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Weakly Almost-Periodic Solutions of Some Abstract Differential Equations (**) (***)

Soluzioni debolmente quasi periodiche di certe equazioni differenziali astratte

RIASSUNTO. — In questa Nota è studiata la quasi-periodicità debole per le soluzioni di certe equazioni differenziali assegnate in spazi di Hilbert.

INTRODUCTION

In this note we study the problem of weak almost-periodicity for solutions of some differential equations in Hilbert spaces (see, for previous work on this argument [2], [3], [7], [8], [9], [10]).

Although almost-periodic functions with range in a locally convex space were introduced long time ago, in Bochner-von Neumann's fundamental paper [5], the theory of the specific case where the range space is a Banach space with its *weak topology* was investigated by Amerio in [1] and subsequently applied by him and others to obtain weakly (and also strongly) almost-periodic solutions of some abstract or partial differential equations. In this paper we concentrate ourselves on the study of weakly almost-periodic solutions in themselves, and obtain some relations in this direction.

1. - Our first result concerns a Bohr-Neugebauer type theorem of the weak type, in an abstract situation suggested by Bochner's Note [4].

One considers a Hilbert space H which can be written as a direct sum $H = \oplus \sum_{n=1}^{\infty} H_n$ where H_n are finite-dimensional closed subspaces of H, mu-

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tually orthogonal, and related to a given closed linear operator C in H in the following way:

Assume that P_n denotes the orthogonal projection on H_n (so that obviously, the relation $h = \sum_{n=1}^{\infty} P_n h$, $\forall h \in H$-strongly, is satisfied, and also $P_n P_m = 0$ for $n \neq m$). The linear operator C is closed, and its domain $D(C) \subset H$ is dense in H; the range of C is also included in H. Next, we suppose that all H_n are contained in $D(C)$, so that C is linear continuous on H_n, and assume also that $CP_n = P_n C$—which implies that the image of H_n through C is contained in H_n. Call C_n the restriction of C to H_n. Consider now a function $f(t)$, $\mathrm{R} \to H$ which is *weakly almost-periodic* (that is $\langle f(t), h \rangle$ is almost-periodic as a scalar-valued function for all $h \in H$). Next, let $u(t)$, $\mathrm{R} \to H$ be a differentiable function which is *bounded* over R, such that $u(t) \in D(C)$ for all $t \in \mathrm{R}$ and the equality

$$u'(t) - Cu(t) = f(t) \qquad \text{holds, } \forall t \in \mathrm{R}\,.$$

Then $u(t)$ is weakly almost-periodic.

In order to establish this result we note firstly that the function $(P_n f)(t)$ is weakly almost-periodic, $\mathrm{R} \to H_n$: for all $h \in H_n$ we have $\langle P_n f(t), h \rangle_{H_n} = \langle f(t), h \rangle_H$ is almost-periodic. Due to the finite-dimensionality of the space H_n, we obtain strong almost-periodicity (also strong continuity) of $P_n f$, as a function from R into H_n, and this for $n = 1, 2, \dots$. Define now a function $u_n(t)$ by the relation: $u_n(t) = (P_n u)(t)$; it obviously is a solution to the equation

$$u_n'(t) = P_n u'(t) = P_n(Cu(t) + f(t)) = Cu_n(t) + (P_n f)(t) = C_n u_n(t) + (P_n f)(t).$$

This is a differential equation in H_n and by one of the assumptions, the operator C when restricted to H_n is linear and continuous. Hence, the function $u_n(t)$ which is bounded over R, is almost-periodic, $\mathrm{R} \to H_n$ by a classical result of Bohr (¹). Next, we have pointwise (i.e. for any $t \in \mathrm{R}$) the equality $u(t) = \sum_{n=1}^{\infty} (P_n u)(t)$, in the strong sense. Therefore, taking any $h \in H$, we obtain still in pointwise sense, the relation

$$\langle u(t), h \rangle = \left\langle \sum_{n=1}^{\infty} (P_n u)(t), \sum_{n=1}^{\infty} P_n h \right\rangle = \sum_{n=1}^{\infty} \langle P_n u(t), P_n h \rangle$$

which is a series of scalar-valued almost-periodic functions. *This is an uniformly convergent series over the real line.*

(¹) See for instance: C. Corduneanu, *Almost periodic functions*, Interscience, 1968.

Consider in fact the simple estimates:

$$\left|\sum_{N}^{N+p}\langle P_n u(t), P_n b\rangle\right| < \sum_{N}^{N+p}\|P_n u(t)\|\,\|P_n b\| < \left(\sum_{N}^{N+p}\|P_n u(t)\|^2\right)^{\frac{1}{2}}\cdot$$

$$\left(\sum_{N}^{N+p}\|P_n b\|^2\right)^{\frac{1}{2}} < \|u(t)\|\left(\sum_{N}^{N+p}\|P_n b\|^2\right)^{\frac{1}{2}} < \left(\sup_{t\in\mathbb{R}}\|u(t)\|\right)\left(\sum_{N}^{N+p}\|P_n b\|^2\right)^{\frac{1}{2}}$$

which establishes the result.

2. - First we note that, under the stated assumptions, if the solution $u(t)$ has precompact range, it is strongly almost-periodic ([3], [7], [10]).

Next, we remark that it is possible to extend somewhat the previously obtained result, considering a certain class of weak solutions of the above differential equation.

Remember that the linear closed operator with dense domain C has an adjoint C^*, where the domain $D(C^*)$ is characterized by the relation:

$$D(C^*) = \{y \in H;\ \langle Cx, y\rangle = \langle x, z\rangle,\ \forall x \in D(C),\ \text{for some } z \in H\}$$

and that for $y \in D(C^*)$, $C^*y = z$ (where z is uniquely defined). In the special case that we are considering in this paper, we see that the whole space H_n (for any natural number n) is contained in $D(C^*)$ and that C^* maps continuously H_n into itself.

Take in fact any element y in H_n and $x \in D(C)$; write the obvious equalities $x = \sum_1^\infty P_n x$ and $Cx = \sum_1^\infty P_j Cx$. It follows that

$$\langle Cx, y\rangle = \left\langle\sum_1^\infty P_j Cx, y\right\rangle = \sum_1^\infty\langle P_j Cx, y\rangle = \sum_1^\infty\langle P_j Cx, P_n y\rangle =$$

$$= \langle P_n Cx, y\rangle = \langle CP_n x, y\rangle = \langle P_n x, C_n^* y\rangle$$

(remember that C_n is the restriction of C to H_n, and it is linear continuous, $H_n \to H_n$). Consequently, one obtains the equality

$$\langle Cx, y\rangle = \langle x, P_n C_n^* y\rangle = \langle x, C_n^* y\rangle\,.$$

Thus, y belongs to $D(C^*)$ and $C^*y = C_n^* y$, which also means that C^* maps H_n into itself.

At this stage, we may define the weak solutions that are here considered as (strongly) continuous functions $\mathbb{R} \to H$, $u(t)$, verifying the equation (see also [2]):

$$(d/dt)\langle u(t), b\rangle = \langle u(t), C^*b\rangle + \langle f(t), b\rangle$$

over the real line, for any $b \in H_n$, where $n = 1, 2, \ldots$ (thus, it is assumed that the scalar-valued function $\langle u(t), b\rangle$ has a derivative for all $b \in \bigcup_{n=1}^{\infty} H_n$). Let us fix now a certain natural number n, and let us consider again the (continuous) function $u_n(t) = (P_n u)(t)$. If b is an element in H_n, it follows that $P_n b = b$, hence

$$\frac{d}{dt}\langle u(t), b\rangle = \frac{d}{dt}\langle u(t), P_n b\rangle = \langle u(t), C^* P_n b\rangle + \langle f(t), P_n b\rangle = \\ = \langle u(t), C_n^* P_n b\rangle + \langle P_n f(t), b\rangle .$$

Now, from the commutativity hypothesis $C_n P_n = P_n C_n$ we deduce obviously that $C_n^* P_n = P_n C_n^*$ and it follows that

$$\frac{d}{dt}\langle P_n u(t), b\rangle = \langle u(t), P_n C_n^* b\rangle + \langle P_n f, b\rangle = \langle P_n u(t), C_n^* b\rangle + \langle P_n f, b\rangle = \\ = \langle C_n P_n u(t), b\rangle + \langle P_n f(t), b\rangle .$$

Hence the function $u_n(t) = P_n u(t)$ has a weak derivative in the finite-dimensional space H_n, that is it has a strong derivative too. It follows that

$$\frac{d}{dt} P_n u(t) = C_n P_n u(t) + P_n f(t)$$

and now we can continue as in the previous section.

3. - Example (equation with a diagonal operator): Consider a bounded diagonal operator A in a separable Hilbert space H with orthonormal basis $\{e_j\}_1^\infty$; thus $Ae_j = \alpha_j e_j$ for some complex numbers α_j ($j = 1, 2, \ldots$). Here the spaces H_n in previous considerations will be the unidimensional spaces $\{\lambda e_n\}_{\lambda \in \mathbb{C}}$. Hence, the above result applies, and the bounded over $\mathbb{R}$ solutions of the equation

$$u' = Au + f$$

where f is weakly almost-periodic, are also weakly almost-periodic. However, in the special case of the homogeneous equation: $u' = Au$ we obtain more: *bounded solutions are strongly almost-periodic* (such a result is not obvious in the more general situation discussed in the beginning when restricted to the homogeneous equation). To demonstrate this statement, we first deduce from the above considerations, the equality

$$\frac{d}{dt}\langle u(t), e_k\rangle = \alpha_k \langle u(t), e_k\rangle .$$

for $k = 1, 2, \ldots$ and accordingly the equality:

$$\langle u(t), e_k\rangle = \exp(\alpha_k t)\langle u(0), e_k\rangle .$$

If the function $u(t)$ is bounded over R, it follows that all the scalar-valued functions $\langle u(t), e_k\rangle$ are also bounded over R, which implies that $\alpha_k = i\sigma_k$ with some $\sigma_k \in \mathbb{R}$ and therefore we obtain

$$u(t) = \sum_{k=1}^{\infty} \langle u(0), e_k\rangle \exp(i\sigma_k t) e_k, \qquad \forall t \in \mathbb{R}$$

which is certainly a pointwise convergent series of almost-periodic functions. However, this is also uniformly convergent over the real line, as easily follows:

$$\left\| \sum_{N}^{N+p} \langle u(0), e_k\rangle \exp(i\sigma_k t) e_k \right\|^2 = \sum_{N}^{N+p} |\langle u(0), e_k\rangle|^2 < \varepsilon^2 \qquad \text{for } N > N(\varepsilon),\ \forall t \in \mathbb{R}.$$

Hence, we derive strong almost-periodicity of $u(t)$—and we also see, from the obvious equality: $\|u(t)\|^2 = \sum_{k=1}^{\infty} |\langle u(0), e_k\rangle|^2 = \|u(0)\|^2$ that the range of $u(t)$ lies on a sphere in H.

Next, let us specialize the Hilbert space; and assume H to be the classical l^2 space of all complex-valued sequences $(\xi_n)_0^\infty$ with $\sum_0^\infty |\xi_n|^2 < +\infty$; in this space consider the action of the « unilateral shift » U, which is defined (see for ex. [6]) by the relation $U(\xi_n)_0^\infty = (0, \xi_0, \xi_1, \ldots)$. It is known, and elementary, that the operator U is bounded and linear, even an isometry of l^2. Consider now the differential equation $u'(t) = Uu(t)$, where $u(t)$ is a differentiable function $\mathbb{R} \to l^2$; this readily translates into the infinite sequence of scalar differential equations $u_0'(t) = 0$, $u_1'(t) = u_0(t)$, $u_2'(t) = u_1(t)$, ...

We derive, by successive integration, the equalities

$$u_0(t) = u_0(0)\,, \; u_1(t) = u_0(0)\,t + u_1(0)\,, \; u_2(t) = u_0(0)\frac{t^2}{2} + u_1(0)\,t + u_2(0)\,, \; \ldots .$$

Thus, each of the functions $u_k(t)$ will be a polynomial of degree k; if $u(t)$ has a bounded range in the space l^2 the same property holds for all functions $u_k(t)$; all these functions must be constant and $u(t)$ reduces to the null function in l^2.

Let us consider now a function $f(t)$, from the real line into l^2, which is weakly almost-periodic. It follows that all the scalar-valued function $f_k(t)$ (where $f(t) = (f_k(t))_0^\infty$) are almost-periodic. Next, if the function $u(t)$, $\mathbb{R} \to l^2$, verifies the differential equation:

$$u'(t) = U u(t) + f(t)$$

it results that the infinite system of ordinary differential equations:

$$u_0'(t) = f_0(t)\,, \; u_1'(t) = u_0(t) + f_1(t)\,, \; u_2'(t) = u_1(t) + f_2(t)\,, \; \ldots$$

is also verified in $\mathbf{R}$. If $u(t)$ is a bounded function over $\mathbf{R}$, we deduce, by the classical theorem of Bohr-Bohl, that all the functions $u_k(t)$ are almost-periodic. This implies weak almost-periodicity of $u(t)$: take in fact any element b in l^2, that is any sequence $(b_i)_0^\infty \subset l^2$. The inner product $\langle u(t), b\rangle$ becomes the infinite series $\sum_0^\infty u_i(t)\bar{b}_i$ and this is uniformly convergent over the real line because of the estimate:

$$\sum_N^{N+p} |u_i(t)\bar{b}_i| < \|u(t)\|^2 \Big(\sum_N^{N+p} |b_i|^2\Big)^{\frac{1}{2}} < \sup_R \|u(t)\|^2 \Big(\sum_N^{N+p} |b_i|^2\Big)^{\frac{1}{2}} .$$

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