A Characterization of Basic Sequences in Banach Spaces (***)

Una caratterizzazione delle Successioni Basiche negli spazi di Banach.

Sunto. — Una successione di uno spazio di Banach è basica se e solo se ogni sua blocco-perturbazione è $M$-basica forte.

Let $(x_n)$ be a sequence of a Banach space $B$ and let $(f_n)$ be a sequence of $B^*$ (the dual of $B$), we say that $(x_n, f_n)$ is biorthogonal if $f_m(x_n) = \delta_{mn}$ (Kronecker indices) for every $m$ and $n$. Let $(x_n, f_n)$ be biorthogonal, we say that $(x_n)$ is

i) M-basis of $B$ if $B = \{x_n\} = \overline{\text{span}}(x_n)$ and if $\{0\} = \{f_n\} \subseteq \{x \in B; f_n(x) = 0 \text{ for every } n\}$;

ii) strong M-basis of $B$ if $B = [x_n]$ and if $[f_n] \subseteq [x_n]$ for every complementary subsequences $(\eta_n)$ and $(\eta'_n)$ of $(n)$;

iii) basis of $B$ if $x = \sum_{n=1}^{\infty} f_n(x_n) x_n$ for every $x$ of $B$.

Moreover we say that $(x_n)$ is M-basic (basis) if it is M-basis (basis) of $[x_n]$.

If $(x_n) \subseteq B$, $q_0 = 0$ and $(g_n)$ is an increasing sequence of positive integers, we say that $(y_n)$ is

block perturbation of $(x_n)$ if $[y_n]_{n=q_0}^{q_0} = [x_n]_{n=q_0}^{q_0} + 1$ for every $m$.

Our aim is to prove the following

**Theorem I:** A sequence is basic if and only if every block perturbation is strong M-basic.
In §1 we shall prove the theorem by means of a geometric characterization of the basic sequences and by a recent result of Plans.

In §2 we shall give another direct proof of Th. I by means of the concept of unitary position.

1. - A NEW GEOMETRIC CHARACTERIZATION OF THE BASIC SEQUENCES

We shall prove the following

**Theorem II:** Let \((x_n, f_n)\) be biorthogonal in \(B\), then

\((x_n)\) is basic \(\Leftrightarrow \cap \left\{ \left\{ \sum_{n=1}^{\infty} f_n(x) x_n \right\}_{n=1}^{\infty} ; (m_\xi) \subseteq (u) \right\} = [x]\) for every \(x\) of \([x_n]\).

In [4] Th. II is proved for the reflexive case.

In [3] it is proved that Th. II implies Th. I.

**Proof of Th. II:** Obviously it is sufficient to prove the inverse implication. Let \(x \in [x_n]\), by [6] (p. 146, Th. 13.1, a), \(1^a \Rightarrow 3^a\) it is sufficient to prove that

\((v_m) \) weakly converges to \(x\), where \(v_m = \sum_{n=1}^{\infty} f_n(x) x_n\) for every \(m\).

We affirm that \((v_m)\) is bounded, because if \(\lim_{n \to \infty} [v_{m_n}] = + \infty\) for a subsequence \((m_k)\) of \((m)\), \((v_{m_k} / |v_{m_k}|)\) would not have subsequence weakly convergent to \(x' \neq 0\) if \((m_k) \subseteq (m)\) with \((v_{m_k} / |v_{m_k}|)\) weakly convergent to \(x' \neq 0\), it would be \(f_n(x') = 0\) for every \(n\), which is impossible since \((x_n)\) is obviously \(M\)-basic and \(x' \in [x_n]\), therefore it would follow by [7] (p. 172) that

\[(1) \cap \left\{ \left\{ [v_{m_k}]_{k=1}^{\infty} ; (m_k) \subseteq (m) \right\} = \{0\} \right\},

which contradicts the hypothesis.

Then \((v_m)\) weakly converges to \(x\), otherwise by the "third Fréchet's axiom" [1] there would exist a subsequence \((v_{m_k})\) without subsequences weakly convergent to \(x\), hence without weakly convergent subsequences (since if \((m_k) \subseteq (m)\) with \((v_{m_k})\) weakly convergent to \(x'\), it would be \(f_n(x - x') = 0\) for every \(n\) hence \(x = x'\)); therefore, since \((v_m)\) is bounded, it would follow by [7] (p. 172) (1) again, which contradicts the hypothesis. This completes the proof of Th. II.

2. - ON THE CONCEPT OF UNITARY POSITION

Given \((x_n)\) we say that \(x \in [x_n]\) is in unitary position as regards \((x_n)\) if there exists a subsequence \((s_k)\) of \((s)\) such that \(x \in [x_{s_k}], x \notin [x_{s_{k'}}]\) for every \(k'\).

In [5] (see also [2] p. 121-126) it is proved, in the frame of biorthogonal
sequences, that

\((x'_n)\) is strong \(M\)-basic \(\iff\) every element of \([x_n]\) is in unitary position as regards \((x_n)\).

Then in order to prove Th. I it is sufficient to prove that, if \((x_n)\) is not basic, there exist \(\bar{x} \in [x_n]\) and a block perturbation \((\xi_n)\) of \((x_n)\) so that \(\bar{x}\) is not in unitary position as regards \((\xi_n)\).

Firstly we prove the following

**Proposition I:** Let \((\eta_n)\) be a basic sequence of \(B\) and let \((u_n)\) be a sequence of a finite dimensional subspace \(U\) of \(B\); then, if \(x_n = u_n + v_n\) for every \(n\), it follows that:

\((x_n)\) is \(M\)-basic \(\iff\) \((x_n)\) is basic.

**Proof:** Indeed suppose that \((x_n)\) is \(M\)-basic but not basic.

By [6] (p. 58) there exists a block sequence \((x'_n)\) of \((x_n)\) (that is there exists an increasing sequence \((q_n)\) of positive integers so that, setting \(q_0 = 0\), \(x'_n \in \text{span} (x'_{n+1})_{n=\ldots,q_n-1} \) for every \(n\)) such that

\[(2) \quad x'_n = u'_n + v'_n \text{ and } |x'_n| = 1 \text{ for every } n, \text{ with } (u'_n) \subset U, (v'_n) \text{ basic and } x'_{2n-1} - x'_{2n} \to 0.\]

Since \((x'_n)\) is bounded (otherwise \((u'_n|v'_n)\)) would have a convergent subsequence, while it is basic), there exist \((\eta'_n) \subset (u'_n)\) and \((\xi'_n) \subset (v'_n)\) so that

\[
h'_{2n-1} \to h', \quad v'_n \to v';
\]

then \(u' = u'' = u\) (otherwise by (2) it would be \(v'_{2n-1} - v'_{2n} \to u'' - u' \neq 0, \) impossible since \((u'_n)\) is basic); hence by (2) \(v'_{2n-1} - v'_{2n} \to 0,\) that is \(v'_{2n-1} \to 0\) and \(v'_{2n} \to 0\) since \((u'_n)\) is basic; therefore by (2) \(x'_{2n} \to u,\) impossible since \((x'_n)\) (block sequence of \((x_n)\)) is \(M\)-basic. This completes the proof of Proposition I.

**Proposition II:** Every \(M\)-basic but not basic sequence has a block perturbation with a subsequence which is basic with brackets but not basic.

Where \((x_n)\) is basic with brackets if there exists an increasing sequence of positive integers \((q_n)\) such that, setting \(q_0 = 0,\)

\[
\mathcal{X} = \sum_{\alpha = 1}^{\infty} \left( \sum_{\alpha = \varepsilon_{n+1}}^{\infty} a_\alpha x_\alpha \right),
\]

with \((a_{\alpha})\) unique, for every \(\varepsilon \) of \([x_n]\).

**Proof:** Let \((x_n)\) be \(M\)-basic and not basic.
We shall prove that there exist a sequence \( \langle y_n \rangle \) of \( B \), \( 1 < k < \infty \) and an increasing sequence \( \langle q_n \rangle \) of positive integers so that, setting \( q_0 = 0 \), for every \( m > 1 \)

\[
\begin{align*}
    (i) \quad & (y_{km-1}, y_{km}) \text{ is block sequence of a block perturbation of} \\
    & (x_k)_{k=\infty}^{\infty}; \\
    (ii) \quad & |y_{km+1} + y_{km+2}| < 1/2^{m+1} \text{ and } |y_m| > K2^m, \\
    & \text{ moreover } |y_1 + y_2| = 1; \\
    (iii) \quad & \text{dist}(y, [y_n]_{n \geq 2m}) > \|y\|/K \text{ for every } y \text{ of } [y_n]_{n=1}^{\infty}.
\end{align*}
\]

We can suppose \( \langle x_n \rangle \) in \( C_{0^{\infty}} \), then, if \( \langle \epsilon_n \rangle \) is the Schauder basis of \( C_{0^{\infty}} \),

\[
\sum_{n=1}^{\infty} \epsilon_n x_n \quad \text{ for every } m.
\]

Since \( \langle x_n \rangle \) is not basic, by [6] (p. 58) there exist \( y_1, y_2 \) so that \( i \) for \( m = 1 \) is true moreover \( |y_1| > 2^2 \) and \( |y_1 + y_2| = 1. \)

Fix a positive integer \( p > 1. \)

Suppose to have \( \langle Z_r \rangle \) of positive integers and \( \langle y_n \rangle \) of \( B \) which verify \( i \) and \( ii \) of (3) for \( 1 < m < p \); moreover for every \( 1 < m < p \) and for every \( y \) of \( [y_n]_{n=1}^{\infty} \)

\[
\text{dist}(y, [y_n]_{n=2m+1}^{2m+2} + [\epsilon_n]_{n \geq m}) > (1 - 1/2^{m+1}) \|y\|
\]

where \( [y_n]_{n=2m+1}^{2m+2} \) does not appear if \( m = p. \)

By (4) it is easy to see that there exist a positive integer \( r_p \) and numbers \( d_{p,n} \) so that, if

\[
X_{p,m} = x_n + \sum_{n=r_p+1}^{r_p} d_{p,n} x_n \quad \text{for } m > r_p,
\]

then \( (x_{p,n})_{n \geq r_p} \subset [\epsilon_n]_{n \geq r_p} \). By Propos. I \( (x_{p,n})_{n \geq r_p} \) is not basic, hence there exist \( q_{p+1} \) and \( (y_{2p+1}, y_{2p+2}) \) block sequence of \( (x_{p,n})_{n \geq r_p} \) so that

\[
|y_{2p+1} + y_{2p+2}| < 1/2^{p+1}, |y_{2p+1}| > 2^{p+1}, (y_{2p+1}, y_{2p+2}) \subset [x_n]_{n=r_p+1}^{r_p};
\]

finally there exists a positive integer \( r_{p+1} \) so that

\[
\text{dist}(y, [\epsilon_n]_{n \geq r_{p+1}}) > (1 - 1/2^{p+1}) \|y\| \quad \text{for every } y \text{ of } [y_n]_{n=1}^{2p+2}.
\]

So proceeding we get \( \langle y_n \rangle \) as in (3) for \( k = 2 \); which completes the proof of Propos. II.

2nd Proof of Th. I: We consider the particular case of \( \langle x_n \rangle \) norming, that is (see [7], p. 174) there exist \( 1 < k < \infty \) and a not decreasing sequence \( \langle l_n \rangle \) of positive integers so that \( |x| < K \text{ dist}(x, [x_n]_{n \geq l_n}) \) for every \( x \) of \( [x_n]_{n=1}^{\infty} \) and for every \( m. \) Suppose that every block perturbation of \( \langle x_n \rangle \) is strong \( M \)-basic. If \( \langle x_n \rangle \) is not basic, by Propos. II we have a sequence \( \langle y_n \rangle \) as in (3)
for (ii) and (iii), moreover with \( y_n = x_n^{\text{me}, n+1} \) for every \( m \), then set
\[
\bar{x} = \sum_{n=1}^{m} (y_{2n-1} + y_{2n}), \quad \bar{v}_1 = y_1, \quad \bar{v}_{n+1} = y_{2n} + y_{2n+1} \quad \text{for every } n > 1.
\]

If \( \bar{x} \notin [v_n] \), the theorem will be proved; indeed setting by (3)
\[
\bar{z}_1 = y_1 \quad \text{and} \quad \bar{z}_n = x_n \quad \text{for } 2 < n < q_1;
\]

moreover for \( m > 1 \)
\[
\bar{z}_{m(n+1)} = y_{2m} + y_{2m+1} = v_{m+1} \quad \text{and} \quad \bar{z}_n = x_n \quad \text{for } q_{2m-1} + 2 < n < q_{2m+1},
\]

then it follows
\[
\bar{x} \notin [\bar{z}_1 \cup (\bar{z}_{4m-1})^{\text{me}}], \quad \text{while} \quad \text{dist}(\bar{x}, [\bar{z}_1 \cup (\bar{z}_{4m-1})^{\text{me}}] + [\bar{z}_1, (\bar{z}_{4m-1})^{\text{me}}]) < \frac{1}{2^n}
\]
for every \( m \); that is \( \bar{x} \) is not in unitary position as regards the block perturbation \( (\bar{z}_n) \) of \( (x_n) \), which contradicts the hypothesis.

In order to prove that \( \bar{x} \notin [v_n] \) it is sufficient to prove that
\[
\| \bar{x} - \sum_{n=1}^{m} \bar{v}_n \| < 1 \quad \text{implies} \quad |a_n| < 1/2^n \quad \text{for } 1 < n < m;
\]

indeed, if \( (y_n, h_n) \) is biorthogonal, by (5) \( h_n(\bar{x}) = 1 \) for every \( n \), while by (5)
\[
h_1(\sum_{n=1}^{m} a_n v_n) = a_1 \quad \text{and} \quad h_{2k}(\sum_{n=1}^{m} a_n v_n) = h_{2k+1}(\sum_{n=1}^{m} a_n v_n) = a_{2k+1} \quad \text{for } 1 < k < m.
\]

Fix \( 1 < p < m \), setting \( a_{m+1} = 0 \) we can suppose true the thesis of (6) for \( p + 1 < n < m + 1 \) and we shall prove the thesis for \( n = p \). By (3) and (5) we have that
\[
\| a_{p+1} y_{2p-1} + \sum_{n=p+2}^{m} a_n v_n \| < \| \sum_{n=1}^{m} a_n v_n \| + \| \sum_{n=1}^{p} a_n v_n + a_{p+1} y_{2p} \| < K.
\]

Hence by the hypothesis of (6) and by (3)
\[
\| a_p y_{2p-1} + a_{p+1} y_{2p} \| < \left( \sum_{n=1}^{m} a_n v_n \right) + \left( \sum_{n=1}^{p-1} a_n v_n + a_{p+1} y_{2p-2} \right) + \left( a_{p+1} y_{2p+1} + \sum_{n=p+2}^{m} a_n v_n \right) \leq \left( \sum_{n=1}^{m} a_n v_n \right) (2 + 2K) < (\| \bar{x} \| + 1)(2 + 2K) < 4K(\| \bar{x} \| + 1).
\]
Moreover, since the thesis of (6) is true for $a_{n+1}$, by (3) we have that

\[
|a_n y_{n-1} + a_{n+1} y_n| \geq (|a_n| - |a_{n+1}|) |y_{n-1}| - |a_{n+1}| |y_{n-1} + y_n| >
\]

\[
> (|a_n| - 1/2^{n+1}) |y_{n-1}| - 1/2^{n+1} > (|a_n| - 1/2^{n+1}) 2^{n+4} K - 1/2^{n+1}.
\]

By (3) and (5) $1/2 < |x| < 3/2$; hence by (7) and (8) we have that

\[
|a_n| < (4K(|x| + 1) + 1/2^{n+1})/2^{n+4} K + 1/2^{n+1} < 1/2^n + 1/2^{n+1} < 1/2^n.
\]

This completes the proof.

**BIBLIOGRAPHY**


