilon = \bigcap_i \epsilon_i \ (i \in I)$
in $\Prj \ (A)$; e.g.:

b) the intersection $\epsilon = \bigcap_i \epsilon_i$ is $D$-regular if, for each $f \in \Prj \ (A)$, the
condition $f \mathbin{Cl} \epsilon_i (\forall i \in I)$ implies $f \mathbin{Cl} \epsilon$.

However we shall see in 1.7 that $P$-regular intersections are always $D$-regular,
and coincide with the PD-regular ones. This does not hold for unions;
there are cases which are $P$- and $D$-regular, but not PD-regular (2.11). The
applications to the convergence of spectral sequences (ch. 5 and [G9]) will
show that the good notion is the strongest one: PD-regularity.

1.6. Elementary properties:
a) PD-regularity implies $D$- and $P$-regularity, for unions as well as
for intersections.

b) Associativity and commutativity hold for all these kinds of unions
and intersections, as well as the cancellability of repeated elements.

c) If $\epsilon = \bigcup_i \epsilon_i$ is $D$- or $P$- or PD-regular and $\epsilon_i \prec \epsilon_i \prec \epsilon$ for every $i$,
then $\epsilon = \cup f_i$ is again so; dually for intersections (for the proof, use 1.4 a), b)).

d) If $\epsilon = \max \epsilon_i$ then $\epsilon = \bigcup \epsilon_i$ is PD-regular; dually for intersections,
e) The transfer mappings preserve \( P \)- and \( PD \)-regular unions or intersections.

f) If \( e = \bigcup \varepsilon_i \) is a \( D \)-regular union and \( e_i \) is null for each \( i \neq j \), then \( e = e_i \) (use 1.1.9 and the fact that \( e_i \cap e_j \) for all \( i \)).

1.7. PROPOSITION: If \( e, e_i \in \text{Prj}(A) \ (i \in I; \ I \neq \emptyset \ (\ast)) \), \( e \) is the \( P \)-regular intersection of the family \( \{e_i\} \) iff:

\[
f \& e = \bigcap_i f \& e_i, \quad \text{for every } f \in \text{Prj}(A).
\]

In this case, if \( f \) commutes with all \( e_i \), it also commutes with \( e \).

\( P \)-regular intersections are also \( D \)-regular, and coincide with the \( PD \)-regular ones.

PROOF: The necessity of (1) is obvious, since \( f \& e = f \& e_i \). Conversely, assume (1) and take \( a \in A(A, A') \). First, \( e = \bigcap_i e_i \) (take \( f = 1_A \) in (1)). Moreover, \( e < e_i \) implies \( a \& e < a \& e_i \); if \( e' < a \& e_i \) for all \( i \):

\[
(2) \quad a \& e' \& a \& e_i = (a \& e_i) \& a \quad (i \in I),
\]

\[
(3) \quad a \& e' \& e = a \& e_i \& e = a \& (e_i \& e) .
\]

Now, as \( e' < a \& e_i < a \& e = a \& e \) for some \( i \in I \) \((I \neq \emptyset)\), by 1.4 (\( d \)) applied to \( a \) it follows:

\[
(4) \quad e' < a \& a \& e = a \& e .
\]

Assume now that \( e = \bigcap e_i \) is \( P \)-regular in \( \text{Prj}(A) \). If \( f \) commutes with all \( e_i \), then (1.7) \( f \& e_i < f \& e_i \) \((i \in I)\) and \( f \& e = \bigcap f \& e_i < \bigcap e_i = e \), i.e. \( f \) commutes with \( e \). In order to prove that \( e = \bigcap e_i \) is also \( D \)-regular, take \( f \& e_i \) \((i \in I)\); then \( f = f \& e_i \) and \( f \& e = \bigcap f \& e_i = \bigcap f = f \), i.e. \( f \& e \). Finally, since \( P \)-regular intersections are preserved by transfer mappings (1.6 (\( e \))), it follows that \( P \)-regular and \( PD \)-regular intersections coincide.

1.8. Commuting projections: In order to give a similar characterization for \( P \)- and \( PD \)-regular unions we need some considerations on the commutativity relation \( \ll \) of projections (1.1).

\( \ast \) The intersection of the empty family in \( \text{Prj}(A) \) is \( 1_A \). This intersection is \( P \)-regular iff every morphism \( a \) of domain \( A \) is epic \((a_\ast(1) = 1)\), which is equivalent to asking that every morphism of domain \( A \) is in \( \text{Im}(a_\ast: A \to A \text{ has to be epic}) \). Instead our intersection satisfies (1) iff \( 1_A \) is the unique projection of \( A \), iff every morphism of domain \( A \) is monic, which is a weaker condition.

It may be easily checked that the mapping \( \& f \) does not preserve \( \ll \); a fortiori it does not preserve intersections or unions.
Consider on $A$ the following equivalent properties concerning parallel projections (always satisfied for categories of relations over exact categories: 2.3):

(1) if $g < e$ and $g < f$, then $e! f$,

(2) $e! f$ iff there is some projection $g$ such that $g < e$ and $g < f$,

(3) if $e! g$ and $g < f$, then $e! f$.

Actually: (1) $\Rightarrow$ (2): if $e! f$, take $g = e \& f = ef = fs$; (2) $\Rightarrow$ (3): if $e! g$ and $g < f$, then $eg = geg$ precedes both $e$ and $g < f$, hence $e! f$; (3) $\Rightarrow$ (1): trivial.

Such a category $A$ also verifies (use 1.4.e)):

(4) if $e! f$ then $a_p(e) \uparrow a_p(f)$, \hspace{1cm} \text{for } e, f \in \Prj (\Dom a), \hspace{1cm}

(5) a finite family of parallel projections $(e_i)$ has intersection iff these projections commute; then the intersection is the product and is $P$-regular.

1.9. PROPOSITION: Assume that the $RI$-category $A$ satisfies the equivalent conditions 1.8.1-3 and let $e, e_i \in \Prj (A) \ (i \in I; \ I \neq \emptyset)$. Then:

a) $e$ is the $P$-regular union of $(e_i)$ iff for every $f \in \Prj (A), f \& e = \bigcup f \& e_i$;

b) $e$ is the $PD$-regular union of $(e_i)$ iff for every $f \in \Prj (A), f \& e = \bigcup f \& e_i$ is $D$-regular.

Moreover, if $e = \bigcup e_i$ (simple union) and there is some $f \in I$ such that $f \uparrow e_i$, then $f \uparrow e$.

PROOF:

a) The beginning of the proof is similar to the one of 1.7: we just reverse all $\prec, \cap$ until we reach 1.7.3, which becomes:

(1) $a_p(e') > a_p a_p(e)$.

Now $e' > a_p(e_i)$ and $a_p(e_i) < a_p(1) = a_\emptyset$ for some $i \in I$; by 1.8.2, $e'$ commutes with $a_\emptyset$ and:

(2) $e' > a_p(a_\emptyset) > a_p a_p(e') > a_p a_p a_p(e) = a_p(e)$.

b) Assume that the right-hand condition holds (its necessity is trivial) and take some morphism $a \in A(A, A')$; then $a_p(e) = \bigcup a_p(e_i)$ by $a$, and we have to prove this union is $D$-regular. Let $a_p(e_i) \uparrow e'$ in $\Prj (A')$, for each $i \in I$. Then (1.4.f)):

(3) $a_\emptyset \& e_i = a_p a_p(e_i) \uparrow a_p(e') \hspace{1cm} (i \in I)$,
and, by the hypothesis of $D$-regularity w.r.t. $\partial a$:

\[(\partial a) \& e \emptyset a^p(e').\]

By 1.4d):

\[a_p(e) = a_p a^p a_p(e) = a_p((\partial a) \& e) \emptyset e'.\]

The last remark follows at once from 1.8.3: $f \vdash e_i$ and $e_i \preceq e$.

1.10. A counterexample: In order to build a semigroup $S$, with regular involution, which does not satisfy the conditions 1.8.1-3, just consider the semigroup with regular involution generated by three projections $e, f, g$ under the conditions $g \preceq e, g \preceq f$. The multiplication table of $S$ is the following:

\[
\begin{array}{ccc|ccc|ccc}
  & e & f & g & ef & e f & e f & \varepsilon f & j f & j f \\
\hline
  e & e & e & e & e f & e f & e f & \varepsilon f & j f & j f \\
  f & f & e & f & f & f & f & f & f & f \\
  g & g & g & g & g & g & g & g & g & g \\
 ef & ef & ef & ef & ef & ef & ef & ef & ef & ef \\
 e f & e f & e f & e f & e f & e f & e f & e f & e f & e f \\
 e f & e f & e f & e f & e f & e f & e f & e f & e f & e f \\
 j f & j f & j f & j f & j f & j f & j f & j f & j f & j f \\
 j f & j f & j f & j f & j f & j f & j f & j f & j f & j f \\
\end{array}
\]

while the involution is obvious: $(ef)^\sim = f$, etc.

By adjoining a unit $1$ we get a regular involution category $A$ (on one object) which does not satisfy 1.8.1 (since $e$ and $f$ do not commute). Notice that $A$ is idempotent.

2. - Unions and Intersections of Projections in RE-Categories

RE-categories, introduced in [G6], are a slight extension of the categories of relations over exact categories (see also 3.1). We recall here the definitions and some properties, and study the unions and intersections of their projections.

2.1. RO-categories: An RO-category $A = (A, \sim, \preceq)$ is a category $A$ provided with a regular involution $\sim$ and a (partial) order relation $\preceq$ on parallel morphisms, consistent with composition and involution. All sets $\text{Prj}(A)$ are thus provided with the canonical order $\preceq$ and with the order $\preceq$, which are different (2.3.1-2); unless otherwise stated, all the order notions (e.g., unions and intersections) we consider for these sets concern the canonical order $\preceq$.

An endomorphism $\kappa: A \to A$ verifying $\kappa \preceq 1$ is called a restriction of $A$,
Every restriction is a projection; two projections \( x, x' \) always commute and:

\[ x \ll x' \quad \iff \quad x \ll x' \, . \]  

Analogously for corestrictions, i.e. endomorphisms \( \zeta : A \to A \) such that \( \zeta \gg 1 \); except that, if \( \zeta, \zeta' \) are so:

\[ \zeta \ll \zeta' \quad \iff \quad \zeta \gg \zeta' \, . \]

We write \( \text{Rst} (A) \) and \( \text{Crs} (A) \) the subsets of \( \text{Prj} (A) \) of restrictions and corestrictions.

The proper morphisms \( h \) (verifying \( h \ll 1 \) and \( h \gg 1 \)) from a subcategory, \( \text{Prp} A \); if \( h \) and \( v \) are so, it is easy to check that \( h \ll v \) implies \( h = v \) ([G6], § 1.7).

2.2. \( \text{RE-categories} \): An \( \text{RE-category} A \) is a \( \text{RO-category} \) satisfying:

\( \text{(RE.1)} \) for every \( e \in \text{Prj} (A) \) there exists a unique restriction \( n(e) \) such that \( n(e) \gg e \) and \( n(e) \ll e \) (the numerator of \( e \)), together with a unique corestriction \( d^*(e) \) such that \( d^*(e) \gg e \) and \( d^*(e) \ll e \) (the \( \sim \)-denominator of \( e \)) \(^4\);

\( \text{(RE.2)} \) for every object \( A \) there is a null restriction \( \omega_A \) and a null corestriction \( \Omega_A \) (equivalently: the ordered set \( (A(A, A), \ll) \) has minimum \( \omega_A \) and maximum \( \Omega_A \)) \(^4\).

Then ([G6], § 4.7) there is a canonical bijection (\( \sim \)-duality) between restrictions and corestrictions, which preserves \( \ll \) and reverses \( \ll \):

\[ \begin{align*}
\text{Rst} (A) & \to \text{Crs} (A), && x \mapsto x^* = d^*(x \Omega x), \\
\text{Crs} (A) & \to \text{Rst} (A), && \zeta \mapsto \zeta_\omega = n(\zeta \omega \zeta),
\end{align*} \]

and allows to define the denominator \(^4\) of a projection \( e \) of \( A \):

\[ d(e) = (d^*(e))_e = n(d^*(e) \omega d^*(e)) = n(e \omega e) \in \text{Rst} (A) \, . \]

Every morphism \( a : A \to A' \) determines four restrictions, the definition, annihilator, values and indetermination (corresponding to the usual subobjects

\(^4\) If \( A = \text{Rel} E \) is a category of relations and \( e \) is associated to the subquotient \( H \uparrow K \) of \( A \):

\[ \begin{align*}
n e &= (A \ll H \gg A), & d^*(e) &= (A \to A \uparrow K \\ n e) = (A \ll (0 \gg A), & \omega_A &= (A \ll (0 \gg A), \\
\Omega_A &= (A \to 0 \ll A), & d(e) &= (A \ll (K \gg A).
\end{align*} \]

The restrictions \( n \) of \( A \), characterized by \( d n = n \), correspond to subjects, while the corestrictions \( \zeta \), characterized by \( n \zeta = 1 \), correspond to quotients. The \( \sim \)-duality described below corresponds to the kernel-cokernel duality between subobjects and quotients of \( E \).
when \( A \) is a category of relations):

\[
\begin{align*}
\text{(4)} & \quad \text{def } a = n(\bar{a}a), \quad \text{an} n a = d(\bar{a}a) \quad \text{in } \text{Rst}\,(A), \\
\text{(5)} & \quad \text{val } a = n(\bar{a}a), \quad \text{ind } a = d(\bar{a}a) \quad \text{in } \text{Rst}\,(A').
\end{align*}
\]

which characterize in the usual way monos, epis, isos and proper morphims.

In this chapter, from now on, \( A \) will always be an RE-category.

2.3. Calculus of projections: The following results hold for \( e, f \in \text{Prj}\,(A) \) ([G6], §§ 4.4, 6.9):

\[
\begin{align*}
\text{(1)} & \quad e < f \iff (ne < nf \text{ and } d' e < d' f) \iff (ne < nf \text{ and } d e > df), \\
\text{(2)} & \quad e < f \iff (ne < nf \text{ and } d' e > d' f) \iff (ne < nf \text{ and } d e < df), \\
\text{(3)} & \quad \text{Rst}\,(A) \text{ and Crs}\,(A) \text{ are modular lattices with 0 and 1}, \\
\text{(4)} & \quad n(e \& f) = ne \cap (nf \cup de) = (ne \cap nf) \cup de, \\
\text{(5)} & \quad d(e \& f) = ne \cap (df \cup de) = (ne \cap df) \cup de, \\
\text{(6)} & \quad e \downarrow f \iff (ne > df \text{ and } nf > de), \\
\text{(7)} & \quad e \uparrow f \iff (ne < nf \cup de \text{ and } de > df \cap ne), \\
\text{(8)} & \quad e \text{ is null iff } ne = de.
\end{align*}
\]

In particular, by (6), every RE-category satisfies the conditions 1.8.1-3.

There is an isomorphism of ordered sets (actually, modular lattices):

\[
\text{(9)} \quad (\text{Prj}\,(A), <) \to \text{Rst}_2\,(A) = \{(x, y) \in \text{Rst}\,(A) \times \text{Rst}\,(A) | x > y\}, \quad e \mapsto (ne, de),
\]

whose reciprocal isomorphism will be written:

\[
\text{(10)} \quad (x, y) \mapsto x'y = x \cdot y^* = y^* \cdot x, \quad (x > y).
\]

Every morphism \( a: A \to A' \) of \( A \) defines increasing transfer mappings for restrictions:

\[
\begin{align*}
\text{(11)} & \quad a_\uparrow : \text{Rst}\,(A) \to \text{Rst}\,(A'), \quad a_\uparrow (x) = n(a_\downarrow (x)) = n(ax\bar{a}), \\
\text{(12)} & \quad a_\downarrow : \text{Rst}\,(A') \to \text{Rst}\,(A), \quad a_\downarrow (x') = n(a_\uparrow (x')) = n(dx'\bar{a}),
\end{align*}
\]

satisfying the obvious functoriality conditions and:

\[
\text{(13)} \quad a_\uparrow (x/y) = a_\uparrow (x)/a_\downarrow (y), \quad (x > y \text{ in } \text{Rst}\,(A)).
\]
2.4. **Proposition (Unions and intersections in RE-categories):** Let $e_i$ be projections of $\mathcal{A}$ in the RE-category $\mathcal{A}$ ($i \in I$).

a) The non-empty family $(e_i)$ has a (simple) union in $\text{Prj}(\mathcal{A})$ iff the families $(ne_i)$, $(de_i)$ have respectively union and intersection in $\text{Rst}(\mathcal{A})$; in such a case: $\bigcup e_i = (\bigcup ne_i)(\bigcap de_i)$.

b) The family $(e_i)$ has a (simple) intersection in $\text{Prj}(\mathcal{A})$ iff the families $(ne_i)$, $(de_i)$ have respectively intersection and union in $\text{Rst}(\mathcal{A})$ and moreover $\bigcap ne_i > \bigcup de_i$; in such a case: $\bigcap e_i = (\bigcap ne_i)(\bigcup de_i)$.

**Proof:**

a) Fix some $j \in I$ ($I \neq \emptyset$). If the families $(ne_i)$, $(de_i)$ have respectively union and intersection in $\text{Rst}(\mathcal{A})$, then $(\bigcup ne_i) > ne_j > de_j > (\bigcap de_i)$, and it is easy to see that the projection $((\bigcup ne_i)(\bigcap de_i))$ is the union of $(e_i)$ in $\text{Prj}(\mathcal{A})$ (use 2.3.1). Conversely let $e = x'y = \bigcup e_i$; then $x = ne_j > ne_i$ and $y = de_j < de_i$ for each $i$; moreover if $x' < ne_i$ and $y < de_i$ in $\text{Rst}(\mathcal{A})$, for every $i$, then $x' > ne_i > de_i > y'$ and we can consider $e' = x'y' \in \text{Prj}(\mathcal{A})$; trivially, $e' > e$, for all $i$, hence $e' > e$, i.e. $x' > x$ and $y' < y$.

b) The right-hand condition is clearly sufficient. Conversely, let $e = x'y = \bigcap e_i$; then $x < ne_i$, and $y > de_i$ for each $i$. Assume that $x' < ne_i$ and $y' > de_i$ in $\text{Rst}(\mathcal{A})$, for all $i$; as $x \cup x' > x = ne_i > de_i = y > y' \cap y'$, we may consider the projection $f = (x \cup x')(y \cap y') < e_i$ ($i \in I$); it follows that $f < e_i$, i.e. $x \cup x' < x$ and $y \cap y' < y$, i.e. $x' < x$ and $y' < y$.

2.5. **Proposition (Unions and intersections of restrictions):** Let $e \in \text{Prj}(\mathcal{A})$ and $x, y \in \text{Rst}(\mathcal{A})$, for $i \in I$. Then.

1) $e = \bigcup x_i$ in $\text{Prj}(\mathcal{A})$ iff $e \in \text{Rst}(\mathcal{A})$ and $e = \bigcup x_i$ in $\text{Rst}(\mathcal{A})$.

2) if $x = \bigcup x_i$ in $\text{Rst}(\mathcal{A})$, the union is $D$-regular in $\text{Prj}(\mathcal{A})$ iff for every $y \in \text{Rst}(\mathcal{A})$, $y \cap x_i = m$ for all $i$ implies $y \cap x = m$.

Further, the following conditions are equivalent:

a) $x = \bigcup x_i$ is $P$-regular in $\text{Prj}(\mathcal{A})$,

b) $x = \bigcup x_i$ is $PD$-regular in $\text{Prj}(\mathcal{A})$,

c) $x = \bigcup x_i$ is a distributive union in the lattice $\text{Rst}(\mathcal{A})$ (i.e. $x \cap y = \bigcup (x_i \cap y)$, for all $y \in \text{Rst}(\mathcal{A})$),

d) $a_n(x) = \bigcup a_n(x_i)$ in $\text{Rst}(\mathcal{A})$, for every $a \in A(\mathcal{A}, \mathcal{A})$.

A dual result holds for intersection of restrictions. Distributive unions and intersections of restrictions are preserved by the transfer mappings $a_n$, as well as by $\cap y$ and $\cup y$ ($y \in \text{Rst}(\mathcal{A})$).
PROOF: Property (1) follows from 2.4 a), with \( \varepsilon = \nu \), \( \nu = \nu / \omega \); (2) is also trivial, since \( x \uplus y \) iff \( x \prec x' \) and \( x \cap y = \emptyset \) (2.3.7).

a) \( \Rightarrow d) \) If \( x = \bigcup x_i \) is \( P \)-regular, \( a_p(x) = \bigcup a_p(x_i) \) in \( \text{Prj}(A) \). By 2.3.12 and 2.4: \( a_p(x) = n(a_p(x)) = \bigcup n(a_p(x_i)) = \bigcup a_p(x_i) \).

d) \( \Rightarrow b) \) Take a morphism \( a : A \to A' \); then, by 2.4, \( a_p(x) = \bigcup a_p(x_i) \) in \( \text{Prj}(A') \):

\[
\begin{align*}
\eta a_p(x) &= a_p(x_i) = \bigcup a_p(x_i) = \bigcup \eta a_p(x_i), \\
\delta a_p(x) &= a_p(\omega) = \bigcap a_p(\omega) = \bigcap \delta a_p(x_i).
\end{align*}
\]

In order to prove that the previous union is \( D \)-regular in \( \text{Prj}(A') \), assume that \( a_p(x_i) \uplus f = \nu' \) in \( \text{Prj}(A') \) \((i \in I)\); by the characterization of domination in 2.3.7 this means that:

\[
\begin{align*}
a_p(x_i) &< \nu' \cup a_p(\omega), \\
a_p(\omega) &> \nu' \cap a_p(x_i), & (i \in I), \\
a_p(x) &= \bigcup a_p(x_i) < \nu' \cup a_p(\omega), \\
y' \cap a_p(x) &= y' a_p(x) = \bigcup y' a_p(x_i) = \bigcup (y' \cap a_p(x_i)) < a_p(\omega),
\end{align*}
\]

so that, again by 2.3.7, \( a_p(x) \uplus f \).

b) \( \Rightarrow a) \) Obvious.

c) \( \Rightarrow a) \) We use the characterization 1.9 of \( P \)-regular unions (the condition 1.8.1 holds, as remarked in 2.3); if \( f \in \text{Prj}(A) \), by 2.3.4-5:

\[
\begin{align*}
n(f \cup f) &= n \cap (\nu \cup df) = n \cap (\bigcup x_i \cup df) = \\
&= \bigcup (n \cap (\nu \cup df)) = \bigcup n(f \cap f), \\
d(f \cup f) &= df \cup (\omega \cap nf) = df, \\
d(f \cap f) &= df \cup (\omega \cap nf) = df,
\end{align*}
\]

so that, by 2.4: \( f \uplus x = \bigcup f \uplus x_i \).

Now a)-d) are equivalent. For the intersection case, use the \('\cap \cap\)-duality, which preserves projections and restrictions but reverses \( < \).

The last remark follows from the functoriality of \( a \mapsto a_p \), together with the following facts: \(- \cap y = y_\uparrow, - \cup y = (1/y)_\downarrow \) (from 2.3).

2.6. THEOREM (\( P \)-regular unions and intersections): Let \( \varepsilon_i \in \text{Prj}(A) \), \( i \in I \neq \emptyset \).

a) The family \( \{\varepsilon_i\} \) has a \( P \)-regular union iff \( \{\varepsilon_i\} \) and \( \{\delta_i\} \) have respectively distributive union and distributive intersection in \( \text{Rst}(A) \); in such a case: \( \bigcup \varepsilon_i = (\bigcup \varepsilon_i) \cap (\bigcap \delta_i) \).
b) The family \((e_i)\) has a \(P\)-regular intersection iff \((ne_i)\) and \((de_i)\) have respectively distributive intersection and distributive union in \(\text{Rst} (A)\) and moreover the former is greater than the latter; in such a case: \(\bigcap e_i = \left(\bigcap ne_i\right) / \left(\bigcup de_i\right)\).

c) \(P\), \(D\), and \(PD\)-regular intersections of projections coincide.

d) If there exist \(x = \max ne_i\) and \(y = \min de_i\), then \(x' y\) is the \(P\)-regular union of \((e_i)\) (this union need not be \(PD\)-regular: see 2.11).

e) If there exist \(x = \min ne_i\) and \(y = \max de_i\), with \(x > y\), then \(x' y\) is the \(PD\)-regular intersection of \((e_i)\).

**Proof:**

\(a)\) If \((e_i)\) has a \(P\)-regular union \(e\), for each \(a: A \to A', a_p(e) = \bigcup a_p(e_i);\) by 2.4, this means that:

\[ a_n(ne) = \bigcup a_n(ne_i), \quad a_n(de) = \bigcap a_n(de_i) \quad \text{in} \ \text{Rst} (A'), \]

hence, by 2.5, \(ne = \bigcup ne_i, \ de = \bigcap de_i\) distributively in \(\text{Rst} (A)\). Conversely, when the last property holds \((e_i)\) has a union \(e = (\bigcup ne_i) / (\bigcap de_i)\) by 2.4, which is easily seen to the \(P\)-regular by reversing the above argument.

\(b)\) Reverse all unions and intersections in the proof of \(a)\).

c) We already know that, as to intersections of projections, \(P\)-regular implies \(D\)-regular and coincides with \(PD\)-regular (1.7). Therefore, let \(\varepsilon = \bigcap e_i\) be \(D\)-regular in \(\text{Prj} (A)\): by 2.4, \(x = \bigcap x_i\) and \(y = \bigcup y_i\) in \(\text{Rst} (A)\), where \(x = ne, y = de\) and so on; by \(b)\), we just need to prove this intersection and union to be distributive.

Let \(\varpi \in \text{Rst} (A)\): first we want to show that \(x \cup \varpi = \bigcap (x_i \cup \varpi)\); we can assume that \(\varpi > x\): otherwise, by setting \(\varpi' = x \cup \varpi > x\), it follows that

\[ x \cup \varpi = \varpi = \bigcap (x_i \cup \varpi) = \bigcap (x_i \cup x \cup \varpi) = \bigcap (x_i \cup \varpi). \]

Take also some restriction \(\varpi' < x_i \cup \varpi\) \((i \in I)\). Then \((\varpi \cup \varpi')/\varpi \bigcap e_i, (i \in I)\) by 2.3.7, since:

\[ \varpi \cup \varpi' < x_i \cup \varpi, \quad y_i \cap (\varpi \cup \varpi') < y < x < \varpi. \]

By the hypothesis of \(D\)-regularity of \(\varepsilon = \bigcap e_i, (\varpi \cup \varpi')/\varpi \bigcap e_i\), hence \(\varpi \cup \varpi' < x \cup \varpi\) and \(\varpi' < x \cup \varpi\).

Second, we have to prove that \(y \cap \varpi = \bigcup (y_i \cap \varpi)\); hence we assume \(\varpi < y\) (otherwise, consider \(\varpi = y \cap \varpi\) and choose some \(\varpi' < y_i \cap \varpi\), for all \(i\). Now we have \(\varpi/(\varpi \cap \varpi') \bigcap e_i, (i \in I)\), since:

\[ \varpi < y < x < x_i \cup (\varpi \cap \varpi'), \quad y_i \cap \varpi < \varpi \cap x \cap \varpi'. \]
again by the $D$-regularity of $e = \bigcap e_i$, $x/(z \cap z') \subseteq e$. Finally $y \cap z < z \cap z'$ and $z' > y \cap z$.

d) Clearly $x/y = \bigcup e_i$, (2.4); since transfer mappings preserve $<$, hence maxima and minima of indexed families, the union is $P$-regular.

e) Analogous to the proof of d), combined with e).

2.7. Corollary (Intersection of decreasing filtering families): Let $(e_i)$ be a decreasing family of projections of $A$, indexed on a non-empty, filtering set $I$ (i.e., an ordered set where every finite subset has some upper bound). If $x = \bigcap ne_i$ and $y = \bigcup de_i$ exist in $\text{Rst} (A)$, then $x > y$ and $x/y = \bigcap e_i$; the last intersection is $P$-regular iff it is $D$-regular, iff it is $PD$-regular, iff $x = \bigcap ne_i$ and $y = \bigcup de_i$ are distributive in $\text{Rst} (A)$.

Proof: We only need to prove the inequality $x > y$, as the rest follows from 2.4(b) and 2.6(b), e). Actually, if $i < j$ in $I$: $ne_j > ne_i > de_i$; thus, for all $i$, $ne_i > \bigcup de_i = \bigcup \bigcup_i de_i$ (the set $\{i: j > i\}$ is cofinal in $I$), and $x = \bigcap ne_i > y = \bigcup de_i$.

2.8. Theorem (Union of increasing filtering families): Let $I$ be a non-empty, filtering set $I$. For increasing $I$-families of projections, $P$-regular unions are $D$-regular and coincide with the $PD$-regular ones.

Proof: Let $(e_i)$ be an increasing $I$-family of projections of $A$ and assume that $e = \bigcup e_i$ is $P$-regular: by 2.6, $ne = \bigcup ne_i$ and $de = \bigcap de_i$ are distributive in $\text{Rst} (A)$; consider now a projection $f = x'/y'$ such that $e \Delta f (i \in I)$, i.e.:

$$ne_i < x' \cup de_i, \quad y' \cap ne_i < de_i, \quad (i \in I).$$

Now, for $i < j$ in $I$:

$$ne_i < ne_j < x' \cup de_j, \quad de_j > de_i > y' \cap ne_j;$$

from the cofinality of $\{i: j > n\}$ in $I$, it follows that:

$$ne_i < \bigcap_{j > i} x' \cup de_j = \bigcap_i x' \cup de_i = x' \cup (\bigcap de_i) = x' \cup de,$$

$$de_i > \bigcup_{j > i} y' \cap ne_j = \bigcup_i y' \cap ne_i = y' \cap (\bigcup ne_i) = y' \cap ne,$$

that is, $e \Delta f$.

Finally, since transfer mappings preserve $P$-regular unions and increasing families (1.4 e)), the above argument also proves that every $P$-regular union is also $PD$-regular.
2.9. *Theorem (Union of telescopic families)*:

a) Let:

\[ x_0 < x_1 < \ldots < x_n, \quad (n \geq 1), \]

be a finite increasing family in \( \text{Rst} (A) \) and consider the associated telescopic family of projections \( e_i = x_i / x_{i-1} \) \((i = 1, 2, \ldots, n)\). Then:

\[ x_n / x_0 = \bigcup e_i \quad \text{(PD-regular in \text{Proj} (A))}. \]

b) Let \((x_i)_{i \in I}\) be an increasing family in \( \text{Rst} (A) \), indexed on an interval \( I \) of \( \mathbb{Z} \) (with at least two elements); consider the associated telescopic family of projections \( e_i = x_i / x_{i-1} \) \((i \in \bar{I} = \{i \in I : i - 1 \in I\})\).

The following conditions are equivalent:

(3) \((e_i)\) has a \( P \)-regular union \( e \),

(4) \((e_i)\) has a \( PD \)-regular union \( e \),

(5) \((x_i)\) has union \( x^+ \) and intersection \( x^- \), distributively in \( \text{Rst} (A) \),

and in such a case: \( e = x^+ / x^- \).

Telescopic unions are preserved by transfer mappings \( a \), and in particular by mappings \( f \& c \).

**Proof:**

a) By 2.6.4, \( x_n / x_0 = \bigcup e_i \) is \( P \)-regular; since transfer mappings of projections trivially preserve telescopic families, we only need to prove that this union is \( D \)-regular. Assume that \( e_i \triangleq f \triangleq x / y \) \((i = 1, \ldots, n)\); then:

\[ x_i < x \cup x_{i-1} \quad \text{and} \quad y \cap x_i < x_{i-1}, \]

\[ x_n < x \cup x_{n-1} \quad < x \cup (x \cup x_{n-2}) \quad < \ldots < x \cup x_0, \]

\[ x_0 > y \cap x_i > y \cap (y \cap x_i) > \ldots > y \cap x_n, \]

which precisely means that \( x_n / x_0 \triangleq f \).

b) We may assume that \( I = \mathbb{Z} \) \((= \bar{I})\), possibly by repeating elements in \((x_i)\). By 2.6, the family \((e_i)\) has \( P \)-regular union \( e = x^+ / x^- \) iff:

\[ x^+ = \bigcup_{i \in \mathbb{Z}} x_i = \bigcup_{i \in \mathbb{Z}} x_i, \quad x^- = \bigcap_{i \in \mathbb{Z}} d_i = \bigcap_{i \in \mathbb{Z}} x_{i-1} \quad \text{(distributive)}. \]

If this is the case, the union is also \( PD \)-regular, by the following argument: for every \( i \in \mathbb{N} \), we have a projection:

\[ f_i = x_i / x_{i-1} = \bigcup_{i+1} \quad \text{(PD-regular union by } a\)). \]
and we get an increasing family \((f_i)\) on \(N\), which has (2.6, 2.8) PD-regular union \(\bigcup_{i \in \mathbb{N}} x_i / \bigcap_{i \in \mathbb{N}} x_i = x' / x = e;\) by associativity and cancellability of repeated elements (1.6), the union of \((e_i)\) is PD-regular.

2.10. Some examples in categories of modules:

a) If \(E\) is the abelian category of modules on some unitary ring \(R\), every increasing union \(H = \bigcup H_i\) of submodules of \(A\) is clearly distributive:

\[
(1) \quad H \cap K = \bigcup (H_i \cap K), \quad \text{for all submodules } K;
\]

indeed, if \(x \in H \cap K\) then \(x \in H_i\) for a suitable index \(i\) and \(x \in \bigcup (H_i \cap K)\).

b) In the category of abelian groups there exist decreasing intersections of subgroups which are not distributive.

E.g. in \(\mathbb{Z}\), the family \(H_n = 2^n \cdot \mathbb{Z} (n \geq 0)\) has null intersection, while for each \(n\), \(2^n \cdot \mathbb{Z} / 3\mathbb{Z} = \mathbb{Z}\).

Notice also that, for decreasing families \((H_n)\) of subgroups of the abelian group \(A\), the conditions:

i) \(\bigcap H_n = 0\), distributively in \(\text{Sub}(A)\),

ii) \(\lim_{\to} (A/\bigcap H_n) = A\),

are independent. First, take:

\[
(2) \quad A = \mathbb{Z}^N, \quad H_n = \{(k_i) \in \mathbb{Z}^N; k_0 = k_1 = \ldots = k_n = 0\},
\]

so that ii) holds and i) does not: for the subgroup \(K = \mathbb{Z}^{\infty}\) of quasi-null sequences of integers, \(K \cap H_n = A\) for all \(n\).

On the other hand, let \(p_i\) denote the sequence of all prime integers and set:

\[
(3) \quad A = \mathbb{Z}, \quad H_n = p_1 \cdot p_2 \cdot \ldots \cdot p_n \cdot \mathbb{Z};
\]

since \(H_n\) is eventually contained in each subgroup of \(\mathbb{Z}\), the condition i) is satisfied, while it is not difficult to see that ii) is not.

2.11. A counterexample: Last we build an example, in the category \(A\) of relations over abelian groups, of a \(P\)-regular, \(D\)-regular union \(1 = e_1 \cup e_2\) which is not PD-regular.

Thus, for unions of projections in \(RE\)-categories: \(P\)- and \(D\)-regular does not imply PD-regular; moreover \(P\)-regular does not imply \(D\)-regular (otherwise \(P\)-regularity should coincide with PD-regularity, as it happens for intersections: 1.7).

Let \(A\) be an abelian group whose lattice of subgroups is a four-element chain: \(0 < Y < X < A\); e.g. \(A = \mathbb{Z}/8\mathbb{Z}\). Thus the lattice \(\text{Rst}(A)\) of restrictions of \(A\) in \(A\) is the chain: \(0 < y < x < 1\), where \(y = (A \ll Y \gg A)\) and
analogously for $x$. Consider the projections of $A$:

(1) \[ y = y / y, \quad x = 1 / x, \]

corresponding to the subquotients $Y$ and $A / X$.

By 2.6 d), $y \cup x = 1 / y = 1$ is $P$-regular. It is easy to see that the union is also $D$-regular: the unique projection $f$ of $A$ such that $y \cup x \subseteq f$ is 1.

Finally we show that $y \cup x = 1$ is not $P\!\!D$-regular. Consider the projection $f = x / y$ and the transfer mapping $f_p = f \& \rightarrow$:

(2) \[ f \& y = (x \cap (y \cup y)) / (x \cap (y \cup y)) = y / y, \]

(3) \[ f \& (1 / x) = (x \cap (1 \cup y)) / (x \cap (1 \cup y)) = x / x, \]

now the union $(y / y) \cup (x / x) = f \& 1 = f$ is not $D$-regular, since the projections (2) and (3) are null (i.e. dominated by $a$), while $f$ is not so.

3. Induction in RI-categories

We treat here the morphisms induced on projections by the morphisms of $A$ and their connections with unions and intersections of projections; in order to avoid the condition that $A$ be factorizing, the induced morphisms will live in the associated factorizing category $B = \text{Fct} A$ recalled in 3.1. $A$ is always an RI-category, with $a \in A(\mathcal{A}, A', \epsilon \in \text{Prj} (\mathcal{A})$ and $\epsilon \in \text{Prj} (\mathcal{A})$.

3.1. Factorizing RI-categories. The RI-category $A$ is said to be factorizing when each morphism has an essentially unique epi-mono factorization (actually the uniqueness follows easily from the existence of a regular involution). Equivalent condition: for each projection $\epsilon$ there exists a mono $e$ such that $\epsilon = \epsilon e$.

In particular, an RE-category $A$ is factorizing if its category of proper morphisms $E$ is componentwise exact; in such case $A = \text{Rel} E$ ([G6], §§ 6.1, 6.4).

Every RI-category $A$ has an associated factorizing RI-category $B = \text{Fct} A$ ([G6], § 3.5): its objects are all the projections of $A$, while a morphism $a : e \rightarrow f$ is given by any $a \in A(\text{Dom} e, \text{Dom} f)$ such that $a = fae$ (or equivalently: $a = a \epsilon = fa$); the composition and involution are those of $A$ and $1_a = (e : e \rightarrow e)$. Thus:

(1) \[ a : e \rightarrow f \text{ is monic in } B \text{ iff } \delta \epsilon = e \text{ (in } A) ; \text{ it is epic iff } a \delta = f. \]

There is an obvious fully faithful embedding:

(2) \[ \text{A} \rightarrow \text{Fct} A, \quad A \mapsto 1_a. \]
If \( s: S \to A \) is monic in \( A \), \( 1_s \) is isomorphic in \( B \) to the associated projection \( e = \pi: A \to A \), through:

\[
s: 1_s \to e.
\]

Thus (1) is an equivalence of categories iff \( A \) itself is factorizing; in this case the reciprocal equivalence is: \( e \mapsto \text{Im} e \) (for some choice of images in \( A \)).

If \( A \) is an RO-category, \( B \) is also so (with the «same» order \(<\) ); \( A \) is an RE-category iff \( B \) is so ([G6], § 6.5).

3.2. Definition: We say that the morphism \( a: A \to A' \) induces from \( e \in \text{Prj} (A) \) to \( f \in \text{Prj} (A') \) the following morphism of \( B (e, f) \):

\[
fae: e \to f.
\]

Analogously, if \( s: S \to A \) and \( t: T \to A' \) are monic in \( A \), we say that \( a \) induces from \( s \) to \( t \) the morphism \( a^*: s_\pi: S \to T \) of \( A \). The second notion is a particular case of the first one (3.1.3) and the two notions are equivalent when \( A \) is factorizing. Therefore we develop mainly the former, but we shift to the latter in the applications of ch. 5.

Beware, this induction, generally, does not agree with composition ([Ma], p. 53; see also 6.3).

We say that \( a \) induces a mono (resp. an epi) from \( e \) to \( f \) whenever the (trivially) equivalent properties (2)-(5) (resp. (2')-(5')) are satisfied:

\[
(2) \quad fae: e \to f \text{ is mono in } B,
\]

\[
(2') \quad fae: e \to f \text{ is epi in } B,
\]

\[
(3) \quad s \Phi a \pi = e \text{ in } A,
\]

\[
(3') \quad sa \Phi e = f \text{ in } A,
\]

\[
(4) \quad e = e \& a^* (f) \text{ in } A,
\]

\[
(4') \quad f = f \& a_\pi (e) \text{ in } A,
\]

\[
(5) \quad e \Phi a^* (f) \text{ in } A,
\]

\[
(5') \quad f \Phi a_\pi (e) \text{ in } A;
\]

accordingly, we say that \( a \) induces an iso from \( e \) to \( f \) when (2) and (2') both hold; by 1.4(e) this is equivalent to:

\[
(6) \quad e \Phi a^* (f) \text{ and } f \Phi a_\pi (e) \text{ in } A.
\]

Notice that the properties (3)-(5), (3')-(5') and (6) do not require the construction of \( \text{Fct} A \). Thus, from now on, we work directly in \( A \).

Trivially, \( a \) induces a mono from \( e \) to \( f \) iff \( a \) induces an epi from \( f \) to \( e \). If \( e \) and \( f \) are null projections, \( a \) induces always an iso from \( e \) to \( f \).

For two parallel projections \( e_1, e_2 \in \text{Prj} (A) \):

\[
(7) \quad e_1 \Phi e_2 \text{ iff } 1_A \text{ induces a mono from } e_1 \text{ to } e_2,
\]

\[
(8) \quad e_1 \Phi e_2 \text{ iff } 1_A \text{ induces an iso from } e_1 \text{ to } e_2,
\]
and in the last case \( e_i \) and \( e_j \) may be said to be canonically isomorphic. Recall that this is not an equivalence relation, generally (1.2; 6.3-6.4).

3.3. Mapping theorem for \( D \)-regular unions: Let \( e_i < e \) in \( \text{Prj}(A) \) and \( f_i < f \) in \( \text{Prj}(A') \), for \( i \in I \).

a) If \( a \) induces a mono from \( e_i \) to \( f_i \) (for all \( i \)) and \( e = \bigcup e_i \) is \( D \)-regular in \( \text{Prj}(A) \), then \( a \) induces a mono from \( e \) to \( f \).

b) If \( a \) induces an epi from \( e_i \) to \( f_i \) (for all \( i \)) and \( f = \bigcap f_i \) is \( D \)-regular in \( \text{Prj}(A') \), then \( a \) induces an epi from \( e \) to \( f \).

Proof: a) By hypothesis, for each \( i: e_i \in \mathcal{A} \), \( a^{e_i}(f_i) \subseteq a^{f_i}(f) \); by 1.4.a), \( e \in \mathcal{A} \) \( a^{e}(f) \) and, by \( D \)-regularity, \( e \in \mathcal{A} \). The property b) follows from a) and \( \sim \)-duality.

3.4. Mapping theorem for \( D \)-regular intersections: Let \( e < e_i \) in \( \text{Prj}(A) \) and \( f < f_i \) in \( \text{Prj}(A') \), for \( i \in I \neq \emptyset \).

a) If \( a \) induces a mono from \( e_i \) to \( f_i \) (for all \( i \)) and \( f = \bigcap f_i \) is \( D \)-regular in \( \text{Prj}(A') \), then \( a \) induces a mono from \( e \) to \( f \).

b) If \( a \) induces an epi from \( e_i \) to \( f_i \) (for all \( i \)) and \( e = \bigcup e_i \) is \( D \)-regular in \( \text{Prj}(A) \), then \( a \) induces an epi from \( e \) to \( f \).

Proof: Also here it suffices to check a). For each \( i: e < e_i \in \mathcal{A} \), hence (1.4.b)): \( e \in \mathcal{A} \), \( a^{e_i}(f_i) \subseteq a^{f_i}(f) \); by 1.4.a) it follows that \( a^{e}(e_i) \subseteq a^{f_i}(f) \), for all \( i \). By \( D \)-regularity, \( a^{e}(e_i) \subseteq a \). Now, for some \( i \in I \neq \emptyset \): \( e \in \mathcal{A} \), \( a^{e}(f_i) \subseteq a^{f_i}(f) \subseteq a^{f_i}(f) \), hence \( e \in \mathcal{A} \). And, again by 1.4.d), \( a^{e}(e) \subseteq a \) implies \( e \in \mathcal{A} \).

3.5. Mapping theorem for filtered unions in \( RE \)-categories: Let \( A \) be an \( RE \)-category and \( a: A \to A' \). Let be given a filtered, non-empty set \( I \) together with projections:

\[
\begin{align*}
(1) & \quad e = \bigcup e_i, \quad e_i = x_i \cdot y_i \in \text{Prj}(A), \quad (i \in I), \\
(2) & \quad f = \bigcup f_i, \quad f_i = x_i \cdot f_i \in \text{Prj}(A'), \quad (i \in I),
\end{align*}
\]

with \( (e_i) \) and \( (f_i) \) increasing and \( e_i < e, f_i < f \) \( (i \in I) \).

a) If \( a \) induces a mono from \( e_i \) to \( f_i \) (for all \( i \)) and \( x = \bigcup x_i, y = \bigcap y_i \) distributively in \( \text{Rst}(A) \), then \( a \) induces a mono from \( e \) to \( f \).

b) If \( a \) induces an epi from \( e_i \) to \( f_i \) (for all \( i \)) and \( x = \bigcup x_i, y = \bigcap y_i \) distributively in \( \text{Rst}(A') \), then \( a \) induces an epi from \( e \) to \( f \).

Proof: By 2.6.a), 2.8 and 3.3.

3.6. Mapping theorem for filtered intersections in \( RE \)-categories: In the same general hypotheses of 3.5, assume now that \( (e_i) \) and \( (f_i) \) are decreasing and \( e_i < e, f_i < f \) \( (i \in I) \).
a) If \( a \) induces a mono from \( e_i \) to \( f_i \) (for all \( i \)) and \( x' = \bigcap x_i, y' = \bigcup y_i \) distributively in \( \text{Rst} (A) \), then \( a \) induces a mono from \( e \) to \( f \).

b) If \( a \) induces an epi from \( e_i \) to \( f_i \) (for all \( i \)) and \( x = \bigcap x_i, y = \bigcup y_i \) distributively in \( \text{Rst} (A) \), then \( a \) induces an epi from \( e \) to \( f \).

PROOF: By 2.7 and 3.4.

3.7. Mapping theorem for telescopic unions and differences in \( RE \)-categories: Let \( A \) be an \( RE \)-category, \( a: A \to A' \) and \( I \) an interval of \( Z \) (with at least two elements); let \( \langle x_i \rangle \) and \( \langle y_i \rangle \) be increasing families of \( \text{Rst} (A) \) and \( \text{Rst} (A') \), and assume that:

\[
\begin{align*}
1. & \quad x' = \bigcup x_i, \quad x = \bigcap x_i \quad \text{are distributive in} \quad \text{Rst} (A), \\
2. & \quad y' = \bigcup y_i, \quad y = \bigcap y_i \quad \text{are distributive in} \quad \text{Rst} (A').
\end{align*}
\]

Write \( I' = \{ i \in I : i - 1 \in I \} \), choose a fixed \( j \in I' \) and set:

\[
\begin{align*}
3. & \quad e_i = x_i / x_{i-1} \in \text{Prj} (A), \quad f_i = y_i / y_{i-1} \in \text{Prj} (A) \quad (i \in I').
\end{align*}
\]

a) If \( a \) induces a mono (resp. epi, iso) from \( e_i \) to \( f_i \) for all \( i \in I' \), then the same holds from \( e = x' / x \) to \( f = y' / y \).

b) If \( a \) induces a mono from \( e \) to \( f \) and an epi from \( e_i \) to \( f_i \) for all \( i \in I' \), then it induces a mono from \( e_i \) to \( f_i \).

b') If \( a \) induces an epi from \( e \) to \( f \) and a mono from \( e_i \) to \( f_i \) for all \( i \in I' \), then it induces an epi from \( e_i \) to \( f_i \).

b'') If \( a \) induces an iso from \( e \) to \( f \) as well as from \( e_i \) to \( f_i \) for all \( i \in I' \), then the same holds from \( e_i \) to \( f_i \).

PROOF: First notice that \( e = \bigcup e_i \) and \( f = \bigcup f_i \) are \( PD \)-regular unions, by 2.9. Thus a) is a straightforward consequence of 3.3 and we only need to prove b) the rest follows by \( \sim \)-duality.

By hypothesis \( a \) induces a mono from \( e \) to \( f \), that is:

\[
e_i \leq e \bigcup_{i \in I'} a^R (f_i) = \bigcup_{i \in I'} a^R (f_i);
\]

thus \( e_i \bigcup_{i \in I'} a^R (f_i) \), i.e.:

\[
\begin{align*}
3. & \quad e_i = e_i \& \left( \bigcup_{i \in I'} a^R (f_i) \right) = \bigcup_{i \in I'} (e_i \& a^R (f_i))
\end{align*}
\]

We prove now that \( e_i \) is disjoint from \( a^R (f_i) \) (i.e. \( e_i \& a^R (f_i) \) is null) for all \( i \in I', i \neq j \). This will complete the proof, since by (3) and 1.6 f) it implies that \( e_i = e_i \& a^R (f_i) \), i.e. our thesis: \( e_i \bigcup a^R (f_i) \).
Actually, take $i \in I$, $i \neq j$: the morphism $a$ induces an epi from $e_i$ to $f_i$, i.e. $f_i \cong a_e(e_i)$, hence (1.4.9) $a^p(f_i) \cong e_i$, and (2.3.7):

(4) \[ a^p(y_i) \prec x_i \cup a^p(y_{i-1}) , \]

(5) \[ a^p(y_{i-1}) \succ x_{i-1} \cap a^p(y_i) , \]

so that, if $i < j$, by (4) and 2.3.4-2.3.5:

(6) \[ n(e_i \& a^p(f_j)) = x_i \cap (a^p(y_j) \cup x_{i-1}) \prec x_i \cap (x_i \cup a^p(y_{i-1}) \cup x_{i-1}) = \]

\[ = x_i \cap (a^p(y_{i-1}) \cup x_{i-1}) = d(e_i \& a^p(f_j)) , \]

and the projection $e_i \& a^p(f_j)$ is null (2.3.8).

Analogously, if $i > j$, by (5) and 2.3.4-2.3.5:

(7) \[ d(e_i \& a^p(f_j)) = (x_i \cap a^p(y_{i-1})) \cup x_{i-1} > (x_i \cap x_{i-1} \cap a^p(y_j)) \cup x_{i-1} = \]

\[ = (x_i \cap a^p(y_j)) \cup x_{i-1} = n(e_i \& a^p(f_j)) . \]

4. Regular Induction

We consider here a notion of « regular induction » which agrees with composition and naturally appears in transformations of models in homological algebra. It should be noticed, however, that non-regular induction may compose with the regular one in regular and useful ways (e.g. in RO-inductive squares: 4.8, 5.2), so that we may not restrict our attention to the regular situation.

$A$ is always an RO-category (2.1) and $B = \text{Fct } A$ the associated factorizing RO-category (3.1).

4.1. RO-squares: In the theory of RO-categories the following square diagrams, called RO-squares in [G6], § 2.2, are of more interest than the commutative ones:

\[ A \xrightarrow{u} B \]
\[ \downarrow v \prec \downarrow w \]
\[ A' \xrightarrow{u'} B' \]

(2) $u$ and $v$ are proper and $va < bu$ (or equivalently: $\bar{u}a < \bar{b}v$),

e.g. they appear in the definition of RO-transformations [G6] or, here, of regular induction. The RO-squares of $A$ form a RO-category, w.r.t. the obvious vertical composition and involution.
4.2. Definition: We say that \( \alpha: A \rightarrow B \) induces regularly \((R\text{-induces for short})\) from \( e \in \text{Prj}(A) \) to \( f \in \text{Prj}(B) \) (the morphism \( \alpha' = f \circ e \) of \( B \) (see 3.2)) if:

\[
\alpha e < f e.
\]

\( R \)-induction agrees with composition: if also \( b: B \rightarrow C \) \( R \)-induces from \( f \) to \( g \in \text{Prj}(C) \) (the morphism \( b' = g \circ f \) of \( B \)), then \( ba: A \rightarrow C \) \( R \)-induces from \( e \) to \( g \), the composed \( b \alpha' \) of the induced morphisms, since:

\[
\begin{align*}
(1) & \quad (ba)e < b(ga) < g(ba), \\
(2) & \quad g(ba)e = gb(a)e < gb(fa)e = (gbf)(fae) = b'a', \\
(3) & \quad g(ba)e = g(gb)a e' < g(bf)a e = (gbf)(fae) = b'a'.
\end{align*}
\]

Moreover, if a \( R \)-induces also from \( e' \) to \( f' \), it is easy to see that the same happens from \( e \& e' \) to \( f \& f' \).

4.3. \( RO \)-induction: Since we are mostly interested in the proper case, we say that \( \eta: A \rightarrow B \) \( RO \)-induces from \( e \) to \( f \) (the morphism \( f \circ e \) of \( B \)) whenever \( \eta \) is proper and \( R \)-induces from \( e \) to \( f \) \((\alpha e < f \alpha)\). This condition, which amounts to saying that a suitable square diagram is \( RO \), will be characterized in 4.5 for categories of relations.

a) If \( \eta: A \rightarrow B \) is proper, it \( RO \)-induces from \( e \) to \( f \) iff the following equivalent conditions hold:

\[
\begin{align*}
(1) & \quad \eta e < f, \\
(2) & \quad e < \eta^e(f);
\end{align*}
\]

indeed, if \( \eta e < f \) we have: \( \eta e = \eta \tilde{u} < f \tilde{u} < f \); if (1) holds: \( e < \tilde{u} \alpha \cdot e \tilde{u} = \eta \alpha < \eta^e(f) \); last, if (2) is satisfied: \( \eta < \eta \tilde{u} = \tilde{u} \tilde{u} < f \).

b) The morphism \( f \circ e \) \( RO \)-induced by \( \eta \) is itself proper (in \( B \)), as:

\[
(f \circ e)^e \cdot (f \circ e) = \tilde{u} \tilde{u} = \tilde{u}(f \circ e) > \tilde{u} \tilde{u} = e,
\]

and analogously \((f \circ e) \cdot (f \circ e) \leftarrow < f \).

4.4. \( RO \)-induction in \( RE \)-categories: If \( A \) is an \( RE \)-category and \( \eta: A \rightarrow B \) is proper, the following conditions are equivalent (by 2.5-2.6):

a) \( \eta: A \rightarrow B \) \( RO \)-induces from \( e \) to \( f \),

b) \( \eta(\eta e) < nf, \eta(\eta e) < df \),

c) \( ne < \eta(nf), de < \eta(df) \);
moreover the induced (proper) morphism is mono (resp. epi) iff \( d \) (resp. \( c \)) holds:

\[ d) \quad df = \mathfrak{n} \cap \mathfrak{u}(df), \]
\[ e) \quad \mathfrak{n}f = df \cap \mathfrak{u}(\mathfrak{n}). \]

4.5. **RO-induction in categories of relations**: If \( A = \text{Rel}E \) is the category of relations over \( E \) (exact), \( u: A \to B \) is proper and \( t: H/K \to A, t: H'/K' \to B \) are subquotients of \( A \) and \( B \) (i.e., monorelations (0.2)), the following conditions are equivalent:

- \( a) \) \( u \) RO-induces a (proper) morphism \( w \) from \( s \) to \( t \),
- \( b) \) \( u(H) < H' \) and \( u(K) < K' \) in \( \text{Sub}_E(B) \),
- \( c) \) there is a RO-square in \( A \):

\[
\begin{array}{c}
A \xrightarrow{u} B \\
\uparrow s \quad \uparrow t \\
H/K \xrightarrow{s} H'/K'.
\end{array}
\]

- \( d) \) there is a commutative diagram of \( E \):

\[
\begin{array}{c}
A \xrightarrow{u} B \\
\downarrow s \\
H \xrightarrow{t} K \\
\downarrow t \\
H/K \xrightarrow{t} H'/K'.
\end{array}
\]

W.r.t. projections, the induced morphism from \( e = \tilde{s}t \) to \( f = \tilde{t}t \) is \( tw\tilde{s} : e \to f \).

**Proof**: \( a) \Rightarrow b) \) by 4.4. \( b) \Rightarrow a) \) by well-known properties of exact categories. \( d) \Rightarrow c) \) the commutative squares of (2) are trivially RO-squares; by vertical involution and composition one gets (1). \( c) \Rightarrow a) \) : \( uw = u\tilde{s}t < tw\tilde{s} < \tilde{t}u = we \).

Finally, the proper morphism \( w \) in (1) and (2) is the induced one:

\[
\begin{align*}
\text{(3)} & \quad w = w\tilde{s}t < (tw)t = w, \\
\text{(4)} & \quad w = w\tilde{s}t = q\tilde{s}t = q\tilde{u}w\tilde{p} = q\tilde{u}w = q\tilde{u}uw = tw = tw.
\end{align*}
\]

4.6. **RO-induction and \( P \)-regular unions of intersections**: Let \( A \) be an \( RE \)-category; if \( e = \bigcup e_i \) and \( f = \bigcup f_i \) are \( P \)-regular, respectively in \( \text{Prj}(A) \) and \( \text{Prj}(B) \) and \( u: A \to B \) RO-induces from \( e_i \) to \( f_i \), for all \( i \), the same holds from \( e \) to \( f \). Analogously for \( P \)-regular intersections of projections.
The proof is a straigntforward application of 4.4 and 2.5-2.6. E.g., in the case of unions:

\[ (1) \quad \nu_a(x) = \nu_a(\bigcup x_i) = \bigcup \nu_a(x_i) < \bigcup x_i = x', \]
\[ (2) \quad \nu_b(y) = \nu_b(\bigcap y_j) = \bigcap \nu_b(y_j) < \bigcap y_j = y'. \]

4.7. RO-inductive squares: Let \( A \) be an RO-category; the following is again a situation typically produced by transformations of models of homological theories.

The diagram of \( A \), equipped with the projections \( e, f, e', f' \):

\[
\begin{array}{ccc}
A & \xrightarrow{a} & B \\
\downarrow e & & \downarrow f \\
A' & \xrightarrow{e'} & B'
\end{array}
\]

will be said to be a RO-inductive square if:

\[ (2) \quad u'a < bu, \]
\[ (3) \quad u \text{ RO-induces from } e \text{ to } f, \]
\[ (4) \quad u' \text{ RO-induces from } e' \text{ to } f', \]
\[ (5) \quad e'ae: e \to e \text{ and } f'bf: f \to f \text{ are proper in } B. \]

Then the morphisms induced by \( bu: A \to B' \) and by \( u'a: A \to B' \) (from \( e \) to \( f' \)), are proper and equal and coincide with the composition of the morphisms induced by \( u \) and \( b \), as well as with the composition of the morphisms induced by \( a \) and \( u' \):

\[ (6) \quad f'(bu)e = f'(u'ae)e = (f'bf)(fua) = (f'f)(e'ae). \]

Indeed, by (2)-(4):

\[ (7) \quad (f'u'e')(e'ae) < f'u'ae < f'bu < f'bf)(fua); \]

now, by 4.3 \( b \) and (3)-(5), the first and the last term of (7) are proper morphisms (in \( B \)), hence coincide (2.1).

5. - CONVERGENCE FOR SPECTRAL SEQUENCES

We give here an application, concerning the spectral sequence of the \( Z \)-filtered complex.

\( E \) is an exact category and \( A = \text{Rel}E \) its \( RE \)-category of relations. By the usual abuse of notation, if \( H \triangleright K \) in the lattice \( \text{Sub}(A) \) of subobjects
of \( A \) in \( E \), the subquotient \( H/K \) will stand also for the \( A \)-subobject \( \iota: H/K \to A \)
described in 0.2, or equivalently for the projection \( \iota = \iota: A \to A \).

5.1. The spectral sequence of a filtered complex: Let be given a \( \mathbb{Z} \)-filtered complex in the exact category \( E \):

\[
A_n = \langle (A_n), (\partial_n), (F_p A_n) \rangle, \quad n, p \in \mathbb{Z},
\]

where, for each \( n \), \( \partial_n: A_n \to A_{n-1} \) is a morphism of \( E \), \( \partial_{n-1} \partial_n = 0 \) and 
\( (F_p A_n) \) is an increasing filtration of \( A_n \), with the coherence condition:
\( \partial_n (F_p A_n) < F_p A_{n-1} \) (*)

Consider the following subquotients of \( A_n \) corresponding to the homology, the graduations associated to the filtrations and the terms of the spectral sequence (use 2.3.1, 2.3.4, 2.3.5):

\[
\begin{align*}
(2) & \quad H_n = \text{Ker} \partial_n / \text{Im} \partial_{n+1}, \\
(3) & \quad G_{np} = F_p A_n / F_{p-1} A_n, \\
(4) & \quad (E^m_{np} = F_p A_n \cap (\partial^* F_{p-r} A_{n-1} \cup F_{p-1} A_n))/\quad (F_p A_n \cap (\partial^* F_{p-r} A_{n-1} \cup F_{p-1} A_n)) = \\
& \quad = G_{np} \& (\partial^* F_{p-r} A_{n-1}) / (\partial_n F_{p+r} A_{n+1}) < G_{np} \quad (r > 0), \\
(5) & \quad E^m_{np} = (F_p A_n \cap (\partial^* 0 \cup F_{p-1} A_n))/\quad (F_p A_n \cap (\partial^* A_{n+1} \cup F_{p-1} A_n)) = \\
& \quad = G_{np} \& H_n < G_{np}, \\
(6) & \quad 'E^m_{np} = H_n \& G_{np} < H_n, \\
(7) & \quad 'E^m_{np} \Phi E^m_{np}.
\end{align*}
\]

5.2. Morphisms: A morphism \( u: A_n \to B_n \) of filtered complexes is a family of morphisms \( u_n: A_n \to B_n \) in \( E \), commuting with the differentials and RO-
inducing from \( F_p A_n \) to \( F_p B_n \):

\[
u_n (F_p A_n) < F_p B_n.
\]

It is easy to see that the morphisms \( u_n \) RO-induce w.r.t. the subquotients considered above in 5.1.2-5.1.6 (in particular, use the preservation of RO-
induction by the \&-product: 4.2). Moreover, since \( 'E^m_{np} \Phi E^m_{np} \) and because of the RO-inductive square lemma (4.7), the morphism \( u: A_n \to B_n \) induces a mono, or an epi, or an iso from \( E^m_{np} (A_n) \) to \( E^m_{np} (B_n) \) iff it does from \( E^m_{np} (A_n) \) to \( E^m_{np} (B_n) \).

(*) \( \partial_n \) denotes the direct images of subobjects by \( \partial \); \( \partial^* \) the counterimages of subobjects.

The degree \( n \) is dropped when no confusion may arise.
5.3. Definition: Say that the filtered complex \( A_n \) converges in degree \( n \) if:

\[
\bigcap_{s \in \mathbb{Z}} F_s A_n = 0, \quad \bigcup_{s \in \mathbb{Z}} F_s A_n = A_n, \quad \text{distributively in } \text{Sub}(A_n).
\]

5.4. Convergence Theorem for Filtered Complexes:

a) If the filtered complex \( A_n \) converges in degree \( n \), then:

\[
H_n = \bigcup_{s \in \mathbb{Z}} E_{ns}^\infty \quad \text{(PD-regular, telescopic union in Sub}_d(A_n)).
\]

b) If \( A_n \) converges in degrees \( n-1 \) and \( n+1 \), then:

\[
E_{ns}^\infty = \bigcap_{s \in \mathbb{Z}} E_{ns}^\infty \quad \text{(PD-regular, decreasing intersection in Sub}_d(A_n)).
\]

Proof:

a) By 2.9 and the hypothesis, there is a PD-regular, telescopic union of subquotients of \( A_n \):

\[
A_n = \bigcup_p (F_p A_n) / F_{p-1} A_n = \bigcup_p G_{ns},
\]

which is preserved by applying \( H_n \).

b) By 2.5 and the hypothesis, there is a PD-regular, decreasing intersection of subquotients of \( A_n \):

\[
H_n = \bigcap_{s \in \mathbb{Z}} (E_{p-s} A_{n-1}) / (E_{p-s} A_{n+1}),
\]

which gives the thesis, by applying \( G_{ns} \).

5.5. Mapping Lemma for Filtered Complexes: Let \( \psi : A_n \to B_n \) be a morphism of filtered complexes. Let \( r, n \in \mathbb{Z} \).

a) If, for all \( p \in \mathbb{Z} \):

\[
E^i_{\mathbb{N}}(\psi) : E^i_{\mathbb{N}}(A_n) \to E^i_{\mathbb{N}}(B_n)
\]

is mono for \( i = n-1 \) and epi for \( i = n \), then the «same» holds for all \( s > r \) and \( p \in \mathbb{Z} \).

b) If, for all \( p \in \mathbb{Z} \):

\[
E^i_{\mathbb{N}}(\psi) : E^i_{\mathbb{N}}(A_n) \to E^i_{\mathbb{N}}(B_n)
\]

is mono for \( i = n-1 \), iso for \( i = n \) and epi for \( i = n+1 \), then the «same» holds for all \( s > r \) and \( p \in \mathbb{Z} \); in particular \( E_{ns}^\infty(\psi) \) is iso \((p \in \mathbb{Z})\).
Proof: We just have to prove (e). Consider the differential of the spectral sequence of $A_n$: 

$$\delta = \partial_n(A_n): E^n_{r,p}(A_n) \to E^n_{r-1,p+1}(A_n) \quad (r > r),$$

together with its «cycles» and «bords»:

$$Z^n_{r,p}(A_n) = \text{Ker } \partial, \quad B^n_{r,p}(A_n) = \text{Im } \partial.$$

We prove now that, for $r > r$:

$$E^n_{p,q}(f) \text{ is epi and } E^n_{p-1,q-1}(f) \text{ is mono for all } p \in \mathbb{Z},$$

by induction on $s$. Since for $s = r$ this is the hypothesis (1), we assume it holds for some $s > r$ and check it for $s + 1$.

The commutative diagram:

$$E^n_{r,p}(A_n) \to E^n_{r,p}(A_n) \to E^n_{r-1,p-1}(A_n)
\downarrow \gamma_{r,p}
\downarrow \gamma_{r,p}
\downarrow \gamma_{r-1,p-1}
E^n_{r,p}(B_n) \to E^n_{r,p}(B_n) \to E^n_{r-1,p-1}(B_n),$$

shows that $E^n_{r,p}(f)$ is iso (for all $p \in \mathbb{Z}$). Thus, the commutative diagrams (with exact rows):

$$Z^n_{r,p}(A_n) \to E^n_{r,p}(A_n) \to B^n_{r,p}(A_n) \to Z^n_{r,p}(B_n) \to E^n_{r,p}(B_n) \to B^n_{r,p}(B_n),$$

show, respectively, that $Z^n_{r,p}(f)$ is epi and $E^n_{r+1,p+1}(f)$ too. Dually, working on the right-hand side of (6), one proves that $E^n_{r+1,p+1}(f)$ is mono; by the arbitrariness of $p \in \mathbb{Z}$, the proof is complete.

5.6. Mapping theorem for filtered complexes: Let $u: A_n \to B_n$ be a morphism of filtered complexes and $r, s \in \mathbb{Z}$.

a) If $A_n$ converges in degrees $n - 1$ and $n + 1$, $B_n$ converges in degree $n$ (5.3) and:

$$E^n_{r-1,p}(u) \text{ is mono and } E^n_{r,p}(u) \text{ is epi, for all } p \in \mathbb{Z},$$

then $H_n(u): H_n(A_n) \to H_n(B_n)$ is epi, as well as $E^n_{r,p}(u)$, for all $p \in \mathbb{Z}$.

b) If $A_n$ converges in degree $n$, $B_n$ converges in degrees $n - 1$ and $n + 1$ and:

$$E^n_{r,p}(u) \text{ is mono and } E^n_{r-1,p}(u) \text{ is epi, for all } p \in \mathbb{Z},$$

then $H_n(u): H_n(A_n) \to H_n(B_n)$ is mono, as well as $E^n_{r,p}(u)$, for all $p \in \mathbb{Z}$,
r) If the filtered complexes $A_\mu$ and $B_\mu$ both converge in degrees $n=1$, $n, n+1$ and, for all $\mu \in \mathbb{Z}$:

\[(3) \quad E^i_\mu(n) : E^i_\mu(A_\mu) \to E_i(B_\mu)\]

is mono for $i = n - 1$, iso for $i = n$ and epi for $i = n + 1$.

then $H_\phi(n) : H_\phi(A_\mu) \to H_\phi(B_\mu)$ is iso (as well as $E^i_\mu(n)$, for all $\mu \in \mathbb{Z}$).

**Proof:** We prove $a)$, since $b)$ is its dual and $c)$ follows from both. By 5.5 $a)$, $E_\mu(n)$ is epi for all $\mu > r$ and all $\mu \in \mathbb{Z}$. By 5.4 $b)$, the subquotient $E_\mu^\mu(A_\mu)$ is the PD-regular intersection of $E_\mu^\mu(A_\mu)$, for $\mu > r$; by the mapping theorem for $D$-regular intersections (3.4 $b)$), $E_\mu(n)$ is epi for all $\mu \in \mathbb{Z}$. Finally, by 5.4 $a)$, $H_\phi(B_\mu)$ is the telescopic, PD-regular union of the subquotients $(E_\mu^\mu(B_\mu))$, for $\mu \in \mathbb{Z}$; by the mapping theorem for $D$-regular unions (3.3 $b)$), $H_\phi(n)$ is epi.

6. **Appendix: The Orthodox and Distributive Case**

In order to clarify the behaviour of the relations $\land$ and $\lor$, we recall here briefly some results from papers on orthodox involutive categories and distributive exact categories ($[G1, G2, G3, G4]$).

6.1. **Orthodox and Inverse RI-categories:** An RI-category $\mathcal{A}$ is said to be orthodox $[G2]$ when its idempotent endomorphisms are stable for composition ($^\circ$).

A category $\mathcal{K}$ is inverse ($[G1]; [Sc]$) whenever each morphism $a : A \to A'$ has a unique generalized inverse $\bar{a}$ ($^\circ$); then the mapping $a \mapsto \bar{a}$ defines a regular involution in $\mathcal{K}$ (clearly the only one). It is not difficult to prove (extending a well-known result for semigroups) that a category is inverse iff it is $\circ$-regular ($^\circ$) and its idempotents commute. iff it has a regular involution and its projections commute.

In an inverse category $\mathcal{K}$ all sets $\text{Prj}(\mathcal{A})$ are semilattices, the meet being the product. The operation $\&$ and the relation $\land$ between projections coincide respectively with the product and the canonical order $<$.

6.2. **The Canonical preorder:** An orthodox RI-category is provided with a canonical preorder (or domination) $a \land b$ on parallel morphisms, consistent with

($) More generally, extending orthodox semigroups, a category is said to be orthodox $[G1]$ if it is $\circ$-regular (each morphism $a : A \to A'$ has some generalized inverse $a' : A' \to A$, with $aa' = a$ and $a'a'' = a'$) and its idempotent endomorphisms are stable for composition. Obviously, an RI-category is always $\circ$-regular.
composition and involution, defined by the following equivalent properties [G2]:

\[ a = aba, \]
\[ a = a\bar{a}\cdot b\cdot \bar{a}a, \]
\[ \text{there exist idempotents } e, f \text{ such that: } a = fbe, \]
\[ \text{there exist projections } e, f \text{ such that: } a = fbe. \]

This preorder extends the domination of projections, which is thus, in the orthodox case, a preorder.

The associated congruence \( \Phi \) yields a quotient \( A/\Phi \) which is an inverse category, provided with its canonical order \( \mathcal{C} \) (extending the well-known canonical order of inverse semigroups).

6.3. Theorem [G3]: The RI-category \( A \) is orthodox iff in \( A \) induced isomorphisms are preserved by composition. The last properties means that, if \( a \in A(A, B) \) induces an iso from \( s \in \text{Prj} (A) \) to \( f \in \text{Prj} (B) \) and \( b \in A(B, C) \) induces an iso from \( f \) to \( g \in \text{Prj} (C) \) then \( ba \) induces the composed iso from \( e \) to \( g: (gbf)(f\epsilon e) = g \cdot ba \cdot e. \)

6.4. Theorem ([G4], § 3.17): Let \( A = \text{Rel} E \) be the category of relations over an exact category \( E \); the following conditions are equivalent:

a) \( A \) is orthodox;

b) for every object \( A \), the modular lattice \( \text{Rst} (A) \) is distributive;

c) \( E \) is distributive (i.e. all its lattices of subobjects are so);

d) the relation of domination \( \mathcal{C} \) is transitive (a preorder) on each set \( \text{Prj} (A) \);

e) the relation \( \Phi \) is transitive (hence an equivalence) on each set \( \text{Prj} (A) \);

f) the operation \& is associative on each set \( \text{Prj} (A) \);

g) for every \( a: A \rightarrow B \) in \( A \), the mapping \( a_\# \) preserves the operation \&;

b) for every \( a: A \rightarrow B \) in \( A \), the mapping \( a_\# \) preserves binary meets and joins.

Analogous results ([G6], § 7.4) hold more generally for any RE-category \( A \) and its associated componentwise exact category \( E = \text{Prp Fct} A \).

6.5. Distributive RE-categories: Thus an orthodox RE-category is also called a distributive one, when we want to stress the properties b) and c) of the previous characterization.
Examples of distributive exact categories: cyclic groups; sets and partial bijections; the distributive expansion $\text{Dst} \, E$ of any exact category ([G6], § 7.10). Their categories of relations are distributive.

6.6. Unions and intersections of projections in inverse categories: Let $K$ be an inverse category (with its unique RI-structure). Then:

a) every intersection of projections is PD-regular,

b) every union of projections is D-regular,

c) P-regular unions of projections coincide with PD-regular unions, as well as with distributive unions (w.r.t. the intersection, or product, in the semilattice $\text{Pr}_I (A)$).

Indeed a) and b) follow from 1.7 and from the last remark in 6.1. As to c), P-regular unions are also PD-regular by a), and coincide with distributive ones by 1.9 (the condition 1.8.1 being trivially satisfied when all projections commute).

6.7. Unions and intersections of projections in orthodox RI-categories: Let $A$ be an orthodox RI-category; write $K = A/\Phi$ the associated inverse category and $a \mapsto A$ the quotient functor. Then:

a) if $e = \bigcap e_i$ in $\text{Pr}_I (A)$, the intersection is D-regular iff $\bar{e} = \bigcap \bar{e}_i$ in $\text{Pr}_K (A)$,

b) if $e = \bigcup e_i$ in $\text{Pr}_I (A)$, the union is D-regular iff $\bar{e} = \bigcup \bar{e}_i$ in $\text{Pr}_K (A)$,

c) if $e = \bigcup e_i$ is PD-regular in $\text{Pr}_I (A)$, then $\bar{e} = \bigcup \bar{e}_i$ is distributive in $\text{Pr}_K (A)$.

Indeed, a) and b) follow trivially from the fact that $e \text{Cf} f$ in $A$ iff $\bar{e} \text{Cf} \bar{f}$ in $B$. As to c), it is easy to see that, if $e = \bigcup e_i$ is PD-regular in $\text{Pr}_I (A)$, then $\bar{e} = \bigcup \bar{e}_i$ is P-regular in $\text{Pr}_K (A)$, hence (6.6c) distributive.

6.8. Remark: Last we notice that the example considered in 2.11 (a P-regular, D-regular union $1 = e_1 \cup e_2$ which is not PD-regular) actually lives in the (orthodox) category of relations over the distributive exact category $E$ of cyclic groups. Its image $1 = \bar{e}_1 \cup \bar{e}_2$ in $\text{Rel} (E)/\Phi$ shows a (D-regular) union in an inverse category which is not P-regular.

REFERENCES


