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# Lattice-Theoretic Aspects of Doctrines and Hyperdoctrines (**)

## Aspetti reticolari di dottrine e iperdottrine

SOMMARIO. — Vengono introdotte categorie di insiemi parzialmente ordinati (rispettivamente di semireticoli, reticoli distributivi, algebre di Heyting e di Boole) con mappe connessioni di Galois che verificano la reciprocità di Frobenius [7]. In questo modo si ottengono categorie codominio per le dottrine e iperdottrine prodotte dal trasferimento (diretto e inverso) di sottoggetti in categorie regolari, logiche, Heyting e booleane [8]. Vengono introdotte generalizzazioni delle precedenti ottenendo così « buoni » funtori « Sub »; si prova, infine, che la costruzione di Sub è « universale ».

### INTRODUCTION

Lawvere [6] introduced doctrines and hyperdoctrines on a base category $\mathcal{C}$ as contravariant functors $P\colon \mathcal{C} \to \mathcal{CAT}$ such that for every morphism $f$ the functor $P(f)$ has a left adjoint $\exists_f$ and—in the case of hyperdoctrines—also a right adjoint $\forall_f$. In common examples $P(A)$ is the poset of subobjects of $A$ and $P(f)$ the inverse image operator along $f$.

In this paper we introduce categories of posets which are codomain categories for doctrines and hyperdoctrines produced by (direct and inverse) transfer of subobjects for regular, logical, Heyting and boolean categories (see [8]).

Actually we introduce generalizations of the latter forgetting products or replacing them by a monoidal structure; these will be called *subregular, sublogical, subHeyting, subboolean and regular monoidal categories.* Corresponding to such categories we introduce *categories of posets* (respectively semilattices, distributive lattices, Heyting algebras and boolean algebras) and *convenient Galois connections.* As in these categories the *subobjects correspond to principal ideals* we prove them to be respectively subregular, sublogical, subHeyting and subboolean and their « Sub » *functors* to be respectively subregular and so on.

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Moreover we show that *every* semilattice, distributive lattice, Heyting algebra and boolean algebra is *isomorphic to a poset of subobjects* in the above categories. Finally we prove that *the construction of Sub is « universal »*.

Analogous results were obtained by Grandis [3] (see also [4]) for exact categories (in the sense of Puppe-Mitchell).

General conventions: This paper concerns pairs $(\mathcal{C}, \mathcal{M})$ where $\mathcal{C}$ is a category and $\mathcal{M}$ a subcategory of *« distinguished monos »* such that $\mathrm{Iso}_{\mathcal{C}} \subseteq \mathcal{M} \subseteq \mathrm{Mono}_{\mathcal{C}}$; for each object $A$ of $\mathcal{C}$, we write $\mathrm{Sub}(A)$ the poset of $\mathcal{M}$-subobjects of $A$, which we always assume to be small. Actually, for the sake of brevity, a *« category »* $\mathcal{C}$ will mean such a pair and a *« functor »* will be assumed to preserve distinguished monos. In such a category a morphism $f$ will be said to be surjective if $f = mg$ with $m \in \mathcal{M}$ imply $m$ iso. The categories $\mathcal{S}et$ of sets and mapping and $\mathcal{P}os$ of posets and order-preserving mappings have all monos as distinguished ones; for $\mathcal{SL}^{op}$, the opposite of the category of semilattices (= inf-semilattices with 1) and homomorphisms, we choose the regular monos (= surjective homomorphism of $\mathcal{SL}$) as distinguished ones. In categories where maps are certain Galois connections we choose as distinguished monos those $u_* \dashv u^*$ such that $u_*$ is an injective mapping.

## 1. - Direct and inverse images of subobjects

1.1. We shall say that a category $\mathcal{C}$ has *inverse images* if the following pullbacks $(y \in \mathcal{M})$

$$(A) \qquad x = f^{-1}(y) \quad \begin{array}{ccc} \cdot & \xrightarrow{f} & \cdot \\ \uparrow & & \uparrow y \\ \cdot & \xrightarrow{f'} & \cdot \end{array}$$

exists and $f^{-1}(y) \in \mathcal{M}$.

If $\mathcal{C}$ has inverse images then we get the functor $\mathrm{Sub}\colon \mathcal{C} \to \mathcal{SL}^{op}$, setting $\mathrm{Sub}(A \xrightarrow{f} B) = \mathrm{Sub}(A) \xleftarrow{f^{-1}} \mathrm{Sub}(B)$; it clearly preserves inverse images. Moreover an inverse-image-preserving functor $F\colon \mathcal{C} \to \mathcal{D}$ between categories with inverse images yields a natural transformation $\mathrm{Sub}^F\colon \mathrm{Sub}\,F \longrightarrow \mathrm{Sub}\colon \mathcal{C} \to \mathcal{SL}^{op}$ taking $x$ to $F(x)$.

1.2. We say that a category $\mathcal{C}$ has *direct images* if:

i) for each map $f$ of $\mathcal{C}$ there is a smallest subobject $\mathrm{im} f$ through which $f$ factors;

ii) for each $f\colon A \to B$ surjective (¹) and for every $\overset{y}{\rightarrowtail} B \in \mathcal{M}$ there is $\overset{x}{\rightarrowtail} A \in \mathcal{M}$ such that $\mathrm{im}(fx) = y$.

(¹) By i) a morphism $f$ is surjective (according to our conventions) iff $\mathrm{im} f = 1$.

Now if we set $\exists_f(x) = \mathrm{im}\,(fx)$ for all $f\colon A \to B$ and $x \in \mathrm{Sub}\,(A)$ we get an order-preserving function from Sub $(A)$ to Sub $(B)$; by ii), $f$ is surjective iff $\exists_f$ is a surjective mapping. Moreover if $\mathcal{M}$ has the following property

$$m, n \in \mathcal{M} \text{ and } m = na \text{ imply } a \in \mathcal{M}$$

we get that every category $\mathfrak{C}$ with direct images yields a direct-image-preserving functor Sub: $\mathfrak{C} \to \mathfrak{Sets}$; moreover a functor $F\colon \mathfrak{C} \to \mathfrak{D}$ between categories with direct images preserves them iff $\mathrm{Sub}^F\colon \mathrm{Sub} \dashrightarrow \mathrm{Sub}\circ F\colon \mathfrak{C} \to \mathfrak{Sets}$ is natural.

1.3. Definition: A category $\mathfrak{C}$ is *subregular* if it has direct and inverse images. A functor is *subregular* if it preserves direct and inverse images.

1.4. If $\mathfrak{C}$ has inverse images the following conditions are equivalent:

*a*) 1.2 i) is verified,

*b*) every map $f$ of $\mathfrak{C}$ factors as $f = yg$ with $g$ surjective and $y$ in $\mathcal{M}$,

*c*) for every map $f$ of $\mathfrak{C}$, $f^{-1}$ has a left adjoint $\exists_f$,

*d*) for every map $f$ of $\mathfrak{C}$ there is a smallest distinguished subobject $y$ such that $f^{-1}(y) = 1$.

1.5. Let $\mathfrak{C}$ have inverse images and verify 1.2 i); then $\exists_f$ is left adjoint to $f^{-1}$ and the following conditions are equivalent:

*a*) $\mathfrak{C}$ is subregular (namely 1.2 ii) is verified),

*b*) if $f$ is surjective then $\exists_f f^{-1} = 1$,

*c*) if $f$ is surjective, in every inverse image square $fx = yf'$ (1.1 $(A)$) the map $f'$ is so,

*e*) (*Beck condition*) for every inverse image square $fx = yf'$ (1.1 $(A)$) we have $\exists_{f'}x^{-1} = y^{-1}\exists_f$,

*e*) (*Frobenius reciprocity*) for every $f\colon A \to B$, $x \in \mathrm{Sub}\,(A)$, $y \in \mathrm{Sub}\,(B)$ we have $\exists_f(f^{-1}(y) \wedge x) = y \wedge \exists_f(x)$,

*f*) for every $f$, $\exists_f f^{-1} = - \wedge \mathrm{im} f$.

1.6. From a syntactical point of view a subregular category is *characterized* by the following « *doctrinal* » *data*: for all $f\colon A \to B$ we have order-preserving mappings

$$\mathrm{Sub}\,(A) \underset{f^{-1}}{\overset{\exists_f}{\rightleftarrows}} \mathrm{Sub}\,(B)$$

such that:

*a*) $f^{-1}$ is the inverse image operation,

*b*) $f^{-1}\exists_f(x) \wedge x = x$,

*c*) $\exists_f(f^{-1}(y) \wedge x) = y \wedge \exists_f(x)$.

## 2. - Frobenius reciprocity and Galois connections

We introduce the subregular category $\mathfrak{C}n\mathfrak{SL}t$, which « simulates » the direct and inverse images of subobjects in subregular categories.

2.1. Say $\mathfrak{C}n\mathfrak{SL}t$ the category whose objects are (1-inf) semilattices and whose maps are *semilattice connections* (or Frobenius connections [2]) namely

$$u = (u_\bullet, u^\bullet) \colon S \underset{u^\bullet}{\overset{u_\bullet}{\rightleftarrows}} T$$

where $u_\bullet$ and $u^\bullet$ are order-preserving mappings such that:

$$u^\bullet u_\bullet(s) \wedge s = s, \qquad s \in S,$$

$$u_\bullet(u^\bullet(t) \wedge s) = t \wedge u_\bullet(s), \qquad s \in S,\ t \in T.$$

The composition is:

$$u \circ v = (u_\bullet, u^\bullet) \circ (v_\bullet, v^\bullet) = (u_\bullet v_\bullet, v^\bullet u^\bullet).$$

2.2. Now for $s \in S$ the principal ideal $\downarrow(s) = \{x \in S : x < s\}$ yields a semilattice connection $\downarrow(s) \underset{-\wedge s}{\overset{\hookrightarrow}{\rightleftarrows}} S$ (conversely a 1-poset $S$ with this property, for all $s \in S$, is necessarly a semilattice). Given a semilattice connection $u \colon S \to T$ we get a factorization through $u_\bullet(S) = \downarrow(u_\bullet(1))$ (trivially $u_\bullet(S) \subset \downarrow(u_\bullet(1))$, while $t < u_\bullet(1)$ gives $t = t \wedge u_\bullet(1) = u_\bullet u^\bullet(t)$); therefore we choose as distinguished monos the semilattice connections $u$ such that $u^\bullet u_\bullet = 1$ (i.e., $u_\bullet$ is injective).

2.3. Proposition: *Given $S$ semilattice there are (natural) isomorphisms of posets (hence of semilattices):*

$$S \simeq \mathrm{Sub}(S) \simeq \mathfrak{C}n\mathfrak{SL}t(S, 2)$$

*where 2 is the two point (totally) ordered set.*

Proof: The map $\downarrow \colon S \to \mathrm{Sub}(S)$ gives the first iso by 2.2; given $s \in S$ we get a semilattice connection $\chi_s \colon S \to 2$ determined by $\chi_s^\bullet(1) = 1$, $\chi_s^\bullet(0) = s$. Thus we have that $\mathrm{Sub}(S) = \{\downarrow(s) : s \in S\}$ in $\mathfrak{C}n\mathfrak{SL}t$.

2.4. Proposition: *$\mathfrak{C}n\mathfrak{SL}t$ is a subregular category; for every semilattice connection $u \colon S \to T$, we have:*

*a)* $u^{-1}(\downarrow(t)) = \downarrow(u^\bullet(t)) = u_\bullet^{-1}(\downarrow(t))$ *for all* $t \in T$,

*b)* $\exists_u(\downarrow(s)) = \downarrow(u_\bullet(s)) = u_\bullet(\downarrow(s))$ *for all* $s \in S$.

(2) Galois connections satisfying Frobenius reciprocity [7].

*Every subregular category* $\mathcal{C}$ *yields a subregular functor* $\mathrm{Sub}\colon \mathcal{C} \to \mathcal{C}n\mathcal{SL}t$ *setting* $\mathrm{Sub}(f) = (\exists_f, f^{-1})$.

Proof: The right-hand equalities in *a*) and *b*) are obvious. Taking the left-hand ones as definitions, it is easy to prove that the conditions 1.6 *a*)-*b*)-*c*) hold. Last, if $m \in \mathcal{M}$ then $\exists_m = m \circ -$, thus $\mathrm{Sub}(m) = {\downarrow}(m)$; given an inverse image square (1.1 ($\mathcal{A}$)) we have that ${\downarrow}(f^{-1}(y))$ is the inverse image of ${\downarrow}(y)$ along $\mathrm{Sub}(f)$ in $\mathcal{C}n\mathcal{SL}t$, by *a*), hence Sub preserves inverse images; it also preserves direct images: if $f$ is surjective then $\mathrm{Sub}(f)$ is a surjective map in $\mathcal{C}n\mathcal{SL}t$ and we conclude with (1.4 *b*)).

2.5. There are *subregular forgetful functors* $D\colon \mathcal{C}n\mathcal{SL}t \to \mathcal{S}et$, $\Lambda\colon \mathcal{C}n\mathcal{SL}t \to \mathcal{SL}t^{op}$ respectively taking $u$ to $u_\bullet$ and $u^\bullet$; by (2.3) the functor $\mathrm{Sub}\colon \mathcal{C}n\mathcal{SL}t \to \mathcal{C}n\mathcal{SL}t$ is *isomorphic* to the identity functor.

2.6. Now consider an intersection-preserving functor $F\colon \mathcal{C} \to \mathcal{D}$ between subregular categories. As $\mathrm{Sub}^F_A$ need not have a right (or left) adjoint even if $F$ is subregular, we do not get a natural transformation $\mathrm{Sub}^F = (\mathrm{Sub}^F_A)$. Therefore in order to formalize $\mathrm{Sub}^F$ (2.7) we introduce the *double category* **CnSLt** whose cells are the squares

$$\begin{array}{ccc} S & \xrightarrow{\;h\;} & S' \\ {\scriptstyle u_\bullet}\downarrow\uparrow{\scriptstyle u^\bullet} & & {\scriptstyle v_\bullet}\downarrow\uparrow{\scriptstyle v^\bullet} \\ T & \xrightarrow[\;k\;]{} & T' \end{array}$$

where $h$, $k$ are semilattice homomorphisms and $u$, $v$ are semilattice connections such that the square *bicommutes* (i.e. $ku_\bullet = v_\bullet h$, $hu^\bullet = v^\bullet k$).

Similarly to [3, 4.7] we have:

2.7. Proposition: $\mathrm{Sub}^F\colon \mathrm{Sub} \to \mathrm{Sub} \circ F\colon \mathcal{C} \to$ **CnSLt** *is a « horizontal transformation of vertical functors »* (*or a* **CnSLt**-*wise transformation*) *iff* $F$ *is subregular.*

Proof: In fact we get that for every $f\colon A \to B$

$$\begin{array}{ccc} \mathrm{Sub}(A) & \xrightarrow{\;\mathrm{Sub}^F_A\;} & \mathrm{Sub}(FA) \\ {\scriptstyle \exists_f}\downarrow\uparrow{\scriptstyle f^{-1}} & & {\scriptstyle \exists_{Ff}}\downarrow\uparrow{\scriptstyle (Ff)^{-1}} \\ \mathrm{Sub}(B) & \xrightarrow[\;\mathrm{Sub}^F_B\;]{} & \mathrm{Sub}(FB) \end{array}$$

bicommutes iff $F$ is subregular.

## 3. - Distributive, Heyting and closure connections

3.1. For *sublogical* categories, i.e. subregular ones with stable finite sups, we get the same results if we substitute $\mathcal{C}n\mathcal{SL}t$ with the category $\mathcal{C}n\mathcal{D}\mathcal{L}t$ of distributive lattices and *distributive connections*, i.e. semilattice connections whose contravariant mapping preserves finite sups. A lattice $D$ is distributive iff for each $d \in D$, ${\downarrow}(d) \to D$ is a distributive connection; hence we have that

$$D \simeq \mathrm{Sub}\,(D) \simeq \mathfrak{CnDLt}\,(D, 3)$$

and Sub $(D)$ is the lattice of the principal ideals of $D$.

3.2. A further step will be provided by the category $\mathfrak{CnHeyt}$ of Heyting algebras and *Heyting connections* [3], i.e. semilattice connections $(u_\cdot, u^\cdot)$ such that $u^\cdot$ has a right adjoint $u_*$).

Let $H$ be a Heyting algebra; for every $h \in H$, (inclusion, $- \wedge h$, $h \to -$): ${\downarrow}(h) \to H$ is a Heyting connection; conversely a lattice with this property is necessarly a Heyting algebra. Now, given $H$ we have in $\mathfrak{CnHeyt}$

$$H \simeq \mathrm{Sub}\,(H) \simeq \mathfrak{CnHeyt}(H, 3)$$

thus Sub $(H)$ is the Heyting algebra of the principal ideals of $H$.

A category $\mathcal{C}$ is *subHeyting* if it is subregular with finite sups and for every $f\colon A \to B$ the $f^{-1}$ has a right adjoint $\forall_f$ so that we get a *hyperdoctrinal situation*:

$$\mathrm{Sub}\,(A) \underset{\forall_f}{\overset{\exists_f}{\underset{\xleftarrow{\;f^{-1}\;}}{\rightrightarrows}}} \mathrm{Sub}\,(B)$$

where $\exists_f \dashv f^{-1} \dashv \forall_f$ and Frobenius reciprocity hold.

Given a subHeyting category $\mathcal{C}$ we get a functor Sub: $\mathcal{C} \to \mathfrak{CnHeyt}$ (since Sub $(A)$ has implication iff $m^{-1}$ has a right adjoint for all $m \in \mathrm{Sub}\,(A)$). Of course *the category $\mathfrak{CnHeyt}$ is subHeyting* itself, with

$$\begin{aligned} u^{-1}({\downarrow}(k)) &= {\downarrow}(u^\cdot(k))\,,\\ \exists_u({\downarrow}(h)) &= {\downarrow}(u_\cdot(h))\,,\\ \forall_u({\downarrow}(h)) &= {\downarrow}(u_*(h))\,, \end{aligned}$$

where $u$ is a Heyting connection from $H$ to $K$, $h \in H$ and $k \in K$; moreover Sub is *subHeyting*, i.e. preserves finite sups, $f^{-1}$, $\exists_f$ and $\forall_f$ for all map $f$ of $\mathcal{C}$.

3.3. For *subboolean* categories, i.e. sublogical ones with complements of subobjects, we introduce the category $\mathfrak{CnBoole}$ of boolean algebras and distributive connections (or Heyting connections:

$$\text{if } (u_\cdot, u^\cdot) \text{ is a distributive connection then } u_* = \neg u_\cdot \neg$$

(3) A Heyting connection between locales is exactly an open map [5], [7].

is right adjoint to $u^*$); also in this case

$$B \simeq \mathrm{Sub}(B) \simeq \mathcal{C}n\mathcal{B}oole\,(B, \Omega)$$

where $B$ is a boolean algebra and $\Omega$ is the four-point boolean algebra.

3.4. We refer the reader to [2] for the notions of closure and universal closure operator.

Let $\mathcal{C}l\mathcal{SL}t$ be the category whose objects are *closure semilattices*, i.e. pairs $(S, ^-)$ where $S$ is a semilattice and $^-$ a closure operator on $S$, and whose maps are *closure connections*, i.e. $u\colon (S, ^-) \to (T, \hat{\ })$ semilattice connection such that $\overline{u^*(t)} = u^*(\hat{t})$ for all $t \in T$. Given a closure semilattice we can extend the first iso of 2.3: therefore $\mathcal{C}l\mathcal{SL}t$ *is subregular and it has a universal closure operator.* For every $\mathcal{C}$ subregular with a universal closure operator there is a (subregular) *closure functor*, i.e. closure-preserving, $\mathrm{Sub}\colon \mathcal{C} \to \mathcal{C}l\mathcal{SL}t$. Moreover every Heyting algebra has the double-negation operator in such a way that $\mathcal{C}n\mathcal{H}eyt \hookrightarrow \mathcal{C}l\mathcal{SL}t$ is a closure functor.

## 4. - Regular monoidal categories

4.1. Regular categories can be characterized as subregular categories $(\mathcal{C}, \mathcal{M})$ where $\mathcal{M} = \mathrm{Mono}_{\mathcal{C}}$, having finite products and $\times\colon \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ subregular (see [8]).

4.2. Definition: A monoidal category $(\mathcal{C}, \otimes, I)$ will be said to be *regular monoidal* if it is subregular and $\otimes\colon \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ is subregular.

4.3. Thus the product of semilattice connections $S \xrightarrow{u} T$, $S' \xrightarrow{v} T'$

$$u \times v = (u_* \times v_*, u^* \times v^*)\colon S \times S' \to T \times T'$$

defines a (non-cartesian) monoidal structure on $\mathcal{C}n\mathcal{SL}t$, whose identity is the one-point lattice, in such a way that $(\mathcal{C}n\mathcal{SL}t, \times, 1)$ is regular monoidal.

4.4. Recall that the correct notion of morphism

$$F\colon (\mathcal{C}, \otimes, I) \to (\mathcal{C}', \otimes', I')$$

for monoidal categories requires the existence of canonical arrows

$$I' \to FI\,,$$

$$FA \otimes' FB \to F(A \otimes B)\,,$$

satisfying axioms: MF1, MF2, MF3 in [1].

4.5. We say that a vertical functor $F: \mathcal{C} \to \mathbf{CnSLt}$, where $(\mathcal{C}, \otimes, I)$ is a monoidal category, is *horizontally monoidal* if there is $\varrho_{AB}: FA \times FB \to F(A \otimes B)$ horizontal transformation of vertical functors such that for the unique semilattice homomorphism $1 \to FI$ the axioms MF1, MF2, MF3 hold (4).

4.6. Proposition: *If* $\mathcal{C}$ *is regular monoidal then* Sub: $\mathcal{C} \to \mathbf{CnSLt}$ *is vertically subregular and horizontally monoidal.*

Proof: Using 2.7, the subregularity of $\otimes$ yields a horizontal transformation $s^{\otimes}_{AB}$: Sub $(A) \times$ Sub $(B) \to$ Sub $(A \otimes B)$ taking $(x, y)$ to $x \otimes y$; to complete the proof we have to show that $s^{\otimes}_{AB}$ verifies the axioms MF1, MF2, MF3 which is tedious, but obvious!

## 5. - The large Sub functor

To study the universality of Sub constructions we introduce some large categories.

5.1. Say $\mathfrak{SREG}$ the category of subregular categories and subregular functors; say $\mathfrak{REG}$ the category of regular monoidal categories and subregular monoidal functor.

5.2. Remark that given a semilattice homomorphism $S \xrightarrow{h} S'$ the isomorphism of 2.3 yields a homomorphism $\bar{h}$: Sub $(S) \to$ Sub $(S')$ where $\bar{h}(\downarrow(s)) = \downarrow(h(s))$ for $s \in S$; thus Sub: $\mathbf{CnSLt} \to \mathbf{CnSLt}$ is a double functor.

5.3. We introduce the « *lax-comma-wise* » category $\mathfrak{SREG} \int \mathbf{CnSLt}$. The *objects* are the subregular vertical functors of codomain $\mathbf{CnSLt}$ i.e. $F: \mathcal{C} \to Cn\mathfrak{SL}t$ subregular.

The *morphisms* are the triangles $(f, \eta')$

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathbf{CnSLt} \\ {\scriptstyle f}\downarrow & \Downarrow \eta' & \nearrow {\scriptstyle G} \\ \mathcal{D} & & \end{array}$$

where $f$ is subregular and $\eta'$: $F \to Gf: \mathcal{C} \to \mathbf{CnSLt}$ is a horizontal transformation of vertical functors (i.e. $\eta'$ assigns to every object $A$ of $\mathcal{C}$ a semilattice

(4) Notice that these axioms concern diagrams in which all connections are isomorphisms: thus the commutativity assumption is unambiguous.

homomorphism $\eta^f_A\colon FA \to GfA$ so that for each $\alpha\colon A \to B$ in $\mathcal{C}$ the square

$$\begin{array}{ccc} FA & \xrightarrow{\eta^f_A} & GfA \\ {\scriptstyle F\alpha}\downarrow & & \downarrow{\scriptstyle Gf\alpha} \\ FB & \xrightarrow[\eta^f_B]{} & GfB \end{array}$$

is bicommutative); we also *require* that, for each $A$, there is a commutative square

$$(S)\qquad \begin{array}{ccc} \mathrm{Sub}\,(FA) & \xrightarrow{\eta^f_A} & \mathrm{Sub}\,(GfA) \\ {\scriptstyle \mathrm{Sub}^F_A}\uparrow & & \uparrow{\scriptstyle \mathrm{Sub}^G_{fA}} \\ \mathrm{Sub}\,(A) & \xrightarrow[\mathrm{Sub}^f_A]{} & \mathrm{Sub}\,(fA) \end{array}$$

i.e. $\eta^f_A F(m) = Gf(m)$ for all $m \in \mathrm{Sub}\,(A)$.

*Composition*: Given $(\mathcal{C}, F) \xrightarrow{(f,\eta^f)} 0\,(\mathcal{D}, G) \xrightarrow{(g,\eta^g)} (\mathcal{E}, H)$ we define

$$(g, \eta^g)\circ(f, \eta^f) = (gf, \eta^{gf})$$

where

$$\eta^{gf}_A = \eta^g_{fA}\circ\eta^f_A\colon FA \to HgfA\,, \qquad A \text{ object of } \mathcal{C}\,.$$

5.4. We have a forgetful functor « domain » Dom: $\mathfrak{SREG}\!\int\!\mathbf{CnSLt} \to$ $\to \mathfrak{SREG}$ setting $\mathrm{Dom}\,(f, \eta^f) = f$; we intend to find a left adjoint to Dom.

5.5. Thus we define Sub: $\mathfrak{SREG} \to \mathfrak{SREG}\!\int\!\mathbf{CnSLt}$ taking a subregular functor $f\colon \mathcal{C} \to \mathcal{D}$ to the triangle

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\mathrm{Sub}} & \mathbf{CnSLt} \\ {\scriptstyle f}\downarrow & \Updownarrow{\scriptstyle \mathrm{Sub}^f} \nearrow{\scriptstyle \mathrm{Sub}} & \\ \mathcal{D} & & \end{array}$$

In fact it is a consequence of 2.4 and 2.7.

5.6. Proposition: Dom Sub $= 1$ *and* Sub *is left adjoint to* Dom.

Proof: We take the counit to be $\varepsilon_{(\mathcal{C},F)} = (1_{\mathcal{C}}, \widetilde{\mathrm{Sub}^F})\colon (\mathcal{C}, \mathrm{Sub}) \to (\mathcal{C}, F)$ where $\widetilde{\mathrm{Sub}^F}$ is obtained composing $\mathrm{Sub}^F$ with the iso $\mathrm{Sub}\, F \simeq F$ (2.3); the naturality of $\varepsilon$ is exactly the commutativity of 5.3 (S).

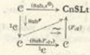
5.7. Last, consider the category $\mathfrak{REG}\int\mathbf{CnSLt}$ whose *objects* are vertically subregular and horizontally monoidal (4.5) functors of codomain **CnSLt** and whose *morphisms* are « triangles » (5.3) $(f, \eta')$ with $f$ in $\mathfrak{REG}$. *Composition* as in $\mathfrak{SREG}\int\mathbf{CnSLt}$.

5.8. Proposition: *There are functors*

$$\mathfrak{REG} \underset{\text{Dom}}{\overset{\text{Sub}}{\rightleftarrows}} \mathfrak{REG}\int\mathbf{CnSLt}$$

*such that* Dom Sub = 1 *and* Sub *is left adjoint to* Dom.

Proof: By 4.6 and 5.5 Sub is defined; following 5.6 the counit $\varepsilon_{(\mathcal{C},F)}$ is the composite




where the horizontal transformation $\varrho_*$ is defined as the composite

$$\mathrm{Sub}\,FA \times \mathrm{Sub}\,FB \xrightarrow{\ \varrho^*_{FA,FB}\ } \mathrm{Sub}\,(FA \times FB) \xrightarrow{\ \varrho_{AB}\ } \mathrm{Sub}\,(F(A \otimes B))$$

for each $A$, $B$ in $\mathcal{C}$.

5.9. A similar global presentation can be given for the results of n. 3.

REFERENCES

[1] S. Eilenberg - G. M. Kelly, *Closed categories*, Proc. La Jolla Conf. on Cat. Alg., Springer (1966), 421-562.

[2] P. Freyd, *Aspects of Topoi*, Bull. Austral. Math. Soc., 7 (1972), 1-76, 467-480.

[3] M. Grandis, *Transfer functors and projective spaces*, Math. Nachr., 118 (1984), 147-165.

[4] M. Grandis, *Concrete representations for inverse and distributive exact categories*, Rendiconti Accademia Nazionale delle Scienze detta dei XL, Memorie di Matematica, 102 (1984), Vol. VIII, fasc. 6, 99-120.

[5] A. Joyal - M. Tierney, *An extension of the Galois theory of Grothendieck* (preprint).

[6] F. W. Lawvere, *Adjointness in foundations*, Dialectica, Vol. 23, No. 3/4 (1969), 281-296.

[7] A. M. Pitts, *Amalgamation and interpolation in the category of Heyting algebras*, J. Pure Appl. Algebra 29 (1983), 155-165.

[8] G. E. Reyes, *From sheaves to logic*, Studies in Algebraic Logic, M.A.A. Studies in Math., Vol. 9, Math. Assoc. of America, (1974), 143-204.

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