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Groups in which All Subgroups are Subnormal (**)

Gruppi con tutti i sottogruppi subnormali

RIASSUNTO. — Si prova che la serie derivata di un gruppo in cui tutti i sottogruppi sono subnormali si arresta dopo un numero finito di passi.

Let $\mathfrak{N}_0$ denote the class of groups in which every subgroup is subnormal. All non-nilpotent $\mathfrak{N}_0$-groups constructed so far ([3], [4], [5], [6], [7]) are metabelian, and it is still an open problem whether there exist non-soluble $\mathfrak{N}_0$-groups. In this Note we prove the following:

THEOREM: *The derived series of a $\mathfrak{N}_0$-group terminates after a finite number of steps.*

This improves previous results in this direction ([1], [2]), and is a consequence of an idea of Brookes, and of a Theorem of Roseblade's [8], which gives a function $\mu(n)$, such that every group in which all subgroups are subnormal of defect at most n, is nilpotent of class at most $\mu(n)$.

An easy corollary of our Theorem is the following.

COROLLARY: *A residually finite $\mathfrak{N}_0$-group is soluble.*

Our first Lemma is a generalization of Theorem B in Brookes [1]; for sake of completeness we give a proof of it. In the statement, $\mathfrak{M}$ is a class of groups, such that no nilpotent group belongs to it.

LEMMA 1: *If $G \in \mathfrak{M} \cap \mathfrak{N}_0$, then there exist a positive integer r, and a $\mathfrak{M}$-subgroup K of G, containing a finitely generated subgroup H, such that every $\mathfrak{M}$-subgroup of K which contains H has defect at most r in K.*

PROOF: Let $G \in \mathfrak{M} \cap \mathfrak{N}_0$ and assume, by contradiction, that for any $\mathfrak{M}$-subgroup K of G, any finitely generated subgroup H of K and any positive

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integer r, there exists a $\mathfrak{M}$-subgroup of K, containing H, whose defect in K is greater than r. By induction on i, we define $\mathfrak{M}$-subgroups K_i and finitely generated subgroups H_i of G such that:

$$K_{i+1} \lneq K_i \qquad \text{and} \qquad H_i < \bigcap_{n \in \mathbb{N}} K_n \qquad \text{for every } i \in \mathbb{N}.$$

We put $K_1 = G$, $H_1 = 1$. Let $i > 1$ and assume we have already defined $H_1, \ldots, H_{i-1}$ and K_{i-1}. Now $K_{i-1} \in \mathfrak{M}$ contains the finitely generated subgroup $\langle H_1, \ldots, H_{i-1} \rangle$, then, by our hypothesis, there exists a $\mathfrak{M}$-subgroup K_i of K_{i-1}, containing $\langle H_1, \ldots, H_{i-1} \rangle$ of defect greater or equal to i in K_{i-1}. Hence there exists a finite subset S of K_i, and a finite subset T of K_{i-1}, such that $[T,_{i-1}S] \not\subseteq K_i$. We put $H_i = \langle S \rangle < K_i$, and observe that $K_i > \langle H_1, \ldots, H_i \rangle$ and that if $L < K_i$ then $\langle H_i, L \rangle$ has defect greater than $i-1$ in K_{i-1}. We put now $H_* = \langle H_i | i \in \mathbb{N} \rangle$, then $H_* < K_n$ for any $n \in \mathbb{N}$, and so, by what above observed, H_* has defect greater than n in K_n; it follows that H_* is not subnormal in G, a contradiction.

Lemma 2: *Let T be a subgroup of a group K. If every subgroup $X > T$ of K is subnormal of defect at most n in K, then $K^{(\tau(n))} < T$, where*

$$\tau(n) = \sum_{i=1}^{n} ([\log_2 \mu(i)] + 1)$$

and $\mu(i)$ is the function of Roseblade's Theorem [8].

Proof: By induction on n. If $n = 1$, then $T \lhd K$ and every subgroup of K/T is normal; in this case K/T is metabelian (in fact K/T is a Dedekind group and $\mu(1) = 2$), and so $K^{(2)} \leq T$.

Let $n > 1$ and S be any subgroup of T^K containing T, then $S^K = T^K$ and, since S has defect at most n in K, it has defect at most $n-1$ in T^K. By inductive hypothesis, $T > (T^K)^{(\tau(n-1))}$. Now, every subgroup of K/T^K is subnormal of defect at most n; therefore, by Roseblade's Theorem, K/T^K is nilpotent of class at most $\mu(n)$, in particular it is soluble of derived length at most $e = [\log_2 \mu(n)] + 1$. Hence $K^{(e)} < T^K$, and so $K^{(\tau(n))} = (K^{(e)})^{(\tau(n-1))} < T$.

Proof of the Theorem: It suffices to prove that hypoabelian groups in $\mathfrak{N}_0$ are soluble. Thus, let G be a hypoabelian group in $\mathfrak{N}_0$, and $\mathfrak{M}$ be the class of non-soluble groups. Assume, by contradiction, that $G \in \mathfrak{M}$. Then, by Lemma 1, there exist a non-soluble subgroup K of G, a finitely generated subgroup H of K, and a positive integer r, such that every non-soluble subgroup of K which contains H has defect at most r in K. If S is such a subgroup of K, then every subgroup of K containing S is non-soluble, and so it has defect at most r in K. By Lemma 2 it follows that $S > K^{(\tau)}$, where $\tau = \tau(r)$. Therefore, if V is any non-soluble subgroup of K, then $\langle V, H \rangle > R$, where $R = K^{(\tau)}$. In particular, for any positive integer n, we have $K^{(n)}H > R$, since K

(and so $K^{(n)}$ for every $n \in \mathbb{N}$) is non-soluble. Now $G \in \mathfrak{N}_*$ is locally nilpotent, and so H is nilpotent. Let t be the derived length of H and put $d = r + t + 1$, then $R \leq K^{(d)}H$, yielding:

$$R^{(t)} \leq K^{(d)}H^{(t)} = K^{(d)} = (K^{(r)})^{(t+1)} = R^{(t+1)} = (R^{(t)})' .$$

But, since G is a hypoabelian group, no non trivial subgroup of G is perfect; therefore $K^{(r+t)} = R^{(t)} = 1$, a contradiction.

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