RITA GIULIANO ANTONINI and PIO ANDREA ZANZOTTO (*)

A General «Geometric» Condition for Measure Derivation (**) (***)

SUMMARY. — In this paper we generalize and set in an «abstract» context, some results on measure derivation contained in [4]. We consider a measurable space $(X, A)$ endowed with a derivation basis $\Phi$ that possesses the property stated in Definition (1.5) below. We study measure derivation with respect to $\Phi$. Some applications to «concrete» derivation bases are also given.

Una condizione generale di tipo geometrico per la derivazione delle misure

SUNTO. — In questo articolo alcuni risultati sulla derivazione delle misure contenuti in [4], vengono generalizzati e riferiti ad una situazione «astratta». Praticamente, si considera un sotto spazio misurabile $(X, A)$ dotato di una base di derivazione $\Phi$ che possiede la proprietà espressa nella Definizione (1.5). Si studia la derivazione, rispetto a $\Phi$, delle misure su $A$. Si considerano anche alcune applicazioni al caso di basi di derivazione «concrete».

INTRODUCTION

In this article, our aim is to study the use, in derivation theory, of a geometric condition suggested by Lemma 3.1, p. 109 of [4]; in the following we shall refer to this condition as the «De Giorgi property».

Although the idea this property is based upon is already present in classical papers by Besicovitch and Morse (see [2], [3], [12], [13]), the use of it in [4]

(*) Indirizzo degli Autori: Dipartimento di Matematica dell'Università, Via F. Buonarroti 2, 56100 Pisa (Italia).
Entrambi gli Autori sono membri del C.N.R.-G.N.A.F.A.
(**) Lavoro eseguito nell'ambito di un progetto nazionale di ricerca parzialmente finanziato dal Ministero della Pubblica Istruzione (40%).
(***) Memoria presentata il 10 aprile 1987 da Giorgio Letta, uno dei XL.

ISSN 0392-4106
is, as far as we know, an unusual and interesting one from various points of view.

In the case of Euclidean space $\mathbb{R}^n$ and with respect to a derivation basis of open balls (such that each point is the center of balls with arbitrarily small diameters), Theorem 3.3 of [4] shows how the De Giorgi property makes it possible in the case of Radon measures to pass from a «local domination» property, connected with the derivation basis, to one of «global domination».

Now, the property enunciated in the above theorem immediately makes it possible to obtain the derivation of Radon measures, thereby avoiding going through the strong Vitali property, which is, on the contrary, what the above-quoted authors do.

In this article we show that the results of [4] may be generalized and set in an «abstract» context.

So, as well as other results, we obtain a unifying view of many concrete examples of derivation bases: see, for instance, sections 4 and 5 below.

In detail, in the sequel we shall consider an abstract measurable space $(X, \mathcal{A})$, endowed with a derivation basis $\Phi$ that possesses the De Giorgi property.

In section 1 we show (Theorem (1.11)) that, if two measures on $\mathcal{A}$ satisfy a local domination property connected to the basis $\Phi$ and the dominating measure fulfills a suitable «regularity» condition with respect to $\Phi$, then a global domination property for the generated outer measures holds good: this theorem is the key result of this article and extends Theorem 3.3 of [4] to the present «abstract» context.

We also point out that the regularity condition required in Theorem (1.11) is satisfied in many practical situations: see the examples in sections 4 and 5.

In section 2 we study the derivation of measures on $\mathcal{A}$ with respect to the basis $\Phi$. The main result in this section is Theorem (2.8). Its proof is based on the key theorem mentioned above and it states the following:

Let $\nu, \mu$ be two measures on $\mathcal{A}$ and let us assume that $\mu$ and $\nu$ are $\sigma$-finite and regular with respect to $\Phi$.

Then the De Possel derivative of $\nu$ with respect to $\mu$ agrees $\mu$-almost everywhere with a version of the Radon-Nikodym density of the $\mu$-absolutely continuous part of $\nu$.

In section 3 we study the relationship between the De Giorgi and the weak Vitali property, via the De Possel’s classical equivalence theorem (which states that a given basis has the density property corresponding to a fixed measure iff it possesses the weak Vitali property corresponding to the same measure). We show, in particular, (see Corollary (3.5)) that a basis $\Phi$ for which the De Giorgi property holds good, possesses the weak Vitali property corresponding to any measure on $\mathcal{A}$ which is $\sigma$-finite and regular with respect to the basis (and, in addition, locally $\Phi$-eventually finite).

Lastly, sections 4 and 5 contain various examples of bases satisfying the De Giorgi property. In the very general case of bases such as A. P. Morse’s «blankets» (see [12], [13]), the results of the foregoing sections allow us to
state two sufficient conditions for a universal derivability of Radon measures on \( \mathbb{R}^n \), which do not appear in [13] and which are complementary to the general theorem of the universal derivability of Morse's star blankets: namely, Theorems (4.10) and (4.13) below.

It is worthwhile noting that here no closure assumption is made, as it is in Morse's theorem, about the elements of our basis.

At the beginning of section 5 we quote an example of derivation basis for which the De Giorgi property holds good, but the strong Vitali property (corresponding to the Lebesgue measure on \( \mathbb{R}^n \)) does not.

Next, we show another general example of a derivation basis which fits into our scheme and was studied by Federer (see [5], 2.8): namely the basis of closed balls in a metric space with a suitable assumption on its metric.

Finally, we would like to thank Professors Ennio De Giorgi and Giorgio Letta for stimulating discussions.

1. - The De Giorgi property

Let \((X, A)\) be a measurable space. In order to define a derivation basis on \((X, A)\), we associate to each point \(x\) of \(X\) a nonempty set of Moore-Smith sequences (families filtering to the right) of \(A\)-sets.

In greater detail:

(1.1) Definition: A derivation basis on \((X, A)\) is a family \(\Phi = (\Phi_x)_{x \in X}\), indexed by \(X\) itself, such that, for each element \(x\) of \(X\), \(\Phi_x\) is a nonempty set of Moore-Smith sequences (families filtering to the right) of \(A\)-sets.

Sequences in \(\Phi_x\) will be called deriving sequences (shortened D.S.) at \(x\).

For every point \(x\) of \(X\), a generic deriving sequence at \(x\) will be denoted by \((A_i(x))_i\), where \(i\) is a typical member of a directed set.

Moreover, by \(D\) we denote the collection of all subsets occurring in the sequences of \(\Phi_x\) for all \(x \in X\); such sets will also be called constituents of the derivation basis. Thus, \(D\) can be defined as the class of all the constituents of the basis \(\Phi\).

Given a derivation basis \(\Phi = (\Phi_x)_{x \in X}\), let \(x_0\) be a point of \(X\) and \(\mathcal{R}(A)\) a relation (i.e. a property) concerning the generic element \(A\) of \(A\).

We say that \(\mathcal{R}(A)\) is eventually verified at \(x_0\) if each D.S. at \(x_0\) eventually satisfies \(\mathcal{R}(A)\); we say that \(\mathcal{R}(A)\) is frequently verified at \(x_0\) if there exists at least one D.S. at \(x_0\) frequently verifying \(\mathcal{R}(A)\).

These definitions may be formalized in a more precise (but less intuitive) way if we replace the relation \(\mathcal{R}(A)\) with the set \(\mathcal{K}\) of the elements of \(A\) verifying it: saying that \(\mathcal{R}(A)\) is eventually verified at \(x_0\) means that each D.S. at \(x_0\) is eventually in \(\mathcal{K}\); saying that \(\mathcal{R}(A)\) is frequently verified at \(x_0\) means that there exists at least one D.S. at \(x_0\) which is frequently in \(\mathcal{K}\).
(1.2) **Definition:** Let \( x_0 \), \( U \) be respectively a point and a subset of \( X \). We say that \( x_0 \) is \( \Phi \)-interior to \( U \) (or that \( U \) is a \( \Phi \)-neighborhood of \( x_0 \)) if the relation \( A \subset U \) is eventually verified at \( x_0 \).

(1.3) **Definition:** We say that \( U \) is a \( \Phi \)-open set if \( U \) belongs to \( \mathcal{A} \) and is a \( \Phi \)-neighborhood of each of its points.

It is quite easy to see that the intersection of two \( \Phi \)-open sets is itself a \( \Phi \)-open set.

(1.4) **Definition:** Let \( X \) be a topological space. A derivation basis \( \Phi \) on \( (X, \mathcal{B}(X)) \) (where \( \mathcal{B}(X) \) is the \( \sigma \)-algebra of Borel sets of \( X \)) is said to be compatible with the topology of \( X \) if every open set, in the topological sense, is \( \Phi \)-open.

(1.5) **Definition:** We say that a derivation basis on \( (X, \mathcal{A}) \) possesses the De Giorgi property (or is a De Giorgi basis) if there exists an integer \( q \geq 1 \) (which we call the De Giorgi constant) such that, for every subset \( E \) of \( X \) and every relation \( \mathcal{R}(A) \) which is frequently verified at every point of \( E \), there exists a countable family \( \{A_i\} \) of \( \mathcal{A} \)-sets verifying the given relation, such that one has

\[
E \subset \bigcup A_i, \quad \sum 1_{A_i} < q.
\]

(1.6) **Remark:** The property stated in the above definition depends on the specific structure of the derivation basis: thus it is a "geometric" property of the basis, of the same kind as the one Besicovitch first pointed out (see [2], [3]) by taking into account the bases of closed balls on \( \mathbb{R}^3 \).

Nevertheless, Besicovitch's covering property is stronger than (1.5) (see [3]) and implies that his bases verify the strong Vitali property with respect to any Radon measure on \( \mathbb{R}^3 \): thus, by virtue of classical results, these bases allow mutual derivation of two arbitrary Radon measures.

In the same period as Besicovitch, A. P. Morse, who studied the case of \( \mathbb{R}^n \) (see [13]), showed that derivation bases which are more general than those of closed balls, enjoy Besicovitch's covering property. However, Morse only considers bases with closed constituents (the so called "star blankets"; see also section 4 below) if this is the case, the strong Vitali property with respect to any Radon measure follows from Besicovitch's property.

Nevertheless, as is remarked in [4], in order to derive two Radon measures, it is enough, for the outer measures generated, to verify a property similar to the one stated in Theorem (1.11) below: in the case considered in [4], this property is obtained for Radon measures on \( \mathbb{R}^n \) and with respect to a basis of open balls, using the fact (as pointed out in [4], Lemma 3.1) that such a basis verifies the covering property we assume here as an axiom.

However the idea contained in [4] is independent of the particular situation considered there, and can be generalized in the way we are about to explain here.
Let us now go back to the general case, and assume $\Phi=(\Phi_i)_{i \in \mathbb{X}}$ is a derivation basis on the measurable space $(X, \mathcal{A})$.

(1.7) **Definition:** A measure (*) $\mu$ defined on $\mathcal{A}$ is said to be regular with respect to the derivation basis, or—more briefly—$\Phi$-regular if, for every element $A$ of $\mathcal{A}$, $\mu(A)$ is the infimum of the set of values taken by $\mu$ on the $\Phi$-open sets including $A$.

(1.8) **Remark:** It is an obvious fact that, if $\mu$ is $\Phi$-regular and $\sigma$-finite, the same property is verified by every measure $\nu$ which is dominated by $\mu$, in the sense that, for each element $A$ of $\mathcal{A}$, one has

$$0 < \nu(A) < \mu(A).$$

(1.9) **Definition:** Let $\lambda$, $\mu$ be two measures on $\mathcal{A}$, $E$ a subset of $X$; $\lambda$ is said to be locally dominated by $\mu$ on $E$ if the relation $\lambda(A) < \mu(A)$ is frequently verified at each point $x$ of $E$.

(1.10) **Lemma:** Let $\Phi$ be a De Giorgi basis (with constant $q$) on the measurable space $(X, \mathcal{A})$.

Let $\mu$, $\nu$ be measures on $\mathcal{A}$, and $E$ a subset of $X$ such that $\lambda$ is locally dominated by $\mu$ on $E$. Then we have

$$q^{-1} \lambda^*(E) < \mu(U),$$

for every $\Phi$-open set $U$ including $E$ ($\lambda^*$ denoting the outer measure generated by $\lambda$ in the collection of subsets of $X$).

**Proof:** The relation

$$A \subset U \quad \text{and} \quad \lambda(A) < \mu(A)$$

is frequently verified at each point of $E$.

By virtue of the De Giorgi property, there exists a countable family $(A_i)$ of elements of $\mathcal{A}$, such that

$$E \subset \bigcup A_i \subset U, \quad \lambda(A_i) < \mu(A_i) \quad \text{for all } i, \quad \sum_1^{\infty} 1_{A_i} < q 1_U.$$

It follows that

$$\lambda^*(E) \leq \sum_1^{\infty} \lambda(A_i) < \sum_1^{\infty} \mu(A_i) < q \mu(U).$$

(1.11) **Theorem:** In addition to the assumptions of Lemma (1.10) above, we suppose that $\mu$ is $\Phi$-regular. Then we have

$$\lambda^*(E) < \mu^*(E).$$

(*) Here and in the following the word «measure» means «positive measure». 
PROOF: Since $\mu$ is $\Phi$-regular, we need only to show that

$$\lambda^*(E) < \mu(U),$$

for every $\Phi$-open set $U$ including $E$.

Let us fix $U$: we claim that

$$\lambda^*(A \cap E) < \mu(A \cap U)$$

for every element $A$ of $A$.

We may, of course, assume that $\lambda^*(E) > 0$, $\mu(U) < +\infty$ (so that $\lambda^*(E) < \mu(U) < +\infty$).

Let us denote by $p$ the greatest of the positive real numbers $r$ such that the (bounded) measure $A \mapsto r\lambda^*(A \cap E)$ (*) is dominated by the (bounded) measure $A \mapsto \mu(A \cap U)$, i.e. such that the (bounded) measure $v_r$ defined by

$$v_r(A) = \mu(A \cap U) - r\lambda^*(A \cap E)$$

is positive.

We have to prove that $p > 1$. We assume the contrary and suppose that $p < 1$. Since $\lambda$ is locally dominated by $\mu$ on $E$ and $U$ is a $\Phi$-open set containing $E$, the relation

$$A \subset U \text{ and } \mu(A) > \lambda(A)$$

is frequently verified at each point of $E$; consequently the same happens for the relation

$$A \subset U \text{ and } v_r(A) = \mu(A) - p\lambda^*(A \cap E) > \lambda(A) - p\lambda(A).$$

It follows that the positive measure $(1 - p)\lambda$ is locally dominated by $v_r$ on $E$.

Let us fix a member $A$ of $A$ and a $\Phi$-open set $V$, such that

$$A \cap E \subset A \cap U \subset V.$$

From Lemma (1.10) above (applied to the pair of positive measures $(1 - p)\lambda$, $v_r$ and to the pair of sets $A \cap E$, $V$), we deduce

$$\frac{1 - p}{q} \lambda^*(A \cap E) < v_r(V) = \mu(V \cap E) - p\lambda^*(V \cap E) < \mu(V) - p\lambda^*(A \cap E),$$

so that

$$\left(\frac{p}{q} + \frac{1 - p}{q}\right) \lambda^*(A \cap E) < \mu(V).$$

(*) It follows from Carathéodory's theorem that $A \mapsto \lambda^*(A \cap E)$ is a measure (see, for example, [7], Th. B, p. 46).
Since $\mu$ is $\Phi$-regular and $V$ is an arbitrary $\Phi$-open set including $A \cap U$, we have

$$\left(p + \frac{1-p}{q}\right) \lambda^\ast(\lambda \cap E) < \mu(A \cap U).$$

In other words, the measure $v_r$, where $r = p + \frac{1-p}{q} > p$ is positive, which leads to a contradiction because of the definition of $p$. This completes the proof. $\blacksquare$

2. - DERIVATION THEOREMS WITH RESPECT TO A DE GIORGI BASIS

Throughout this section $(X, \mathcal{A})$ will denote a fixed measurable space, on which a derivation basis $\Phi = (\Phi_\alpha)_{\alpha \in \mathcal{X}}$ is given.

(2.1) DEFINITION: A measure $\mu$ on $\mathcal{A}$ is said to be $\Phi$-eventually finite if the relation $\mu(A) < +\infty$ is eventually verified at each point of $X$.

(2.2) DEFINITION: Consider two measures $\mu$, $\nu$ on $\mathcal{A}$, and assume that $\mu$ is $\Phi$-eventually finite.

Adopting the convention $1/0 = \infty$ (in addition to the usual one $0 \cdot \infty = 0$), for each point $x$ of $X$, we define

$$D_\nu(x) = \inf \left\{ \liminf_i \nu(A_\alpha(x)) \cdot \frac{1}{\mu(A_\alpha(x))} \right\},$$

where the limit inferior in brackets is referred to the generic D.S. $\sigma = (A_\alpha(x))$, at $x$ and the infimum is taken from among all sequences $\sigma$ belonging to $\Phi_\sigma$; in an exactly similar way we define

$$D_\mu(x) = \sup \left\{ \limsup_i \nu(A_\alpha(x)) \cdot \frac{1}{\mu(A_\alpha(x))} \right\}.$$

The extended real functions $D_\nu$ and $D_\mu$ are called respectively the $\Phi$-lower derivative and the $\Phi$-upper derivative of $\mu$ with respect to $\nu$.

If $D_\nu(x) = D_\mu(x)$, we say that the $\Phi$-derivative $D_\nu(x) = D_\nu(x) = D_\mu(x)$ of the measure $\nu$ exists at $x$, or, alternatively, that $\nu$ is $\Phi$-derivable at $x$ (with respect to $\mu$).

(2.3) THEOREM: Assume that $\Phi$ is a De Giorgi basis.

Let $\nu$, $\mu$ be two $\sigma$-finite measures on $\mathcal{A}$ such that $\mu$ is $\Phi$-eventually finite and $\nu$ is absolutely continuous with respect to $\mu$.

Let $f$ denote a version of the density of $\nu$ with respect to $\mu$.

Then:

(a) If $\mu$ is $\Phi$-regular, we have $f < D_\nu \quad \mu^\ast$-almost everywhere;
(b) If $\nu$ is $\Phi$-regular, we have $D_\nu < f \quad \mu^\ast$-almost everywhere;

($D_\nu$, $D_\nu$ being defined according to (2.2)).
PROOF: (a) In order to prove that $f < Dv$ on the complement of a $\mu^*$-nullset, it is sufficient to show that, for each positive real number $\epsilon$, the set $\{Dv < \epsilon\} \setminus \{Dv < \epsilon\} \cap \{f < \epsilon\}$ is a $\mu^*$-nullset.

Consider a $\mu$-cover of $\{Dv < \epsilon\}$, i.e. an $A$-measurable set $C$ containing $\{Dv < \epsilon\}$ and such that every $A$-measurable set included in $C \setminus \{Dv < \epsilon\}$ is $\mu$-null.

We have to prove that, on $C$, the density $\nu$ with respect to $\mu$ is $\mu^*$-almost everywhere upper-bounded by $\epsilon$, or equivalently, that $\nu(B) < \epsilon \mu(B)$ for every $A$-measurable set $B$ contained in $C$. Fix such a set $B$ and observe that $B$ is an $A$-measurable cover of $B \cap \{Dv < \epsilon\}$ with respect to $\mu$ (and also, with respect to $\nu$, since $\nu$ is absolutely continuous with respect to $\mu$). Recalling the convention we made in (2.2), from the definition of $Dv$ we can deduce that, for each point $x$ in $B \cap \{Dv < \epsilon\}$, there exists a D.S. at $x$ frequently verifying the relation $\nu(A) < \epsilon \mu(A)$.

In other words, $\nu$ is locally dominated by $\epsilon \mu$ on the set $B \cap \{Dv < \epsilon\}$.

From Theorem (1.11) we deduce

$$\nu(B \cap \{Dv < \epsilon\}) < \epsilon \nu(B \cap \{Dv < \epsilon\})$$

i.e.

$$\nu(B) < \epsilon \mu(B).$$

(b) We need only to prove that, for every positive real number $\epsilon$, the set $\{Dv > \epsilon\} \setminus \{Dv > \epsilon\} \cap \{f > \epsilon\}$ is a $\mu^*$-nullset.

We can of course repeat the foregoing proof.

From Theorem (2.3) we obtain the following as a corollary:

(2.4) Theorem: Assume that $\Phi$ is a De Giorgi basis.

Let $\nu$, $\mu$ be two $\sigma$-finite and $\Phi$-regular measures on $A$, such that $\mu$ is $\Phi$-eventually finite and $\nu$ is absolutely continuous with respect to $\mu$.

Then each of the two functions $D\nu$, $D\nu$, defined according to (2.2), agrees $\mu^*$-almost everywhere with a version of the density of $\nu$ with respect to $\mu$.

Now consider the case of two mutually singular measures.

We have the following:

(2.5) Theorem: Assume that $\Phi$ is a De Giorgi basis.

Let $\nu$, $\mu$ be two measures on $A$, such that $\mu$ is $\Phi$-eventually finite and $\nu$ is singular with respect to $\mu$ (i.e. there exists an $A$-measurable set $S$, such that $\mu(S) = 0$, $\nu(S') = 0$).

Then, if $\nu$ is $\Phi$-regular, both functions $D\nu$, $D\nu$, defined according to (2.2), vanish $\mu^*$-almost everywhere.

PROOF: It is sufficient to consider $D\nu$ and prove that, for every strictly positive real number $\epsilon$, we have

$$\mu^*(E) = 0$$

where $E = \{D\nu > \epsilon\} \cap S'$. 
It must be noted that, by the definition of $Dv$, for each point $x$ in $E$, there exists a D.S. at $x$ frequently verifying the relation $v(A) > c_\mu(A)$.

In other words, $c_\mu$ is locally dominated by $v$ on $E$.

From Theorem (1.11) it follows

$$c_\mu^*(E) < v^*(E) < v(Y) = 0.$$  

(2.6) **Remark:** The above theorems can be applied in the special case where $\mu, v$ are Borel regular measures on a topological space $X$ provided with a derivation basis $\Phi$ compatible with the topology of $X$ (see Definition (1.4)).

(2.7) **Definition:** With the same assumptions and notations as in Definition (2.2), the derivation basis $\Phi$ is said to derive the measure $v$ with respect to the measure $\mu$ if both functions $Dv$, $Dv$ agree $\mu^*$-almost everywhere with a version of the density of the $\mu$-absolutely continuous part of $v$.

Using the terminology introduced in the above Definition, the results of Theorems (2.4) and (2.5) may be expressed by saying that the basis $\Phi$ derives the measure $v$ with respect to $\mu$.

In addition we have the following:

(2.8) **Theorem:** Assume that $\Phi$ is a De Giorgi basis.

Let $\mu, v$ be two $\sigma$-finite and $\Phi$-regular measures on $A$, such that $\mu$ is $\Phi$-eventually finite.

Then the basis $\Phi$ derives $v$ with respect to $\mu$.

**Proof:** Write $v$ as the sum of two ($\sigma$-finite) measures $v_*$ and $v_+$ (resp. $v_-$) denoting the absolutely continuous (resp. singular) part of $v$ with respect to $\mu$.

By virtue of Remark (1.8), $v_*$ and $v_+$ are $\Phi$-regular.

We apply Theorem (2.4) (resp. (2.5)) to measures $v_\mu$, $\mu$ (resp. $v_\mu$, $\mu$): if $f$ denotes a version of the density of $v_\mu$ with respect to $\mu$, by taking into account Definition (2.2), we find that the relation

$$f < Dv < Dv < f$$

holds $\mu^*$-almost everywhere.  

(2.9) **Definition:** Let $Y$ be a $\Phi$-open subset of $X$. Then the relation $A \subset Y$ is eventually verified at every point of $Y$. For each point $x$ of $Y$ and each D.S. of at $x$,

let us consider all (countable) subsequences of $\sigma$ whose elements all verify the above relation; when $\sigma$ runs through $\Phi_\sigma$, such subsequences form a set which we denote by $\Phi_\sigma^1$. If $\mathcal{B}$ is the $\sigma$-algebra induced by $A$ on $Y$, the derivation basis $\Phi_\sigma^1 = (\Phi_\sigma^1)_\text{ext}$ on the space $(Y, \mathcal{B})$ will be called the restriction of $\Phi$ to $Y$.

(2.10) **Remark:** With the assumptions made in the above Definition, let $v, \mu$ be two measures on the space $(X, A)$ and let us assume that $\mu$ is $\Phi$-eventually finite.
According to Definition (2.2), for every point \( x \) of \( X \), \( Dv(x) \) (resp. \( Dv(x) \)) denotes the value at \( x \) of the lower derivative (resp. upper derivative) of \( v \) with respect to \( \mu \) and the basis \( \Phi \).

Let \( v_1 \) (resp. \( \mu_1 \)) be the restriction of \( v \) (resp. \( \mu \)) to \((Y, A)\) and, for every point \( x \) of \( Y \), let us denote by \( D_1v_1(x) \) (resp. \( D_1\mu_1(x) \)) the value at \( x \) of the lower derivative (resp. upper derivative) of \( v_1 \) with respect to \( \mu_1 \) and the basis \( \Phi_1 \) that is the restriction of \( \Phi \) to \( Y \).

For every point \( x \) of \( Y \) we then have

\[
Dv(x) = D_1v_1(x), \quad Dv(x) = D_1v_1(x).
\]

3. - The De Giorgi and Vitali Properties

Throughout this section, \((X, A)\) will denote a fixed measurable space on which a derivation basis \( \Phi = (\Phi_\alpha)_{\alpha \in A} \) and a \( \Phi \)-eventually finite measure \( \mu \) are defined.

We quote the following classical

(3.1) Definition: For every \( \Lambda \)-measurable set \( A \), we put \( v_A = 1_A \cdot \mu \).

The derivation basis \( \Phi \) is said to possess the lower \( \mu \)-density property (resp. the upper \( \mu \)-density property) if, for each \( \Lambda \)-measurable set \( A \), we have \( \mu^* \)-almost everywhere

\[
1_A < Dv_A \quad \text{(resp. } Dv_A < 1_A),
\]

\( Dv_A \) (resp. \( Dv_A \)) denoting the lower (resp. upper) derivative of \( v_A \) with respect to \( \mu \), defined according to (2.2).

The derivation basis \( \Phi \) is said to possess the density property with respect to \( \mu \) (or to possess the \( \mu \)-density property) if, for each \( \Lambda \)-measurable set \( A \), both functions \( Dv_A \), \( Dv_A \) agree with \( 1_A \) \( \mu^* \)-almost everywhere.

(3.2) Theorem: Assume that \( \Phi \) is a De Giorgi basis and that the measure \( \mu \) is \( \sigma \)-finite and \( \Phi \)-regular.

Then \( \Phi \) possesses the \( \mu \)-density property.

Proof: It suffices to apply Theorem (2.4) and observe that, for any \( \Lambda \)-measurable set \( A \), the fact \( \mu \) is \( \Phi \)-regular implies the same holds for \( v_A \).

The following Definition states, in our terminology, the classical weak Vitali property with respect to a fixed measure on \((X, A)\) (see [10], Chap. II, Def. 1.3 page 16 and Def. 2.7 page 21).

(3.3) Definition: Denote by \( \mu^* \) the outer measure generated by \( \mu \). Consider any subset \( E \subseteq X \) of finite outer measure and any relation \( R(A) \), frequently verified
at every point of $E$; let us assume that, for any $\mu$-cover $M$ of $E$ and any number $\varepsilon > 0$, there exists a countable family $(A_i)$ of sets verifying the above relation and such that, putting $A = \bigcup A_i$, the following conditions hold:

1) $\mu^*(E \setminus E \cap A) = 0$;
2) $\mu(A \setminus A \cap M) < \varepsilon$;
3) $\sum_i \mu(A_i) - \mu(A) < \varepsilon$.

In this situation, we say that the basis $\Phi$ possesses the weak Vitali property with respect to the measure $\mu$ (or the weak Vitali $\mu$-property).

The following theorem is basically due to De Possel (see [10], Chap. III, Th. 1.2 page 30):

(3.4) Theorem: Assume that the derivation basis $\Phi$ possesses the lower $\mu$-density property.

Then $\Phi$ possesses the weak Vitali $\mu$-property.

Proof: Let $E \subseteq X$ be a subset of strictly positive outer measure and consider a $\mu$-cover $B$ of $E$.

Let $t_0 = 1_{B} \cdot \mu$. Denoting by $Dv_0$ the lower derivative of $t_0$ with respect to $\mu$, by virtue of our assumptions the relation $1_{B} < Dv_0$ is verified $\mu^*$-almost everywhere, so that there exists at least one point $x$ of $E$ such that $1 = 1_{B}(x) < Dv_0(x)$. Choose any real number $x$, with $0 < x < 1$; by the definition of $\overline{Dv_0}$, we have that the relation

$$\frac{\mu(B \cap A)}{\mu(A)} > x$$

is eventually verified at $x$.

Bearing in mind the convention made in Definition (2.2), it follows that the relation

$$\mu(A \cap B) > x \mu(A) \quad \text{and} \quad 0 < \mu(A) < +\infty$$

is eventually verified at $x$.

Now the conclusion follows along exactly the same lines as in the proof of Theorem (1.2), Chap. III of [10].

From Theorems (3.2) and (3.4) we can deduce the following

(3.5) Corollary: Let $\Phi$ be a De Giorgi basis.

Then $\Phi$ possesses the weak Vitali property with respect to any measure on $\mathcal{A}$ which is $\Phi$-eventually finite, $\sigma$-finite and $\Phi$-regular.

Let $\mu, v$ be two measures on the space $(X, \mathcal{A})$ and $\Sigma$ a derivation basis on $(X, \mathcal{A})$. It is well known that a sufficient condition for $\Sigma$ to define
(in the sense of Definition (2.7)) either measure with respect to the other, is the fact that $\Sigma$ enjoys the weak Vitaly property with respect to both measures, provided that they verify a suitable finiteness condition (which holds, for example, for any two positive Radon measures on a locally compact space with a countable base); see [10], Chap. II, §§ 1, 2, 3.

Thus, for a De Giorgi basis $\phi$ on $(X, \mathcal{A})$, the results stated by Theorems (2.4) and (2.8) can be viewed as consequences of Corollary (3.5) above and Theorems (2.3) and (3.2), Chap. II of [10].

4. - EXAMPLES OF DE GIORGI BASES IN $\mathbb{R}^n$

Throughout this section, $(X, \mathcal{A})$ will denote a fixed nonempty open and bounded set $X$ in $\mathbb{R}^n$, provided with the $\sigma$-algebra $\mathcal{A}$ of Borel subsets of $X$.

For every element $A$ of $\mathcal{A}$, we shall denote by $\delta(A)$ the diameter of $A$.

(4.1) Example (a): For each point $x$ of $X$, consider the D.S. $\Phi_x=(A_l(x))_l$ whose elements are the open balls $A_l(x) = \{y: |x-y| < l\}$ with their center at $x$, directed by inclusion (with $l$ varying through any nonempty subset $L(x)$ of the set of strictly positive real numbers such that $\inf \{l: l \in L(x)\} = 0$).

It is clear that the family $\Phi=(\Phi_x)_{x \in X}$ is a derivation basis which is compatible with the topology on $X$.

Moreover, from Lemma (3.1), page 109 of [4], it follows that $\phi$ verifies the De Giorgi property (with constant $g = 3^n$).

Example (b): For each point $x$ of $X$, consider the D.S. $\Sigma_x=(B_l(x))_l$ whose elements are the closed balls $B_l(x) = \{y: |x-y| < l\}$ with their center at $x$, directed by inclusion (with $l$ varying as in Example (a)); the family $\Sigma=(\Sigma_x)_{x \in X}$ is also a De Giorgi basis (see [2], [3]), compatible with the topology on $X$.

Example (c): We obtain another De Giorgi basis on $X$ (compatible with the topology on $X$) by considering the family $\Gamma=(\Gamma_l(x))_{x \in X}$ such that, for each $x$, $\Gamma_l(x) = (C_l(x))_{l \in L}$, is a D.S. of closed cubes $C_l(x)$ with their center at $x$ (see [6], Th. (1.1)), directed by inclusion (with $l$ varying as in Example (a)).

Example (d): The same is true for $\Delta=(\Delta_l(x))_{x \in X}$ such that, for each $x$, $\Delta_l(x) = (D_l(x))_{l \in L}$ is a D.S. of open cubes $D_l(x)$ with their center at $x$ (see [6], Chap. I, Remark (2), page 5) directed by inclusion (with $l$ varying as in Example (a)).

(Actually the bases considered in the above examples possess a stronger property than the one stated in (1.5), that is Besicovitch's property; see [6], page 43).

The examples considered above are particular cases of the derivation bases ("blankets") studied by A. P. Morse (see [12], [13]).

Let us now quote Morse's main definitions, trying to use his terminology and notations as far as possible.
(4.2) **Definition (see [13]):** By a Morse covering $\mathcal{F}$ of the set $X$ we mean a function which associates to every point $x$ of $X$ a nonempty family $\mathcal{F}(x)$ of sets of $A$, in such a way that, for every $x$, the following properties hold:

(a) Every set in $\mathcal{F}(x)$ is bounded;

(b) $x$ is an interior point of every set belonging to $\mathcal{F}(x)$;

(c) For every element $E$ of $\mathcal{F}(x)$, we have

$$\inf_{E \in \mathcal{F}(x)} \delta(E) = 0.$$ 

(4.3) **Definition:** Let $\mathcal{F}$ be a Morse covering of the set $X$. For each point $x$ of $X$, let us order the sets in $\mathcal{F}(x)$ by decreasing diameters and consider the family $\Phi_x$ whose elements are the cofinal subsequences of $\mathcal{F}(x)$.

Then the derivation basis $\Phi = (\Phi_x)_{x \in X}$ on $(X, A)$ is said to be generated by the covering $\mathcal{F}$.

(4.4) **Definition (see [13], Def. 1.4):** If $B$ is a subset of $X$ and $x \in B$, the internal radius of $B$ at $x$ is defined as the supremum of those numbers $r$ for which the closed ball $S_r(x)$, with its center at $x$ and radius $r$, is included in $B$.

(4.5) **Definition (see [13], Def. 5.3):** For every subset $B$ of $X$ the hub of $B$ is the set of those points $y \in B$ such that $(1-t)x + ty \in B$ whenever $x \in B$ and $0 < t < 1$. We denote this set by $\text{hub}(B)$.

If $x \in \text{hub}(B)$, we denote by $b(x, B)$ the internal radius of $\text{hub}(B)$ at $x$.

Clearly $B = \text{hub}(B)$ if $B$ is convex.

(4.6) **Definition:** Let $\mathcal{F}$ be a Morse covering of set $X$.

We use the term regularity function of $\mathcal{F}$ for the function $L$ defined, for every point $x \in X$, by the following relation

$$L(x) = \limsup_{B \in \mathcal{F}(x), \delta B < 0} \frac{\delta(B)}{b(x, B)}.$$ 

(4.7) **Definition (see [13], Def. 5.6):** Let $\mathcal{F}$ be a Morse covering of set $X$. $\mathcal{F}$ is called a star blanket iff it verifies the following conditions:

a) For every point $x$ of $X$, one has $L(x) < + \infty$, with $L$ denoting the regularity function of $\mathcal{F}$;

b) For every point $x$ of $X$, all the elements of $\mathcal{F}(x)$ are closed sets.

The following theorem, due to A. P. Morse, is perhaps the most important and general result obtained in the field of the «concrete» derivation bases considered in this section (see [13], Th. 5.13 page 442; see also [10], Th. 4.9.2, page 119):
Morse's Theorem: Consider for set \( X \) a star blanket \( \mathcal{S} \). Then the derivation basis \( \Phi \) generated by \( \mathcal{S} \), possesses the strong Vitali property with respect to any Radon measure \( \mu \) on \( X \).

It follows that a derivation basis generated by a star blanket on \( X \) enjoys the universal derivability property with respect to Radon measures on \( X \): in other words, if we consider two arbitrary Radon measures on \( X \), such a basis « derives » (in the sense of Definition (2.7)) either of them with respect to the other.

However, the results of the previous sections, which are based on the De Giorgi property, allow us to establish, in the case of the basis generated by a Morse covering, two more sufficient conditions of universal derivability, which are not considered in Morse's paper: these are stated by Theorems (4.10) and (4.13) below.

These conditions are to be considered as complementary to the one stated in Morse's Theorem: in particular, they do not require the sets in every class \( \mathcal{F}(x) \) to be closed.

We are now going to state a property which will be fundamental for the sequel, and which follows directly from certain very significant results obtained by Morse.

(4.8) Proposition: Let \( \mathcal{S} \) be a Morse covering of set \( X \). Suppose that there exists a strictly positive constant \( \lambda \) such that, for any \( x \), \( L(x) < \lambda \), with \( L \) denoting the regularity function of \( \mathcal{S} \).

Then the derivation basis \( \Phi \), generated by \( \mathcal{S} \), is a De Giorgi basis.

Proof: Let \( E \) be a subset of \( X \) and \( \mathcal{A}(A) \) a relation which is frequently verified at each point of \( E \).

By virtue of our assumptions, the relation

\[
\delta(A) < 1 \quad \text{and} \quad \delta(A) < \lambda \ b(x, A)
\]

is eventually verified at each point \( x \) of \( X \).

Let us denote by \( \mathcal{K} \) the set of the constituents of \( \Phi \) verifying the relation \( \mathcal{A}(A) \); then the following relation

\[
A \in \mathcal{K}, \quad \delta(A) < 1, \quad \delta(A) < \lambda \ b(x, A)
\]

is frequently verified at each point of \( E \).

We shall use \( \Omega \) to denote the set of pairs \((x, A)\), where \( x \) is an element of \( E \) and, for each \( x \), \( A \) belongs to class \( \mathcal{F}(x) \cap \mathcal{K} \) and verifies the relation

\[
\delta(A) < 1 \quad \text{and} \quad \delta(A) < \lambda \ b(x, A).
\]

We call \( \mathcal{L} \) the class of all sets occurring as the second element in some pair of \( \Omega \); we have \( \mathcal{L} \subset \mathcal{K} \) (and the set \( E \) is included in the union of the sets belonging to \( \mathcal{L} \)).
By Theorems (4.8) and (5.10) of [13] (see also [10], Coroll. 3.5 page 113 and Lemma 4.8 page 117) \( \mathcal{C} \) has a subclass \( \mathcal{S} \) such that

\[ E \subset \bigcup_{a \in \mathcal{S}} A_a, \]

and which verifies the following property:

if \( q = (9a)^2 \), there exists in \( \mathcal{S} \) \( q \) subclasses \( \mathcal{Q}_i \) such that

\[ \mathcal{S} = \bigcup_{i=1}^{q} \mathcal{Q}_i, \]

each class \( \mathcal{Q}_i \) being countable and disjoint.

Thus

\[ \sum_{a \in \mathcal{S}} 1_a < q. \]

Since class \( \mathcal{S} \) is countable and included in \( \mathcal{X} \), the proof is complete. \( \blacksquare \)

(4.9) **Remark:** Let \( \Phi \) be the derivation basis on \( \mathcal{X} \) generated by a Morse covering \( \mathcal{F} \) of \( \mathcal{X} \); from the assumption \((b)\) of Definition (4.2), it follows directly that every \( \Phi \)-open subset of \( \mathcal{X} \) is open (in the usual topology of \( \mathbb{R}^n \)).

On the other hand, for any point \( x \) of \( \mathcal{X} \) and any (open) ball \( B \) with its center at \( x \), from assumptions \((b)\) and \((c)\) of Definition (4.2), it follows that relation \( B \circ (\mathcal{F}) \) is eventually verified at \( x \).

Thus every open subset of \( \mathcal{X} \) is also \( \Phi \)-open.

Hence, in \( \mathcal{X} \), the class of open sets is the same as the class of \( \Phi \)-open sets.

It follows from the previous Remark that every Radon measure on \( \mathcal{X} \) is \( \Phi \)-regular.

As an immediate consequence of Proposition (4.8) and Theorem (2.8), we obtain the following:

(4.10) **Theorem:** Let \( \mathcal{F} \) be a Morse covering of the set \( \mathcal{X} \) and let us suppose that the regularity function of \( \mathcal{F} \) verifies the assumptions of Proposition (4.8).

Let \( \nu, \mu \) be two arbitrary Radon measures on \( \mathcal{X} \).

Then the derivation basis \( \Phi \), generated by \( \mathcal{F} \), derives \( \nu \) with respect to \( \mu \).

From Proposition (4.8) and Theorem (3.2) we get the following:

(4.11) **Theorem:** Let \( \mathcal{F} \) be a Morse covering of the set \( \mathcal{X} \) and let us suppose that the regularity function of \( \mathcal{F} \) verifies the assumptions of Proposition (4.8).

Then the derivation basis \( \Phi \), generated by \( \mathcal{F} \), possess the density property with respect to any Radon measure \( \mu \) on \( \mathcal{X} \).

From the last theorem, by means of Theorem (3.4), we get the following:
(4.12) Theorem: Let $\mathcal{F}$ be a Morse covering of the set $X$ and let us suppose that the regularity function of $\mathcal{F}$ verifies the assumptions of Proposition (4.8).

Then the derivation basis $\Phi$, generated by $\mathcal{F}$, possesses the weak Vitali property with respect to any Radon measure on $X$.

From Theorem (4.10) we easily obtain the following sufficient condition of universal derivability.

(4.13) Theorem: Let $\mathcal{F}$ be a Morse covering of set $X$ whose regularity function $L$ verifies the following properties:

(a) For every point $x$ of $X$, $L(x) < +\infty$;

(b) $L$ is everywhere upper semicontinuous.

Let $\mu$, $\nu$ be two arbitrary Radon measures on $X$.

Then the basis $\Phi$, generated by $\mathcal{F}$, derives $\nu$ with respect to $\mu$.

Proof: Fix any strictly positive real number $\lambda$ and consider the set

$$Y = \{x \in X : L(x) < \lambda\}$$

by assumption (b), $Y$ is open, hence $\Phi$-open (see Remark (4.9)).

We denote by $r_1$ (resp. $\mu_1$) the restriction to $Y$ of the Radon measure $\nu$ (resp. $\mu$) and by $\Phi^*$ the restriction of the basis $\Phi$ to $Y$.

By Theorem (4.10), the basis $\Phi^*$ derives $r_1$ with respect to $\mu_1$.

Denote by $f$ a version of the density of the $\mu$-absolutely continuous part of $\nu$; thus, for every point $x$ of $Y$ outside a $\mu_1^*$-nullset, we have

$$1_r(x)f(x) = D^-r_1(x) = D^+r_1(x),$$

with $D^-r_1$ (resp. $D^+r_1$) denoting the lower (resp. upper) derivative of $r_1$ with respect to the measure $\mu_1$ and the basis $\Phi^*$.

Now, by Remark (2.10), the relation

$$D^-r_1(x) = D\nu(x) \quad \text{and} \quad D^+r_1(x) = D\nu(x)$$

is verified at each point $x$ of $Y$, with $D\nu$ (resp. $D\nu$) denoting the lower (resp. upper) derivative of $\nu$ with respect to the measure $\mu$ and the basis $\Phi$.

Thus, from the above relations, for every point $x$ of $Y$ outside a $\mu^*$-nullset, we obtain

$$f(x) = D\nu(x) = D\nu(x).$$

For every integer $n > 1$, let us pose

$$Y_n = \{x \in X : L(x) < n\}.$$
From the above, it follows that, for each $n$, the relation

$$f = Dv = Dv$$

is verified $\mu^*$-almost everywhere in the open set $Y_n$.

Since we have $X = \bigcup Y_n$, we can conclude that the above-mentioned relation holds $\mu^*$-almost everywhere in $X$.

From the above Theorem and (3.4), we deduce the following property:

(4.14) Theorem: Let us assume that the Morse covering $\mathcal{F}$ of set $X$ verifies the hypothesis of Theorem (4.13).
Then for every Radon measure $\mu$ on $X$, the derivation basis $\Phi$, generated by $\mathcal{F}$, possesses the $\mu$-density property and (therefore) the weak Vitali $\mu$-property.

5. - More examples

We shall begin this section with an example of a De Giorgi basis which verifies the universal derivability property with respect to Radon measures, but which does not possess the strong Vitali property. This example is derived from C. Hayes [9], page 291.

Let $X$ be the open unit square in $\mathbb{R}^2$ with its center at the origin and whose sides are parallel to the axes.

For each point $x$ of $X$, let $\langle I_k(x) \rangle_k$ be the sequence of sets defined as follows: for every strictly positive integer $k$, $I_k(x)$ is the open (or closed) square centered at $x$, with sides parallel to the axes and a side-length $\frac{1}{4k}$ together with the set of all points having rational coordinates, included in the open (or closed) square centered at $x$, with sides parallel to the axes and a side-length of $\frac{1}{4k}$.

Let us consider the derivation basis $\Phi = (\Phi_k)_{k \in \mathbb{N}}$ such that, for each $x$, we have $\Phi_k = \{I_k(x)\} \subset X$, with sequence $\langle I_k(x) \rangle_k$ being chosen in such a way that, for any $k$, $I_k(x) \subset X$.

Because of the shape of sets $I_k$, it is easy to prove that, for any disjoint sequence $\langle I_n \rangle_n$ whose elements are all included in $X$, we have

$$A\left(X - \bigcup_n I_n\right) > 0,$$

with $A$ denoting the Lebesgue measure in $\mathbb{R}^2$: thus $\Phi$ does not possess the strong Vitali property with respect to $A$.

On the other hand, if we consider the regularity function ((4.6)) $L$ of $\Phi$, we have $L = 4\sqrt{2}$ everywhere in $X$. 

Thus $\Phi$ is a De Giorgi basis (Proposition (4.8)).

By Theorem (4.10), $\Phi$ enjoys the universal derivability property with respect to the Radon measures on $X$ and, by Theorem (4.12), $\Phi$ possesses the weak Vitali property with respect to any Radon measure $\mu$ on $X$.

Now we shall quote another example of a space, which possesses the De Giorgi property; see, for reference, [5], § 2.8.

Let $(X, \varrho)$ be a metric space. Let us denote by $A_r(x)$ the open ball in $X$, with radius $r$ and its center at $x$, and suppose that the metric $\varrho$ has the following property:

\begin{equation}
(5.1) \text{Definition (see [5], 2.8.9): By saying that $\varrho$ is directionally $(\xi, \eta, \zeta)$-limited, we mean that $\xi > 0$, $0 < \eta < \frac{1}{\zeta}$, $\zeta$ is a positive integer and the following condition holds:}
\end{equation}

If $a \in X$, $B \subset A_r(a) - \{a\}$, and if

$$\frac{\varrho(a, x)}{\varrho(a, c)} > \eta \quad \text{whenever } b \in B, c \in B, b \neq a, \varrho(a, b) > \varrho(a, c), x \in X \text{ with}$$

$$\varrho(a, x) = \varrho(a, c), \quad \varrho(x, b) = \varrho(a, b) - \varrho(a, c),$$

then $\text{card } B < \zeta$.

\begin{equation}
(5.2) \text{Remark: In [5], 2.8.9, page 146, it is pointed out that, for suitable constants } \xi, \eta, \zeta, \text{ the condition stated in (5.1) is verified either in the case when } (X, \varrho) \text{ is a finite dimensional normed vector space, provided with the distance } \varrho \text{ induced by its norm, or in the case when } X \text{ is any compact subset of a Riemannian manifold (of class } > 2), \text{ with its usual metric.}
\end{equation}

From Theorem (2.8.14) of [5] we can directly deduce the following:

\begin{equation}
(5.3) \text{Theorem: Let } (X, \varrho) \text{ be a separable metric space such that the metric } \varrho \text{ is directionally } (\xi, \eta, \zeta)\text{-limited, and } r \text{ a constant with } 0 < r < \frac{\xi}{2}.
\end{equation}

\text{Let } E \text{ be any subset of } X \text{ and } \mathcal{F} \text{ a covering of } E \text{ obtained by associating with each point } x \text{ of } E \text{ a closed ball with its center at } x \text{ and with a radius less than } r. \text{ If } q \text{ denotes the integer } 2\zeta + 1, \text{ then there exists } q \text{ subclasses } \mathcal{G}_i \text{ of } \mathcal{F} \text{ such that the following relations hold:}

\begin{enumerate}
\item [(a)] $E$ is included in the union of the balls belonging to $3$, where

$$3 = \bigcup_{i=1}^{\xi} \mathcal{G}_i.$$

\item [(b)] For each $i$, the subclass $\mathcal{G}_i$ is countable and disjoint.
\end{enumerate}
On a metric space $(X, \rho)$ for which the assumptions of the above Theorem hold, let us consider a derivation basis $\Sigma$ formed by closed balls, such that each point of $X$ is the center of some ball of arbitrarily small diameter: from Theorem (5.3) we obtain that $\Sigma$ is a De Giorgi basis.

REFERENCES