Rendiconti
Accademia Nazionale delle Scienze detta dei XL
Memorie di Matematica
104° (1986), Vol. X, fasc. 3, pagg. 39-42

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A Footnote to a Conjecture in Number Theory (**)

Dedicated to the memory of my mother

SUMMARY. — In the present footnote we prove the average form of a conjecture from S. Chowla and H. Walum (when $r > 1$).

A proposito di una congettura in Teoria dei numeri

SUNTO. — In questo lavoro sarà presa in considerazione una congettura di S. Chowla e H. Walum: essa, formulata in termini di media, sarà dimostrata nel caso di $r > 1$.

1. - INTRODUCTION

In 1963, S. Chowla and H. Walum [2] conjectured, as an extension of the question of the correct order of the error term in the Dirichlet's divisor problem, that

$$(1)\qquad \sum_{n\leq\sqrt{x}} n^a B_r\left(\left\{\frac{x}{n}\right\}\right) = O(x^{a/2+1/4+\varepsilon}), \quad x\to\infty,$$

is true for every $\varepsilon > 0$, where the integers $a \geq 0$ and $r \geq 1$ are given and $B_r(y)$ is the r-th Bernoulli polynomial, and $\{v\}$ denotes the fractional part of v. Recently S. Kanemitsu and R. Sita Ramachandra Rao [3] have established the validity of (1) for all *real* $a \geq \frac{1}{2}$, and $r \geq 2$ with $\varepsilon = 0$ even. Their paper (alongwith [4]) contains further investigations concerning (1) and also a detailed introduction to earlier works in this topic. In the present footnote we prove a result, which leads to (cf. [1]) an improved extension of Theorem 2

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(**) Memoria presentata il 29 maggio 1985 da Ennio De Giorgi, uno dei XL.

ISSN-0392-4106

occurring in [4]. Setting

$$G_{a,r}(x,Q)=\sum_{n\leq Q} n^a B_r\left(\left\{\frac{x}{n}\right\}\right),$$

we state our result as a

THEOREM: Let $1<Q<cX^{\frac{1}{2}}$, where c is a constant. Then, for any given $a>-\frac{1}{2}$ and $r\geq 2$,

$$X^{-1}\int_X^{2X}|G_{a,r}(x,Q)|^2\,dx=O(Q^{2a+1}) \tag{2}$$

holds with the O-constant depending at most on a, r and c.

Our proof is simple, using Hilbert's inequality (in the following qualitative form)

$$\sum_{1\leq m\neq n\leq Q}\frac{a_m\bar{a}_n}{m-n}=O\Big(\sum_{m\leq Q}|a_m|^2\Big) \tag{H}$$

with an absolute O-constant. (A proof of (H) can be read from the identity

$$\frac{1}{2\pi i}\sum_{1\leq m\neq n\leq Q}\frac{a_m\bar{a}_n}{m-n}=\frac{1}{2}\sum_{m\leq Q}|a_m|^2-\int_0^1 du\int_0^u\Big|\sum_{m\leq Q}a_m e(mt)\Big|^2 dt\,;$$

cf. Ramachandra [5].) Instead of using (H), in the following proof, if we proceed to estimate straightaway (by taking the absolute values inside) there would be a loss of « log » factor, but we can allow Q to vary with x and there by obtain

THEOREM 0: Let $Q_x(>1)$ be a monotonically increasing function of $x(>1)$ satisfying $Q_x<cx^{\frac{1}{2}}$ with some constant c. Then, for any given $a>-\frac{1}{2}$ and $r\geq 2$,

$$X^{-1}\int_X^{2X}|G_{a,r}(x,Q_x)|^2\,dx=O(Q^{2a+1}(1+X^{-1}Q^2\log Q))\,, \tag{2'}$$

where $Q=Q_X$, holds with the O-constant depending at most on a, r and c.

REMARKS: By taking $r=2$, $Q_x=x^{\frac{1}{2}}$, we would already have an improvement of Theorem 2 occuring in [4], though not to an extent provided by Theorem * in [1]. However, observe that if $Q_x=O((x/\log x)^{\frac{1}{2}})$ the present theorem gives the (required) mean $O(Q_{2X}^{2a+1})$. It will be clear from the proof that one also has results corresponding to (2), (2') when n^a, occurring in the definition of $G_{a,r}(x,Q)$, is replaced by any other « smooth » function; for example, $n^a(\log n)^b$.

2. - Proof of the Theorems

As usual, set $e(y) = \exp(2\pi i y)$. Using the Fourier expansion

$$B_r(\{y\}) = \Big(\sum_{m \neq 0} m^{-r} e(my)\Big) F_r ,$$

where $F_r = -r!(2\pi i)^{-r}$, and that $r > 2$, we obtain for any $Y(>1)$

$$F_r^{-1} G_{a,r}(x, Q) = \sum_{1 \leq |m| < Y} m^{-r} \sum_{n \leq Q} n^a e\left(\frac{mx}{n}\right) + O(Q^{a+1} Y^{1-r}) . \tag{3}$$

Now, consider (for $|m| > 1$)

$$\int_X^{2X} \Big| \sum_{n \leq Q} n^a e\left(\frac{mx}{n}\right) \Big|^2 dx = X \sum_{n \leq Q} n^{2a} + \sum_{1 \leq n_1 \neq n_2 \leq Q} n_1^a n_2^a \int_X^{2X} e\left(mx\left(\frac{1}{n_1} - \frac{1}{n_2}\right)\right) dx .$$

On carrying out the integration the second sum on the right becomes the difference of two sums of the form

$$(2\pi i m)^{-1} \sum_{1 \leq n_1 \neq n_2 \leq Q} n_1^a e(mt/n_1) n_2^a e(-mt/n_2) \Big/ \left(\frac{1}{n_1} - \frac{1}{n_2}\right),$$

with $t = 2X$, and $t = X$ (for Theorem) or t depending on n_1, n_2 (for Theorem 0). We now give the proof of Theorem. Since t is independent of n_1, n_2 the latter sum, according to (H), is

$$O\Big(\sum_{n \leq Q} n^{2(a+1)}\Big) = O(Q^{2a+3}) .$$

Thus, we get (via (3) and Cauchy's inequality) on making also a choice for $Y > Q^{\frac{1}{2}}$,

$$\int_X^{2X} |G_{a,r}(x, Q)|^2 dx = O\Big(\sum_{1 \leq m \leq Y} m^{-r}(m^{-1} Q^{2a+3} + XQ^{2a+1}) + XQ^{2a+1}\Big) =$$
$$= O(Q^{2a+3} + XQ^{2a+1}) ,$$

since $r > 2$. Using $Q < cx^{\frac{1}{2}}$, we see that Theorem is proved. For the proof of Theorem 0, the estimation of the double sums above is done by taking the absolute values inside so that the dependence of t on n_1, n_2 presents no difficulty. This gives the bound $O(Q^{2a+3} \log Q)$, with $Q := Q_{2X}$, and the rest of the proof being as above, we obtain Theorem 0.

Acknowledgement: The inclusion of Theorem 0 above has been consequent to a remark from Prof. R. Sita Ramachandra Rao, and I am very thankful to him for that comment about a step in an earlier version of the present manuscript.

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