Rendiconti
Accademia Nazionale delle Scienze detta dei XL
Memorie di Matematica
102° (1984), Vol. VIII, fasc. 5, pagg. 79-98

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On the Bilateral Boundary Value Problem and the Existence of Global Gevrey Solutions of Linear Differential Equations (**)

Sul problema al bordo bilatero e l'esistenza di soluzioni Gevrey per equazioni differenziali lineari

Sunto. — Per operatori differenziali lineari a coefficienti costanti si considera il problema di Cauchy nelle classi di Gevrey Γ^d, $d>1$, supponendo che la forma caratteristica sia localmente iperbolica (in particolare, reale e di tipo principale). Microlocalizzando opportunamente il problema, si danno teoremi di esistenza ed unicità per le sue soluzioni. Se ne deduce che per tali operatori, $P(D)$, risulta $P(D)\Gamma^d(\mathbb{R}^n)=\Gamma^d(\mathbb{R}^n)$ per $1<d<\mu/(\mu-1)$ dove μ è la massima molteplicità delle caratteristiche (di modo che al numero d non è imposta alcuna limitazione superiore nel caso di caratteristiche semplici).

Introduction

We deal with the non-characteristic Cauchy problem on the Gevrey classes Γ^d, $d>1$ for a partial differential operator $P=P(D)$ in $\mathbb{R}^n$ with constant coefficients. Let S^{n-1} be the $(n-1)$-dimensional real sphere and, with $\nu\in S^{n-1}$ non-characteristic, let us denote by ϱ the projection along the meridians with poles $\pm\nu$. For a point ξ' in the equator S^{n-2}, first we assume that P is micro-hyperbolic to ν at any point of the fibre $\varrho^{-1}(\xi')$ in the sense of (2.1) and denote by μ the largest vanishing order in $\varrho^{-1}(\xi')$ of the principal symbol. In such hypothesis we prove that the Cauchy problem on the hyperplane $N=\{x\cdot\nu=0\}$ has solution in the Gevrey classes Γ^d, $1<d<\mu/(\mu-1)$ modulo solutions of a Plemelj problem which are microanalytic in $(\mathbb{R}^n\setminus N)\times\varrho^{-1}(\xi')$; and we state the microlocal uniqueness of the solution too. Then we strenghten the result and prove the microanalyticity in $\mathbb{R}^n\times(S^{n-1}\setminus\varrho^{-1}(\Gamma))$, for an open set $\Gamma\ni\xi'$

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(**) Memoria presentata il 3 Novembre 1983 da Giuseppe Scorza Dragoni, uno dei XL.

ISSN-0392-4106

in the equator S^{n-2}, and the uniqueness modulo $\mathscr{A}$ of the solution in case we correspondingly require the microanalyticity in $N\times(S^{n-2}\setminus\Gamma')(\Gamma'\subset\subset\Gamma)$ for the data.

Second we treat operators which are microhyperbolic to ν at a μ-characteristic point $\xi\in S^{n-1}$ and consider their Weierstrass decomposition $P=H\cdot E$ in $\Delta\ni\xi$, H still being microhyperbolic of order μ and E invertible. We prove the existence, and the microlocal uniqueness in $\mathbf{R}^n\times\{\xi\}$, of a solution $u\in\Gamma^d$, $1<d<\mu/(\mu-1)$, of $P(D)u=0$, the traces of $E(D)u$ up to the order $\mu-1$ microlocally coinciding in $N\times\{\varrho(\xi)\}$ with prescribed Γ^d functions. When the Cauchy data vanish (as microfunctions) outside $N\times\Gamma''$, $\Gamma''\subset\subset\varrho(\Delta)$, we correspondingly obtain the vanishing of the solution outside $\mathbf{R}^n\times\Delta'$ for suitable $\Delta'\subset\Delta\cap\varrho^{-1}(\Gamma'')$; this gives the unicity modulo $\mathscr{A}$ of the solution in such case.

As an application we deduce

if P is microhyperbolic at all characteristics with vanishing order $<\mu$ then

$$P(D)\Gamma^d(\mathbf{R}^n)=\Gamma^d(\mathbf{R}^n)\qquad \textit{for } 1<d<\frac{\mu}{\mu-1}$$

(and for any $d>1$ if P is in addition weaker than its principal part).

(When $d=1$ the result is classical [1].) In fact, given the equation $Pu=f$, $f\in\Gamma^d$ we first find solutions on compact sets of $\mathbf{R}^n$ and decompose them into terms which are microanalytic outside the elements Δ of a suitable covering of S^{n-1}. By adjusting those terms with microlocal solutions of Cauchy problems we make their boundary values agree. Due to the above uniqueness, they agree mod $\mathscr{A}$ on compact sets of $\mathbf{R}^n$ so that they define a global solution in $\Gamma^d/\mathscr{A}$ of the equation.

Essential in the whole exposition are: the theory of boundary values of hyperfunction solutions of P.D.E. [6], [8]; theorems on propagation of singularity at the boundary [4], [5]; the theory of Fourier hyperfunctions [3], [7]. Results on existence of Γ^d solutions are also present in [2] (especially for real simply-characteristic symbols and in case of d rational) and in [9].

I would like to express my sincere gratitude to Professor A. Kaneko for his invaluable advice during our discussions at Tokyo University.

1. - The case of real principal symbol

Let $P=P(D)$ be a differential operator of order m with constant coefficients (or even analytic in a domain $V\subset\mathbf{R}^n$), let $N\subset V$ be a real analytic hypersurface non-characteristic with respect to P, consider a normal system $\{B_j\}_{j=1,\dots,m}$ of boundary operators (e.g. $B_j(D)=(-i(\partial/\partial n))^{j-1}$), and let $\{C_j\}_{j=1,\dots,m}$ be its dual system defined by formula (32) of [8].

Let f be a hyperfunction on the positive part V_+ of $V\setminus N$, solution of

$Pf = 0$. Then for any extension $\tilde{f}$ to V vanishing on the negative side V_-, we can write in a unique way

$$P\tilde{f} = Pg + \sum_{j=1}^{m} C_j(f_j \otimes \delta_N) \tag{1.1}$$

with $g \in \Gamma_N(V, \mathscr{B})$, $f_j \in \Gamma(N, '\mathscr{B})$ (where $\mathscr{B}$ and $'\mathscr{B}$ denote the sheaves of hyperfunctions in V and N respectively). As shown in [8] this is nothing else but the dual version of the Cauchy-Kowalevsky theorem.

Following [8] we will define f_j to be the boundary values of f and write

$$f_j = B_j f|_{N_+} . \tag{1.2}$$

Besides we will denote by $[f]_+$ the unique extension with support in $\bar{V}_+$ for which $g = 0$ in (1.1) and call it the canonical extension of f. Note that if f is extensible as a section of the kernel sheaf $\mathscr{B}_P$ ($=$ solutions $f \in \mathscr{B}$ of $Pf = 0$) over a neighbourhood of N in V, then the product of f by the characteristic function θ_+ of V_+ makes sense and moreover we obtain by Green's formulas

$$[f]_+ = \theta_+ f, \qquad B_j f|_{N_+} = B_j f|_N . \tag{1.3}$$

Similarly we can define the canonical extension $[f]_-$ and the boundary values $B_j f|_{N_-}$ for solutions in V_-; we only need to put a factor -1 in the right hand side of (1.1).

(In the same way we can define the canonical extensions $[f]_\pm \in \mathscr{D}'(V)$ and the boundary values $B_j f|_{N_\pm} \in \mathscr{D}'(N)$ for distribution solutions in $V_\pm$; in such case we need to assume that f is extensible as a distribution; and if it is extensible as a distribution solution we regain (1.3).)

For $\nu = (0, \ldots, 1)$, let ϱ be the projection of S^{n-1} with poles $\pm\nu$ and let Γ be an open convex set in the equator S^{n-2}; denote by $\zeta = (\zeta', \zeta_n)$ the coordinates in $\mathbf{C}^n$ and by $\xi = (\xi', \xi_n)$ those in $\mathbf{R}^n$. Assume, henceforth, that P has constant coefficients and suppose that the characteristic form $P_m(\zeta)$, $\zeta \in \mathbf{C}^n$ verifies:

$$P_m(\nu) \neq 0; \; P_m(\xi', \zeta_n) = 0 , \; \frac{\xi'}{|\xi'|} \in \Gamma' , \; \Gamma' \subset\subset \Gamma \quad \text{implies either} \tag{1.4}$$

$$\operatorname{Im}\zeta_n = 0 \text{ or } |\operatorname{Im}\zeta_n| > C_{\Gamma'}|\xi'| .$$

It is easy to recognize that (1.4) is equivalent to assume that P_m is locally hyperbolic at any $\xi \in \varrho^{-1}(\Gamma) \cup \{\pm\nu\}$ to the $\pm\nu$-direction (in the sense of subsequent condition (2.1)). And therefore if P_m has real coefficients, simple characteristics, and verifies

$$\operatorname{grad} P_m(\xi) \cdot \nu \neq 0 \quad \text{if } \xi \in \varrho^{-1}(\Gamma) \cup \{\pm\nu\} \text{ and } P_m(\xi) = 0 ,$$

then (1.4) is satisfied.

One can also restate (1.4) in an equivalent condition involving the zeros of P

$$(1.5)\qquad P_m(\nu)\neq 0\,;\ P(\xi',\zeta_n)=0\,,\ \frac{\xi'}{|\xi'|}\in\Gamma'\,,\ \Gamma'\subset\subset\Gamma\,,\qquad \text{implies either}$$

$$|\operatorname{Im}\zeta_n|<C'_{\Gamma'}|\xi'|^{(\mu-1)/\mu}\ \text{or}\ |\operatorname{Im}\zeta_n|>C_{\Gamma'}|\xi'|$$

where μ is the largest multiplicity of the characteristics in $\varrho^{-1}(\Gamma')$.

Last we recall that conditions (1.4), (1.5) are also equivalent to the existence of micro-local fundamental solutions $E_\pm$ in $\mathbf{R}^n\times(\varrho^{-1}(\Gamma')\cup\{\pm\nu\})$, $\forall\Gamma'\subset\subset\Gamma$ (in the sense of Section 2), with sing supp $E_\pm$ contained in proper cones of the half spaces $\pm x\cdot\nu>0$ (see [10]).

In the previous hypotheses we can solve the Cauchy problem in the Gevrey classes modulo microanalytic solutions of the Plemelj problem; and also prove the microlocal uniqueness of the solution

Theorem 1.1. *Assume that the characteristic form $P_m(\zeta)$ fo $P(D)$ verifies (1.4), denote by μ the largest multiplicity of the characteristics in $\varrho^{-1}(\Gamma)$; let N denote the hyperplane $x_n=0$, and take a system of Cauchy data $(u_j)_{j=0,\dots,m-1}$, in $\Gamma_0^d(N)$ (= Gevrey functions with compact support), $1<d<\mu/(\mu-1)$. Then there exists $w\in\mathscr{B}_F(\mathbf{R}^n\setminus N)\cap\mathscr{A}^*((\mathbf{R}^n\setminus N)\times\varrho^{-1}(\Gamma''))$, $\Gamma''\subset\subset\Gamma$, such that the problem*

$$(1.6)\qquad P(D)u=0\,;\quad D_n^j u|_N-(D_n^j w|_{N_+}-D_n^j w|_{N_-}+u_j)=0\,,\quad 0<j<m-1\,,$$

has a solution $u\in\Gamma^d(\mathbf{R}^n)$.

Such w is unique modulo $\mathscr{A}_F^*(\mathbf{R}^n\times\varrho^{-1}(\Gamma''))|_{x_n>0}\oplus\mathscr{A}_F^*(\mathbf{R}^n\times\varrho^{-1}(\Gamma''))|_{x_n<0}$ and correspondingly u is unique modulo $\mathscr{A}_F^*(\mathbf{R}^n\times\varrho^{-1}(\Gamma''))$. ($\mathscr{A}^*(\Omega\times\Delta)$), $\Omega\times\Delta\subset\mathbf{R}^n\times S^{n-1}$ denotes the set of $f\in\mathscr{B}(\Omega)$ which are microanalytic in $\Omega\times\Delta$.

Proof. First let us recall that in view of (1.5), we can find $F_k^h(\xi',x_n)$, $h=0,1,2$, $k=0,\dots,m-1$, for $\xi'/|\xi'|\in\Gamma''$, ($\Gamma''\subset\subset\Gamma$), $|\xi'|>c$, (c large), and for $h(-1)^h x_n<0$, solutions of the problem

$$(1.7)\qquad P(\xi',D_n)F_k^h(\xi',x_n)=0\,;\qquad \sum_k D_n^j F_k^h(\xi',x_n)|_N=\delta_{jk}$$

and with the estimates

$$(1.8)\qquad |D_n^j F_0^k(\xi',x_n)|<c^{j+1}|\xi'|^{m+j-k}\exp\left[c'|\xi'|^{(\mu-1)/\mu}|x_n|\right]\,,$$

$$(1.9)\qquad |D_n^j F_k^h(\xi',x_n)|<c^{j+1}|\xi'|^{m+j-k}\exp\left[-c'|\xi'||x_n|\right]\ \text{if}\ (-1)^h x_n<0\,,\ h=1,2$$

(see [9]).

Similarly for $\xi'/|\xi'| \notin \Gamma''$, $|\xi'| > c$, we can find F_2^k solutions of

$$(1.10) \qquad P(\xi', D_n) F_2^k(\xi', x_n) = 0 ; \qquad D_n^j F_2^k(\xi', x_n)|_N = \delta_{jk} ,$$

satisfying the estimates (1.8) with $|\xi'|$ instead of $|\xi'|^{(n-1)/n}$ in the exponent.

Let us decompose the singularity of the Cauchy data u_j by setting

$$u_j(x') = u_j^1(x') + u_j^2(x') =$$
$$= u_j(x') \underset{x'}{*} \exp[-x'^2] W_0(x', \Gamma') + u_j(x') \underset{x'}{*} \exp[-x'^2] W_0(x', S^{n-1} \setminus \Gamma') ,$$

$\Gamma' \subset\subset \Gamma'' \subset\subset \Gamma$, where

$$W_0(x', \Gamma') = \int_{\Gamma'} W_0(x', \omega') \, d\omega'$$

for the plane wave component $W_0(x', \omega')$ of $\delta' = \delta_N$ (see [3]).

Concerning u_j^1 we claim that the Fourier transform $\mathcal{F}(u_j^1)(\zeta')$ is an entire function which satisfies

$$(1.11) \qquad \mathcal{F}(u_j^1)(\xi') = O(\exp[-c|\xi'|^{1/d}]) \text{ when } \xi' \in \mathbf{R}^{n-1} \text{ and } \mathcal{F}(u_j^1)(\xi') = O\cdot$$
$$\cdot(\exp[-\delta|\xi'|]), \ \forall \delta > 0, \text{ when } \xi' \in \mathbf{R}^{n-1} \setminus \Gamma'^{+}, \ (\Gamma'^{+} = \mathbf{R}^{+}\Gamma') .$$

Remembering the formula

$$\mathcal{F}(u_j * \exp[-x'^2] W_0(x', \Gamma')) = \mathcal{F}(u_j) \mathcal{F}(\exp[-x'^2] W_0(x', \Gamma')) =$$
$$= \mathcal{F}(u_j)\big(\mathcal{F}(\exp[-x'^2]) * \mathcal{F} W_0(x', \Gamma')\big) ,$$

then (1.11) ensues from the following remarks

(a) $\mathcal{F}(u_j)(\zeta')$ is entire and $\mathcal{F}(u_j)(\xi') = O(\exp[-c|\xi'|^{1/d}])$, $\xi' \in \mathbf{R}^{n-1}$,

(b) $\mathcal{F}(\exp[-x'^2])(\zeta')$ is entire rapidly decreasing of any exponential order in $\mathbf{R}^{n-1}$,

(c) $\mathcal{F}(W_0(x', \Gamma'))(\xi')$ coincides with the characteristic function $\theta_{\Gamma'^+}$ of Γ'^{+}.

For (c) observe that if $\Delta \subset S^{n-2}$ and if its dual cone Δ^* coincides with Γ' then $\operatorname{Im} \zeta' \cdot x' > 0$ when $x' = \omega' r$ ($\omega' \in \Gamma'$, $r \in \mathbf{R}^+$), and $\zeta' \in \mathbf{R}^n + i\Delta$ and therefore we have in the sense of hyperfunctions:

$$(2\pi)^{-(n-1)} \int \exp[ix' \cdot \zeta'] \theta_{\Gamma'^+}(x') \, dx' |_{\zeta' \mapsto \xi' + i0} =$$
$$= (2\pi)^{-(n-1)} \int_{\Gamma'} d\omega' \int_0^{+\infty} dr \exp[ir\omega' \cdot \zeta'] r^{n-2} |_{\zeta' \mapsto \xi' + i0} =$$
$$= (n-2)! (-2\pi i)^{-(n-1)} \int_{\Gamma'} \frac{1}{(\zeta' \cdot \omega')^{n-1}} d\omega' |_{\zeta' \mapsto \xi' + i0} = W_0(\zeta', \Gamma')|_{\zeta' \mapsto \xi' + i0} .$$

Using (1.11) and setting $\mathscr{F}(u_j^1) = \hat{u}_j^1$ in the sequel, let us define

$$(1.12) \qquad v_h(x', x_n) = (-1)^h (2\pi)^{-(n-1)} \sum_{k=0}^{m-1} \int_{\Gamma_\varepsilon^{\circ +}} \exp[ix'\cdot\xi'] \hat{u}_k^1(\xi') F_h^k(\xi', x_n)\, d\xi' ,$$

$$0<h<2 , \ (-1)^h h x_n < 0 , \quad (\Gamma_\varepsilon^{\circ +} = \Gamma^{\circ +} \cap \{|\xi'| > \varepsilon\}),$$

$$(1.13) \qquad v_3(x', x_n) = -(2\pi)^{-(n-1)} \sum_{k=0}^{m-1} \int_{(\mathbf{R}^n \setminus \Gamma^{\circ +})_\varepsilon} \exp[ix'\cdot\xi'] \hat{u}_k^1(\xi') F_3^k(\xi', x_n)\, d\xi' .$$

Obviously the integrals in (1.12) converge absolutely for $(-1)^h h x_n < 0$ and define three sections of Γ^d (of $\mathscr{A}$ for $h=1,2$) over the interior of such regions; this follows immediatly from (1.8), (1.9), (1.11) and from the assumption $1/d > (\mu-1)/\mu$. Concerning v_3, observe that for $|x_n| < \varepsilon'$ we have in view of (1.11) and the analogous of (1.8) for F_3^k

$$|D^\alpha v_3| < c^{|\alpha|+1} \int_{(\mathbf{R}^n \setminus \Gamma^{\circ +})_\varepsilon} |\xi'|^{m+j+|\alpha|} \exp[\varepsilon'|\xi'|] c_{\varepsilon'} \exp[-2\varepsilon'|\xi'|]\, d\xi' < c^{m|\alpha|+1} |\alpha| ! ,$$

which shows that v_3 is analytic. It is clear that $Pv_h = 0$, $\forall h$, and moreover, in view of (1.7), (1.10)

$$D_n^j v_0|_N - (D_n^j v_1|_{N_+} - D_n^j v_2|_{N_-} + D_n^j v_3|_N) = (2\pi)^{-(n-1)} \int_{\{|\xi'|>\varepsilon\}} \exp[ix'\cdot\xi'] \hat{u}_j^1(\xi')\, d\xi' = u_j^1 + f_j ,$$

where f_j are entire functions.

Last by taking an entire solution v_4 of the problem

$$Pv_4 = 0 ; \qquad D_n^j v_4|_N = f_j , \qquad j = 0, \ldots, m-1 ,$$

which is given by the Cauchy-Kowalevsky theorem, and by setting

$$u = v_0 , \qquad w' = [v_1]_+ + [v_3]_+ + [v_4]_+ + [v_2]_- ,$$

we solve the problem (1.6) for data $u_j = u_j^1$.

It remains to find a solution $v_5 \in \mathscr{A}_P^*((\mathbf{R}^n \setminus N) \times \varrho^{-1}(\Gamma'))$ of the Plemelj problem

$$(1.14) \qquad D_n^j v_5|_{N_+} - Dv_5|_{N_-} = -u_j^2 ,$$

since then by putting $w = w' + [v_5]_+ + [v_5]_-$, we obtain the desired solution of (1.6). To this end consider the regular fundamental solution $E(x)$ defined by Hörmander; it belongs to $\mathscr{D}'$ and also to $\mathscr{Q}^\delta$ for suitable δ (= Fourier hyperfunctions of exponential growth with type δ). By putting $E_k = D_n^k E$, we then have

$$(1.15) \qquad D_n^j E_k|_{N_+} - D_n^j E_k|_{N_-} = \begin{cases} \dfrac{i}{P_m(\nu)} \delta' & \text{for } j = m-k-1 , \\ 0 & \text{for } j \neq m-k-1 . \end{cases}$$

To prove (1.15) first we note that

$$(1.16)\qquad P([E]_+ + [E]_-) = \sum_{j=1}^{m} C_j^j (D_n^{j-1} E|_{N_+} - D_n^{j-1} E|_{N_-} \otimes \delta_n)\,,$$

$$(1.17)\qquad P(E) - \delta = -iP_m(\nu)\left(\frac{i}{P_m(\nu)}\,\delta' \otimes \delta_n\right)$$

where $C_m^j(D)$ coincides with the constant $-iP_m(\nu)$.

Making the difference of these equalities we obtain for $P(F)$, $\big(F = E - ([E]_+ + [E]_-)\big)$, an expression of type $\sum_{j=0}^{m-1} f_j \otimes D_n^j \delta_n$, $f_j \in \mathscr{D}'(N)$. On the other hand we know that F can be written

$$F = \sum f_j' \otimes D_n^j \delta_n \qquad \text{(since supp } F \subset N\text{)}\,;$$

this gives $F = 0$ and thus (1.15) follows.

We recall now that $u_j^2 \in {}'\mathscr{Q}^{-\delta'}$, $\forall \delta' > 0$, for both terms in the convolution defining u_j^2 belong to ${}'\mathscr{Q}^{-\delta}$; moreover by the rule of S.S. for the convolution

$$(1.18)\qquad \text{S.S. } u_j^2|_{\mathbf{D}^{n-1} \times \Gamma^\circ} = 0\,,$$

where $\mathbf{D}^{n-1} = N \cup S^{n-2}\infty$ is the base space for the sheaf ${}'\mathscr{Q}^{-\delta}$ and S.S. denotes the singular spectrum in the sense of [3]. Since each E_k belongs to $\mathscr{Q}^{\delta}$, $\delta < \delta'$, then we can set

$$(1.19)\qquad v_3 = iP_m(\nu) \sum_{k=0}^{m-1} E_k * (u_{m-k-1}^2 \otimes \delta_n)$$

where all convolutions make sense for the same argument as above.

It is clear that $Pv_3 = 0$ in $\mathbf{R}^n \setminus N$ and that (1.14) holds due to (1.15). Last note that S.S. $u_k^2 \otimes \delta_n \cap \mathbf{D}^n \times \varrho^{-1}(\Gamma^\circ) = \emptyset$, and so by the rule of S.S. quoted above we conclude that v_3 is microanalytic to $\varrho^{-1}(\Gamma^\circ)$ (even in the points at ∞).

If now all data u_j vanish, then setting $b = [u]_+ - ([w]_+ + [w]_-)$, we have $Pb = 0$ in $\mathbf{R}^n \setminus N$ in addition to $D_n^j b|_{N_+} = D_n^j b|_{N_-}$, so that b is extended to a solution in the whole $\mathbf{R}^n$. Since b is microanalytic to $\varrho^{-1}(\Gamma^\circ)$ in $\{x_n < 0\}$ then it is in the whole $\mathbf{R}^n$ due to the propagation of microregularity related to the existence of «good» microlocal fundamental solution quoted above; thus $-b$ provides the desired extension of $w|_{x_n<0}$. All other uniqueness statements are obtained by the same technique. The proof is complete.

If we consider the problem (1.6) for $u_j = u_j^1$, then a solution is given by $u = v_3$ and $w = w'$ defined in the course of the preceding proof. Naturally w is analytic in $\mathbf{R}^n \setminus N$ in such case. In the following we will prove that u and w are microanalytic to $\varrho^{-1}(S^{n-2} \setminus \Gamma)$ even at the boundary N; this will strengthen very much the uniqueness conclusions.

THEOREM 1.2. *In the hypotheses of Theorem 1.1 there exists $w \in \mathscr{A}^*(\mathbf{R}^n \times \varrho^{-1} \cdot (S^{n-2} \setminus \Gamma'))$, analytic solution of $Pw = 0$ in $\mathbf{R}^n \setminus N$ such that the problem (1.6) for $u_j^1 = u_j * \exp[-x'^2] W_0(x', \Gamma')$, has a solution $u \in \Gamma^a(\mathbf{R}^n) \cap \mathscr{A}^*(\mathbf{R}^n \times \varrho^{-1}(S^{n-2} \setminus \Gamma'))$. Such w is unique modulo $\mathscr{A}_P(\mathbf{R}^n)|_{x_n>0} \oplus \mathscr{A}_P(\mathbf{R}^n)|_{x_n<0}$, and consequently u is unique modulo $\mathscr{A}_P(\mathbf{R}^n)$.*

PROOF. First observe that $F_0^k(\xi', \zeta_n)$, $\zeta_n = x_n + iy_n$, are entire functions of ζ_n with the estimates

$$(1.20) \qquad |D_n^j F_0^k(\xi', \zeta)| < c^{j+1} |\xi'|^{m+j-k} \exp\left[c'(|x_n||\xi'|^{(a-1)/a} + |y_n||\xi'|)\right].$$

With an intermediate Γ'' ($\Gamma' \subset\subset \Gamma'' \subset\subset \Gamma$) set $\Sigma^* = \Gamma''$. If $y' \in \Sigma'^+$, $\Sigma' \subset\subset \Sigma$, we have $y' \cdot \xi' > \delta |y'||\xi'|$, $\forall \xi' \in \Gamma''^+$, and so the integral defining v_0 in (1.12) converges even after letting

$$x \mapsto x + iy \in \mathbf{R}^n + i\left\{(y', y_n) : y' \in \Sigma'^+, |y_n| < \frac{\delta}{c'} |y'|\right\}.$$

Therefore S.S. $v_0 \subset \mathbf{R}^n \times \varrho^{-1}(\Gamma'')$.

On the other hand $F_h^k(\xi', x_n)$, $(-1)^h x_n < 0$, $h = 1, 2$, accept analytic continuation to $(-1)^h x_n > 0$ with the estimates

$$(1.21) \qquad |D_n^j F_h^k(\xi', \zeta_n)| < c^{j+1} |\xi'|^{m+j-k} \exp\left[c'(|x_n| + |y_n|)|\xi'|\right].$$

Because of the term $|x_n||\xi'|$ in the esponent, we could not extend v_h as hyperfunction solutions by means of definition (1.12). Nevertheless the traces $D_n^j v_1|_{N_+}$, $D_n^j v_2|_{N_-}$ can be calculated by (1.12) and so they are microanalytic to $S^{n-2} \setminus \Gamma''$ by the above argument. A result of Kashiwara [4] permits to conclude that both v_1, v_2 have extensions (actually the canonical extensions), whose singular spectrum does not intersect $N \times \varrho^{-1}(S^{n-2} \setminus \Gamma'')$; (for analytic traces this would ensue from Holmgren's theorem). To this end, first remember from the definition that

$$P([v_1]_+) = \sum_j C_j (D_n^j v_1|_{N_+} \otimes \delta_n),$$

and so if $\xi' \notin \Gamma''$ then we have $P([v_1]_+) = 0$ in $N \times \varrho^{-1}(\xi')$ (in the sense of microfunctions). Since P is invertible as microdifferential operator in $\mathbf{R}^n \times I$ for some neighbourhood $I \ni v$, then we conclude

$$(1.22) \qquad [v_1]_+ = 0 \quad \text{in } N \times (\varrho^{-1}(\xi') \cap I), \ \xi' \notin \Gamma''.$$

On the other hand we know from [4] that if $(\xi', \xi_n) \in \operatorname{supp}[v_1]_+|_x$, $x \in N$, then $\varrho^{-1}(\xi') \subset \operatorname{supp}[v_1]_+|_x$. Therefore if $\xi' \notin \Gamma''$ we obtain

$$(N \times \varrho^{-1}(\xi')) \cap \operatorname{supp}[v_1]_+ = \emptyset$$

due to (1.22). Since we could similarly handle $[v_2]_-$ and since $w = w' = [v_1]_+ + [v_2]_- + [v_3]_+ + [v_4]_+$ with v_3 and v_4 analytic in $\mathbf{R}^n$, we conclude $w \in \mathscr{A}^*(\mathbf{R}^n \times \varrho^{-1}(S^{n-2} \setminus \Gamma'))$.

When all Cauchy data vanish we know from the propagation of the microanalyticity to $\varrho^{-1}(\Gamma')$ that the hyperfunction $b = -[u]_+ + [w]_+ + [w]_-$ provides an extension of $w|_{x_n<0}$ belonging to $\mathscr{A}^*_P(\mathbf{R}^n \times \varrho^{-1}(\Gamma'))$. Thus by (1.3), $[w]_- = \theta_- w$ and similarly $[w]_+ = \theta_+ w$, $[u]_+ = \theta_+ u$; this gives

$$b \in \mathscr{A}^*(\mathbf{R}^n \times \varrho^{-1}(S^{n-2} \setminus \Gamma')), \ \left(\text{and so } b \in \mathscr{A}^*(\mathbf{R}^n \times \varrho^{-1}(S^{n-2}))\right).$$

for $u, w \in \mathscr{A}^*(\mathbf{R}^n \times \varrho^{-1}(S^{n-2} \setminus \Gamma'))$ by hypothesis. At last by Sato's theorem b is analytic and so all uniqueness statements follow.

Remark. Suppose that u_j are elements of $\mathscr{Q}(\mathbf{D}^{n-1})$ whose Fourier transforms verify

(1.23) $\quad \hat{u}_j \in L^1_{loc}$,

(1.24) $\quad \hat{u}_j(\xi') = O(\exp[-c|\xi'|^{1/d}])$, $\qquad \xi' \in \mathbf{R}^{n-1}$,

(1.25) $\quad$ for some $\Gamma'' \subset\subset \Gamma'$ and for suitable ε, $\hat{u}_j(\xi') = O(\exp[-\varepsilon|\xi'|])$,

$$\xi' \in \mathbf{R}^{n-1} \setminus \Gamma''^{*}.$$

(In particular we can take u_j in the form $u_j = \varphi_j * W_0(\cdot, \Gamma'')$ with $\varphi_j \in \Gamma^d_0$; in such case $\hat{u}_j = \hat{\varphi}_j \cdot \widehat{W_0(\cdot, \Gamma'')}$ where the first factor is entire and verifies (1.24) whereas the second coincides with $\theta_{\Gamma''*}$.)

Put $u = v_0$, $w = [v_1]_+ + [v_2]_- + [v_3]_+ + [v_4]_+$ where in the integrals defining the v_k we can let $\Gamma'' = \Gamma'$ and note that in the present situation the integral (1.13) is convergent only under the condition $|x_n| < \varepsilon'$, $\varepsilon' = \varepsilon/c'$ and consequently v_2 is defined and analytic only for $|x_n| < \varepsilon'$. Thus we obtain w in $\mathscr{A}_P(\{0 < |x_n| < \varepsilon'\})$, u in $\Gamma^d_P(\mathbf{R}^n)$, both microanalytic to $\varrho^{-1}(S^{n-2} \setminus \Gamma')$ even in N, and solutions of (1.6); in such situation w is unique modulo

$$\mathscr{A}(\{|x_n| < \varepsilon'\})|_{x_n>0} \oplus \mathscr{A}_P(\{|x_n| < \varepsilon'\})|_{x_n<0}$$

whereas u is still unique mod $\mathscr{A}_P(\mathbf{R}^n)$.

At last notice that the boundary values of u still belong to $\mathscr{Q}$ and verify (1.23), (1.24), and (1.25) for any $\varepsilon > 0$.

Now we show how important role the preceding results play in the existence of global Gevrey solutions.

Theorem 1.3. *Let P have characteristic form with real coefficients and simple characteristics. Then*

$$P(D)\Gamma^d(\mathbb{R}^n) = \Gamma^d(\mathbb{R}^n) \qquad \forall d > 1 .$$

Proof. By the remarks below (1.4), we can find a family $v_j \in S^{n-1}$, $P_m(v_j) \neq 0$, and a family $\Gamma_j \subset S^{n-1} \cap \{\xi \cdot v_j = 0\}$ such that $\bigcup_j \varrho_j^{-1}(\Gamma_j)$ covers S^{n-1} (ϱ_j being the projections from $\pm v_j$ along the meridians) and such that (1.5) is satisfied $\forall j$. Leaving out the case $d = 1$ which is classical, fix $f \in \Gamma^d$ and take $g_r \in \Gamma_0^d$ solutions of $Pg = f$ in $S_r = \{|x| < r\}$. By taking a closed covering $\bigcup_j \bar\Delta_j = S^{n-1}$ with $\Delta_j \cap \Delta_{j'} = 0$, $\Delta_j \subset\subset \varrho_j^{-1}(\Gamma_j)$, let us decompose

$$g_r = \sum_j g_r^j = \sum_j g_r * \exp[-x^2]\, W(x, \bar\Delta_j), \tag{1.26}$$

where W is the curvilinear wave component of $\delta(x)$ (see [3]). Recall here that the convolution by $W(\cdot, \bar\Delta_j)$ does not give any propagation of singularity due to the estimate S.S. $W(\cdot, \bar\Delta_j) \subset \{0\} \times \bar\Delta_j$. Fix j, let $v_j = (0, \ldots, 1)$, $\Gamma_j \subset \{\xi_n = 0\}$ and pick an intermediate $\Gamma_j' \subset\subset \Gamma_j$, $\Delta_j \subset\subset \varrho_j^{-1}(\Gamma_j')$. Before embarking in the use of the Remark let us check that all hypotheses are fulfilled by the boundary values of the g_r^j.

Lemma 1.4. *$D_n^k g_r^j|_{\mathbb{R}}$ verify* (1.23), (1.24), (1.25).

Proof. Observing that

$$\begin{aligned} {}'\mathcal{F}(D_n^k g_r^j|_{\mathbb{R}}) &= {}'\mathcal{F}(D_n^k g_r * \exp[-x^2]\, W(x, \bar\Delta_j)|_{\mathbb{R}}) = \\ &= {}'\mathcal{F}(D_n^k g_r) \underset{x_n}{*} {}'\mathcal{F}(\exp[-x^2]\, W(x, \bar\Delta_j))|_{\mathbb{R}} , \end{aligned} \tag{1.27}$$

then we gain the two last properties provided that we prove

$$ {}'\mathcal{F}(D_n^k g_r)(\xi', x_n) = O(\exp[-\varepsilon|\xi'|^{1/d}]) \quad \text{uniformly in } x_n, \text{ and } = 0 \text{ for large } x_n , \tag{1.28}$$

$$\left| \begin{aligned} &{}'\mathcal{F}(\exp[-x^2]\, W(x, \bar\Delta_j))(\xi', x_n) = O(\exp[-\varepsilon|\xi'|]) \\ &\qquad\qquad \text{for } \xi' \in \mathbb{R}^{n-1} \setminus \Gamma_j'^{+} \text{ uniformly in } x_n , \\ &{}'\mathcal{F}(\exp[-x^2]\, W(x, \bar\Delta_j))(\xi', x_n) = O(|\xi'| + 1)^N \\ &\qquad\qquad \text{for } \xi' \in \mathbb{R}^{n-1} \text{ uniformly in } x_n . \end{aligned} \right. \tag{1.29}$$

The first is an easy variant of the Paley Wiener theorem for Γ^d functions.

Concerning (1.29) first recall that for a suitable polynomial J

$$(1.30)\qquad {}'\mathcal{F}(\exp[-x^2]W(x,\bar{\Delta}_j))(\xi',x_n)=$$
$$=\int_{\mathbf{R}^{n-1}}\exp[-ix'\cdot\xi'-x^2]\left(\frac{(n-1)!}{(-2\pi i)^n}\int_{\bar{\Delta}_j}\frac{J(x,\omega)}{(x\cdot\omega+i(x^2-(x\cdot\omega)^2)+i0)^n}\,d\omega\right)dx'.$$

Take Σ_j with $\Sigma_j^*=\bar{\Delta}_j$; then Σ_j intersects the equator due to $\pm\nu\notin\Sigma_j^*$ and moreover $\varrho(\Sigma_j^*)=(\Sigma_j\cap S^{n-2})^*$. Therefore from $\varrho(\Sigma_j^*)\subset\subset\Gamma'_j$, we deduce that when $\xi'\notin\Gamma_j'^{+}$ there is $(y',0)\in\Sigma_j^+$ with $y'\cdot\xi'<-\varepsilon|y'||\xi'|$. Thus letting

$$x'\mapsto x'+iy',\qquad |y'|=\varepsilon'$$

in (1.30), we estimate the integrand by $c'\exp[-\varepsilon|\xi'|-x^2]$ which proves the first of (1.29).

Now we pass to the second. For the (global) Fourier transform, first we claim that

(1.31) $\quad\mathcal{F}(\exp[-x^2]W(x,\bar{\Delta}_j))(\xi)$ is exponentially decreasing on relatively compact cones of $\mathbf{R}^n\setminus\bar{\Delta}_j^+$,

$$(1.32)\qquad \mathcal{F}(\exp[-x^2]W(x,\bar{\Delta}_j))(\xi)=0(|\xi|+1)^N.$$

The first can be proven by the same argument as above. Since $\forall\Sigma'_j\subset\subset\Sigma_j$ we can find ε and C such that for $\zeta=x+iy\in\mathbf{R}^n+i\{y:y/|y|\in\Sigma'_j,\ |y|<\varepsilon\}$,

$$\frac{1}{|(\zeta\cdot\omega+i(\zeta^2-(\zeta\cdot\omega)^2))^n|}<\frac{C}{|y|^n}\qquad\forall\omega\in\bar{\Delta}_j,$$

then we know that $\exp[-x^2]W(x,\bar{\Delta}_j)$ is a distribution of order $<n+1$ which is real analytic rapidly decreasing outside 0. Thus we have (1.32) for $N<n+1$.

Now we write

$$|{}'\mathcal{F}(\exp[-x^2]W(x,\bar{\Delta}_j))(\xi',x_n)|=$$
$$=(2\pi)^{-1}\left|\int\exp[ix_n\xi_n]\mathcal{F}(\exp[-x^2]W(x,\bar{\Delta}_j))(\xi)\,d\xi_n\right|=$$
$$=(2\pi)^{-1}\left|\int_{\pi^{-1}(\xi')\cap\Delta_j'^{+}}\exp[ix_n\xi_n]\mathcal{F}(\exp[-x^2]W(x,\bar{\Delta}_j))(\xi)\,d\xi\right|+$$
$$+(2\pi)^{-1}\left|\int_{\pi^{-1}(\xi')\cap(\mathbf{R}^n\setminus\Delta_j'^{+})}\exp[ix_n\xi_n]\mathcal{F}(\exp[-x^2]W(x,\bar{\Delta}_j))(\xi)\,d\xi\right|,$$

where π is the projection $\mathbf{R}^n\ni(\xi',\xi_n)\mapsto\xi'$, and $\Delta'_j\supset\supset\Delta_j$ with $\pm\nu\notin\Delta'_j$.

By observing that $\pi^{-1}(\xi')\cap\Delta_j'^{+}$ is a compact interval of length $c|\xi'|$ for some c and by using (1.32) we estimate the first part of the integral by $c'|\xi'|^{N+1}$.

Since we can estimate the remaining by $c' \exp[-\varepsilon|\xi'|]$ due to (1.31), finally we obtain the second of (1.29). This achieves the proof of the two last conditions of the lemma.

It remains to prove the regularity L^1_{loc} for ${}'\mathscr{F}(D^k_n g^j_\nu|_N)$. First observe that since $\exp[-x^2]W(x, \Delta_j)$ is a real analytic rapidly decreasing function outside the origin, then $\mathscr{F}(\exp[-x^2]W(x, \Delta_j))(\xi)$ accepts analytic continuation to some domain $\{\zeta = \xi + i\eta;\ |\eta| < \varepsilon\}$ [7]; and thus we have the same regularity in ξ' for the partial Fourier transform. On the other hand ${}'\mathscr{F}(D^k_n g_\nu)$ is an entire function of ξ'. Then in view of (1.27) we attain the conclusion.

End of proof of Theorem 1.3. Put $u^j_1 = u^j_1 = 0$ and solve $\forall \nu > 1$ the problems

$$Pu^j_{\nu+1} = 0\,;\ D^k_n u^j_{\nu+1}|_N - (D^k_n u^j_{\nu+1}|_{N_+} - D^k_n u^j_{\nu+1}|_{N_-} + D^k_n(g^j_\nu - g^j_{\nu+1} + u^j_\nu)|_N) = 0\,,$$
$$0 < k < m-1\,,$$

with

$$u^j_{\nu+1} \in \mathscr{A}_F(\{0 < |x_n| < \varepsilon\}) \cap \mathscr{A}^*(\{|x_n| < \varepsilon\} \times \varrho_j^{-1}(S^{n-2} \setminus \Gamma_j))\,,$$
$$u^j_{\nu+1} \in \Gamma^{\mathscr{A}}(\mathbf{R}^n) \cap \mathscr{A}^*(\mathbf{R}^n \times \varrho^{-1}(S^{n-2} \setminus \Gamma_j))\,,$$

and with the boundary values $D^k_n u^j_{\nu+1}|_N$ still fulfilling the conditions of the Remark. This is possible in view of the Remark and the Lemma. Then if we put $b^j_\nu = g^j_\nu + u^j_\nu$, we obtain

(a) $P(b^j_{\nu+1} - b^j_\nu) \in \mathscr{A}(S_\nu)$.

(b) $P([b^j_{\nu+1} - b^j_\nu]_+ - ([u^j_{\nu+1}]_+ + [u^j_{\nu+1}]_-))$ coincides with $\theta^j_+ P(b^j_{\nu+1} - b^j_\nu)$ (θ^j_+ being the characteristic function of $\{x \cdot \nu_j > 0\}$) in a neighbourhood of N in S_ν and thus its S.S. is contained in $N \times \{\pm \nu_j\}$ there.

(c) $[b^j_{\nu+1} - b^j_\nu]_+ - ([u^j_{\nu+1}]_+ + [u^j_{\nu+1}]_-)$ is microanalytic to $\varrho_j^{-1}(S^{n-2} \setminus \Gamma'_j)$ in a neighbourhood of N; moreover, in view of (b) and of the analyticity of $u^j_{\nu+1}$ in $\{0 < |x_n| < \varepsilon\}$, it is microanalytic to $\varrho_j^{-1}(\Gamma''_j)$, $\Gamma'_j \subset\subset \Gamma''_j \subset\subset \Gamma_j$, (and so to the whole $\varrho_j^{-1}(S^{n-2})$ in a neighbourhood of N in S_ν by propagation of regularity.

Therefore again by propagation of regularity and by the fact that ν_j is non-characteristic, we conclude that $b^j_{\nu+1} - b^j_\nu$ is analytic in $S_{\nu'}$, $\nu'/\nu = c$. This shows that there exists $b^j = \lim_\nu b^j_\nu \in \Gamma^{\mathscr{A}}/\mathscr{A}$, $\forall j$, and it is clear that $b = \sum_j b^j$ is a solution of $Pb = f \bmod \mathscr{A}$. Since we know that $P\mathscr{A}(\mathbf{R}^n) = \mathscr{A}(\mathbf{R}^n)$, then we can find a «true» solution $b \in \Gamma^{\mathscr{A}}$ of the equation.

2. - The microhyperbolic case

A hyperfunction E which verifies $PE - \delta \in \mathscr{A}^*(\mathbf{R}^n \times \Delta)$ for some neighbourhood $\Delta \ni \xi^0$ will be called a microlocal fundamental solution at ξ^0.

When P admits microlocal fundamental solutions $E_\pm$ at ξ^0 whose singularities are contained in proper convex cones of the half spaces $\pm x\cdot v > 0$, then we will call $P(D)$ «microhyperbolic at ξ^0 to v».

This is equivalent for some $\Delta \ni \xi^0$ and for some $\varepsilon > 0$ to the algebraic condition

$$(2.1)\qquad P_m(\xi + \tau v) = 0\,,\ \frac{\xi}{|\xi|} \in \Delta\,,\ |\tau| < \varepsilon|\xi| \Rightarrow \operatorname{Im}\tau = 0$$

see [10]; this property for $P_m(\zeta)$ will be called «local hyperbolicity at ξ^0 to v».

Let P_m be locally hyperbolic at ξ^0 to v, let $I = I((P_m)_{\xi^0}, v)$ ($(P_m)_{\xi^0}$ = localization at ξ^0) denote the component of v in the real complement of $(P_m)_{\xi^0}(\xi) = 0$; then all $v' \in I$ make (2.1) fulfilled (with different Δ and ε). Let us assume $P_m(v) \neq 0$ (otherwise choose another $v' \in I$), put $v = (0, \ldots, 1)$, and last denote by μ the multiplicity of ξ^0 as a zero of P_m. By Weierstrass's theorem we can find a complex neighbourhood $D \ni \xi^0$ and a decomposition

$$P(\zeta) = H(\zeta)\cdot E(\zeta)\,,\ \frac{\zeta}{|\zeta|} \in D\,,\ \zeta \text{ large}\,,$$

where H and E are polynomials in ζ_n with analytic coefficients of orders μ and $m - \mu$ respectively, with E non-vanishing for $\zeta \in D$, ζ large, and the principal symbol of H having a root of order μ ay ξ^0. Moreover we have

$$(2.2)\qquad H(\xi', \zeta_n) = 0\,,\ \frac{\xi'}{|\xi'|} \in \Gamma \subset\subset \varrho(D \cap S^{n-1})\,,\ \xi' \text{ large} \Rightarrow |\operatorname{Im}\zeta_n| < c|\xi'|^{(\mu-1)/\mu}\,.$$

(In case $P \prec P_m$ we can replace $c|\xi'|^{(\mu-1)/\mu}$ by c in (2.2)). We can canonically associate with $H(\zeta)$ and $E(\zeta)$ two microdifferential operators (of Weierstrass type) $H(D)$ and $E(D)$ on the sheaf of microfunctions $\mathscr{C}|_{\varrho^{-1}(\Gamma)}$ (see Appendix).

For a hyperfunction f which is microanalytic to $\pm v$ in $N = \{x\cdot v = 0\}$, let us denote by $\gamma^\mu(f)$ the restriction to N of all $D_n^j f$, $0 < j < \mu - 1$. Notice that if f is a microfunction in $\mathscr{C}|_{\varrho^{-1}(\Gamma)}$ and if ϱ is proper when restricted to $\operatorname{supp} f|_N$, then we can again define $\gamma^\mu(f) \in \mathscr{C}^\mu_{N\times\Gamma}$. It suffices to take an extension $\tilde f$ whose support is the closure in $\mathbf{R}^n \times S^{n-1}$ of $\operatorname{supp} f$ due to the flabbiness of $\mathscr{C}$; since $\pm v \notin \operatorname{supp} \tilde f|_N$ then γ^μ is well defined on any representing hyperfunction. It is obvious that the result of such process does not depend as an element of $\mathscr{C}|_{N\times\Gamma}$ of the choice of the extension and of the representative.

Now we state the microlocal version of Theorem 1.1.

Theorem 2.1. *Let P be microhyperbolic at ξ^0 (to v), let μ be the multiplicity of ξ^0 as a root of P_m, assume $P_m(v) \neq 0$, consider the Weierstrass decomposition $P = HE$ at the ray through ξ^0, and take $(u_j)_{j=0,\ldots,\mu-1}$ in Γ_0^d, $1 < d < \mu/(\mu-1)$. Then there* *exists* $u \in \Gamma^{d}_{P}(\mathbf{R}^n)$, *microlocal solution at* $\xi^{0\prime}$ *of the problem*

$$\gamma^{\mu}(E(D)u) = (u_j)\,, \tag{2.3}$$

in the sense that the equality holds in $'\mathscr{C}|_{N\times\Gamma}$ *for some* $\Gamma \ni \xi^{0\prime}$. *Such* u *is unique modulo* $\mathscr{A}^*(\mathbf{R}^n \times \Delta)$, $\Delta \ni \xi^0$.

PROOF. If

$$H(\zeta) = \sum_{j=0}^{\mu} a_j(\zeta')\zeta_n^{\mu-j} \quad\text{and}\quad E(\zeta) = \sum_{j=0}^{m-\mu} b_j(\zeta')\zeta_n^{m-\mu-j}\,,$$

put $H^k(\zeta) = \sum_{j=0}^{k} a_j(\zeta')\,\zeta_n^{k-j}$, $0 < k < \mu$. Let us define for $\xi'/|\xi'| \in \Gamma$, ξ' large

$$F_0^k(\xi', x_n) = (2\pi i)^{-1}\int_{\Gamma_{\xi'}} \frac{\exp[i\zeta_n x_n]\,H^{\mu-1-k}(\xi', \zeta_n)}{P(\xi', \zeta_n)}\,d\zeta_n\,, \tag{2.4}$$

where $\Gamma_{\xi'}$ is a curve surrounding the μ zeros of $H(\xi', \zeta_n) = 0$ for ζ_n. Clearly F_0^k satisfy (1.8) in view of (2.2).

Setting $u_j^1 = u_j * W_0(\cdot, \Gamma'')$, $\Gamma'' \subset\subset \Gamma$, then $u_j^1 = u_j$ as microfunctions in $N \times \Gamma''$. Now if we define u by (1.12) for $b = 0$, then obviously $Pu = 0$. Besides u belongs to $\mathscr{Q}_{x'}\,\mathscr{B}_{x_n}$ (see Appendix) and it has x_n as analytic parameter. By Lemma 2.5 in Appendix we have

$$Eu(x) = \sum_{j=0}^{m-\mu} D_n^{m-\mu-j} b_j(D_{x'})u(x) = \sum_{j=0}^{m-\mu} \int_{\Gamma''^{+}} \exp[ix'\cdot\xi']\,b_j(\xi')\,D_n^{j}\,'\hat{u}(\xi', x_n)\,d\xi' =$$

$$= \sum_{k=0}^{\mu-1} \int_{\Gamma''^{+}} \exp[ix'\cdot\xi']\,\hat{u}_k^1(\xi')(2\pi i)^{-1}\int_{\Gamma_{\xi'}} \frac{\exp[ix_n\zeta_n]\,H^{\mu-1-k}(\xi', \zeta_n)}{H(\xi', \zeta_n)}\,d\zeta_n\,,$$

$$\Gamma' \subset\subset \Gamma'' \subset\subset \Gamma\,, \quad (\Gamma''^{+} \text{ suitably trumcated})\,,$$

and so $D_n^j Eu|_N$ equals u_j^1 in $N \times \Gamma''$ and thus u_j in $N \times \Gamma'$ (due to slight application of the residue theorem [9]).

Concerning the uniqueness of u, note that $\pm\nu \notin \operatorname{supp} u|_N$ and therefore the following equalities hold in $\mathscr{C}(\mathbf{R}^n \times \varrho^{-1}(\Gamma'))$

$$D_n(\theta_+ Eu) = \theta_+ D_n(Eu) - iEu|_N \otimes \delta_n\,, \tag{2.5}$$

$$a_j(D_{x'})(\theta_+ Eu) = \theta_+ a_j(D_{x'})Eu\,. \tag{2.6}$$

The first is obvious whereas the second can be proved by using the kernel $k_j(x', \omega')$ associated to $a_j(D_{x'})$ and the formula

$$\int k_j(x', \omega')\theta_+(x_n)Eu(\omega', x_n)\,d\omega' = \theta_+(x_n)\int k_j(x', \omega')Eu(\omega', x_n)\,d\omega'\,.$$

Therefore if $u_j = 0$ in $N \times \Gamma$, then $\operatorname{supp} H(\vartheta, Eu)|_{\varrho^{-1}(\Gamma)} = \emptyset$; this implies $\operatorname{supp} Eu|_{\varrho^{-1}(\Gamma)} = \emptyset$, due to the microhyperbolicity of H in $\varrho^{-1}(\Gamma)$, and last $\operatorname{supp} u|_{\Delta} = \emptyset$, due to the invertibility of E in Δ.

THEOREM 2.2. *Let P be microhyperbolic to v, $(P_m(v) \neq 0)$, at any point in $\Delta \ni \xi^0$; denoting by μ the multiplicity of ξ^0 let us fix Cauchy data $(u_j^1) = (u_j * W_0 \cdot (\cdot, \Gamma''))$, $(u_j \in \Gamma_0^d$, $1 < d < \mu/(\mu - 1)$, $\Gamma'' \subset\subset \varrho(\Delta))$. (Or else suppose that (u_j^1) satisfy* (1.23), (1.24), (1.25).) *Then for $\Gamma' \downarrow \xi^{0\prime}$ there are neighbourhoods $\Delta' \ni \xi^0$, $\Delta \supset\supset \Delta' \downarrow \xi^0$, and functions $u \in \Gamma_F^d(\mathbf{R}^n) \cap \mathscr{A}^*(\mathbf{R}^n \times (S^{n-1} \setminus \bar{\Delta}'))$ which solve* (2.3) *modulo $\mathscr{A}^*(N \times \Gamma')$. Such u are unique modulo $\mathscr{A}$.*

PROOF. With F_0^b defined as in Theorem 2.1 and F_2^b defined as in Theorem 1.1, let us define u by (1.12) for $b = 0$ and v_2 by (1.13). Obviously v_2 is analytic in a neighbourhood of N and besides $\gamma^\mu(Eu) - \gamma^\mu(v_2) = (u_j^1)$; thus $\gamma^\mu(Eu) = (u_j^1)$ in $\mathscr{C}|_{N \times \Gamma'}$.

Refining the proof of Theorem 1.2, we want to prove that u is microanalytic not only to $S^{n-1} \setminus \overline{\varrho^{-1}(\Gamma'')}$ but even to $S^{n-1} \setminus \bar{\Delta}'$, $\Delta' \subset \varrho^{-1}(\Gamma'')$ with Δ' arbitrarily small depending on Γ''. In fact, since by hypothesis $(P_m)_{\xi^0}(v) \neq 0$, then for

$$\left|\frac{\zeta'}{|\zeta'|} - \xi'\right| < \varepsilon, \qquad \left(\xi = \left(\frac{\xi^{0\prime}}{|\xi^{0\prime}|}, \frac{\xi_n^0}{|\xi^{0\prime}|}\right)\right),$$

there are exactly μ zeros $\zeta_n = \lambda(\zeta')$ of $P_m(\zeta', \zeta_n) = 0$ which verify

$$\left|\frac{\lambda(\zeta')}{|\zeta'|} - \xi_n\right| < c\varepsilon;$$

moreover they verify

$$\left|\frac{\lambda(\zeta')}{|\zeta'|} - \xi_n\right| < c\left|\frac{\zeta'}{|\zeta'|} - \xi'\right|.$$

Since such zeros are the zeros of the principal symbol of H, then we have the following estimate for the zeros $\zeta_n = \lambda^0(\zeta')$ of $H(\zeta', \zeta_n) = 0$

$$\left|\frac{\lambda^0(\zeta')}{|\zeta'|} - \xi_n\right| < \varepsilon \qquad \text{when } \left|\frac{\zeta'}{|\zeta'|} - \xi'\right| < \frac{1}{c_\varepsilon}, \ |\zeta'| > c_\varepsilon. \tag{2.7}$$

Thus by letting $x \mapsto x + iy$ in (1.12) and using (2.2), (2.4), (2.7), we estimate the integrand by

$$c \exp\left[-y' \cdot \xi' - c'|\xi'|^{1/d} + c''|x_n||\xi'|^{(\mu-1)/\mu} - y_n \xi_n |\xi'| + \varepsilon |y_n| |\xi'|\right]$$

provided that

$$\xi' \in \Gamma_\varepsilon^+ = \left\{\left|\frac{\xi'}{|\xi'|} - \xi'\right| < \frac{1}{c_\varepsilon}, |\xi'| > c_\varepsilon\right\}.$$

Let Σ'_ε be the set of all $(y', y_n) \in S^{n-1}$ which satisfy $y' \cdot \xi' + y_n \xi_n |\xi'| - \varepsilon |y_n| |\xi'| > \varepsilon |\xi'| \; \forall \xi' \in \Gamma'^{+}_{\varepsilon}$. Then we see that the part of the integral (1.12) over $\Gamma'^{+}_{\varepsilon}$ converges and defines an analytic function in

$$\mathbf{R}^n + i\left\{y : \frac{y}{|y|} \in \Sigma_\varepsilon\right\}$$

and so the S.S. of such part is contained in the ε'-neighbourhood of ξ^0 with $\varepsilon' \downarrow 0$ for $\varepsilon \downarrow 0$. If $\Gamma'' \subset \Gamma'$, such integral coincides, mod entire functions, with u.

The uniqueness of u modulo functions microanalytic to Δ is due to the already quoted propagation of regularity; however, since by hypothesis u is microanalytic to $S^{n-1} \setminus \bar{\Delta}'$, $\Delta' \subset\subset \Delta$, then it is unique modulo $\mathscr{A}$.

This achieves the proof.

Let us apply the preceding machinery to the theory of the Γ^d solvability.

Theorem 2.3. *Let P be microhyperbolic at any characteristic and let μ be the largest multiplicity of its characteristics. Then*

$$P(D)\Gamma^d(\mathbf{R}^n) = \Gamma^d(\mathbf{R}^n), \text{ for } 1 < d < \frac{\mu}{\mu - 1} \quad (\forall d > 1 \text{ in case } P < P_m).$$

Proof. Setting $V = \{P_m(\xi) = 0\}$, let us choose a finite covering

$$V \cap S^{n-1} \subset \widehat{\bigcup_j \Delta''_j}, \qquad \Delta''_i \cap \Delta''_j = \emptyset, \qquad \Delta''_j \subset\subset \Delta_j, \tag{2.8}$$

in such way that for some $v_j \in V$, P_m is locally hyperbolic to v_j at any point in Δ_j. Let $P = H_j E_j$ be the Weierstrass decomposition in Δ_j, μ_j the degree of H_j, ϱ_j the projection of S^{n-1} from the poles $\pm v_j$, N_j the hyperplane $x \cdot v_j = 0$. Last let us assume that the covering (2.8) is so fine that with an intermediate Δ'_j ($\Delta''_j \subset\subset \Delta'_j \subset\subset \Delta_j$), Δ'_j corresponds to $\Gamma''_j = \varrho_j(\Delta''_j)$ as in Theorem 2.2.

Fix $f \in \Gamma^d$; if $g_\nu \in \Gamma^d_0$, $d > 1$, solves the equation $Pg = f$ in S_ν, let us decompose

$$g_\nu = g^0_\nu + \sum_j g^j_\nu = g_\nu * \exp[-x^2]\, W(x, S^{n-1} \setminus \bigcup_j \bar{\Delta}'_j) + \sum_j g_\nu * \exp[-x^2]\, W(x, \bar{\Delta}'_j).$$

First note that $g^0_{\nu+1} - g^0_\nu$ is microanalytic in $\mathbf{R}^n \times V$ by construction, and in $S_\nu \times (S^{n-1} \setminus V)$ because $P(g^0_{\nu+1} - g^0_\nu) \in \mathscr{A}(S_\nu)$ (remember that the convolution by W does not give propagation of singularity); therefore $g^0_{\nu+1} - g^0_\nu \in \mathscr{A}(S_\nu)$. Concerning the terms with $j \neq 0$ set $v_j = (0, \dots, 1)$ by simplicity, take $u_1 = 0$, and solve in view of Theorem 2.2 the problems

$$Pu_{\nu+1} = 0\,; \qquad \gamma^{\mu_j}(E_j u^j_{\nu+1}) = \gamma^{\mu_j}(E u^j_\nu - g^j_{\nu+1} + g^j_\nu) \qquad \text{in } \mathscr{C}|_{N \times \Gamma_j}$$

with $u^j_{\nu+1}$ belonging to $\Gamma^d(\mathbf{R}^n) \cap \mathscr{A}^*(\mathbf{R}^n \times (S^{n-1} \setminus \bar{\Delta}'_j))$, $(1 < d < \mu_j/(\mu_j - 1))$, and with $\gamma^{\mu_j}(E u^j_{\nu+1})$ still satisfying (1.23), (1.24), (1.25). Setting $h^j_\nu = g^j_\nu + u^j_\nu$, we obtain

$$(2.9)\quad b'_{\nu+1}-b'_{\nu}=0 \qquad \text{in } \mathscr{C}|_{\mathbf{R}^n\times(S^{n-1}\setminus\Delta'_j)},$$

$$(2.10)\quad H_j(\theta,E_j(b'_{\nu+1}-b'_\nu))=\theta,H_jE_j(b'_{\nu+1}-b'_\nu)+\sum_{k=1}^{n}C'_k(D_n^{k-1}E_j(b'_{\nu+1}-b'_\nu)|_{\mathbf{R}}\otimes\delta_n),$$

for suitable microdifferential operators $C'_k=C'_k(D_{x'})$ given by formulas (2.5), (2.6).

Note that the first term in the right hand side of (2.10) is null in $S_\nu\times\varrho_j^{-1}(\Gamma'_j)$ and the second in $\mathbf{R}^n\times\varrho_j^{-1}(\Gamma'_j)$. By the microhyperbolicity of H_j in Δ_j we conclude that $E_j(b'_{\nu+1}-b'_\nu)$ vanishes in $S_\nu\times\Delta_j$, $\nu'/\nu=c$, and by the invertibility of E_j, $b'_{\nu+1}-b'_\nu$ also vanishes. Moreover the last vanishes in $\mathbf{R}^n\times\times(S^{n-1}\setminus\Delta'_j)$, $\Delta'_j\subset\subset\Delta_j$ in view of (2.9) and so $b'_{\nu+1}-b'_\nu\in\mathscr{A}(S_\nu)$.

Remembering the equality $P\mathscr{A}(\mathbf{R}^n)=\mathscr{A}(\mathbf{R}^n)$ which is classical, we then conclude as in Theorem 1.3.

Appendix to section 2

For a real cone Δ^+ and a positive constant δ, let us consider a set in the form

$$D^+=\{\zeta=\xi+i\eta\in\mathbf{C}^n:\xi\in\Delta^+,|\eta|<\delta(1+|\xi|)\}\,;$$

let us denote by $(f)_{x^*}$, $f\in\mathscr{C}|_{\mathbf{R}^n\times\Delta}$, $x^*=(x,\xi)\in\mathbf{R}^n\times\Delta$, the germ of f at x^*. If $F(\zeta)$ is an analytic function on the cone D^+ (even trumcated) with polynomial growth, let us define

$$(2.11)\qquad F(D)(f)_{x^*}=\Big((2\pi)^{-n}\int_{\Delta^+}\exp[ix\cdot\xi]\,F(\xi)\hat{\tilde f}(\xi)\,d\xi\Big)_{x^*}$$

where Δ^+ is possibly trumcated and where $\tilde f$ is a Fourier hyperfunction of $\mathscr{Q}$ the image of $\tilde f$ in $\mathscr{C}$ being f near x^* (due to the flabbiness of $\mathscr{C}$ and $\mathscr{Q}$).

First we prove

LEMMA 2.4. *If* $x_0^*\notin$ S.S. $\tilde f$ *then* $x_0^*\notin$ S.S. $\int_{\Delta^+}\exp[ix\cdot\xi]\,F(\xi)\hat{\tilde f}(\xi)d\xi$.

PROOF. By inversion of the integration order

$$\int_{\Delta^+}\exp[ix\cdot\xi]\,F(\xi)\hat{\tilde f}(\xi)\,d\xi=\int_{\Delta^+}\exp[ix\cdot\xi]\,F(\xi)\Big(\int_{\mathbf{R}^n}\exp[-iw\cdot\xi]\tilde f(w)\,dw\Big)d\xi=$$

$$=\int_{\mathbf{R}^n}\tilde f(w)\Big(\int_{\Delta^+}\exp[i(x-w)\cdot\xi]\,F(\xi)\,d\xi\Big)dw\,.$$

In the part of the integral with w near x_0 we can let $w \mapsto w + i\sigma$, $|\sigma| < \varepsilon$, $\sigma \cdot \xi_0 < 0$ and correspondingly we can make the substitution $x \mapsto x + i\sigma$ without destroying its convergence (in the hyperfunction sense).

In the remaining part we have $|x - w| > \varepsilon(1 + |w|)$ for x near x_0, and thus by setting $\zeta = \xi + i\eta$, $\eta \cdot (x - w) > \varepsilon' |\xi| (1 + |w|)$ with η depending on w and $z = x + iy$, $|y| < \varepsilon''$, we estimate the second integrand by

$$\exp\left[-\varepsilon'|\xi| - \varepsilon'|\xi||w| + \varepsilon''|\xi|\right] < \exp\left[-\varepsilon'|\xi| - \varepsilon''|w| + \varepsilon''|\xi|\right] \qquad \text{for } |\xi| > \varepsilon$$

and so both integrals converge for suitably small ε''. This show that this part defines an analytic function near x_0. The proof is complete.

Let now F be in the form

$$F(\zeta) = \sum_{j=0}^{m} a_j(\zeta')\zeta_n^{m-j}, \tag{2.12}$$

with $a_j(\zeta')$ analytic in some conical complex (truncated) neighbourhood G' of $\Gamma' \subset \mathbf{R}^{n-1}$. When $a_0(\zeta') = 1$, F is said to be a symbol of Weierstrass type with respect to ζ_n. For $f \in \mathscr{C}|_{\mathbf{R}^n \times \varrho^{-1}(\Gamma)}$ and $x^* \in \mathbf{R}^n \times \varrho^{-1}(\Gamma)$, let us define

$$F(D)(f)_{x^*} = \left((2\pi)^{-(n-1)} \sum_j \int_{\Gamma'} \exp\left[ix' \cdot \xi'\right] a_j(\xi') D_n^j \hat{\tilde{f}}(\xi', x_n)\, d\xi' \right)_{x^*} \tag{2.13}$$

where $\tilde{f}$ is an element of the sheaf $\mathscr{Q}_{x'}\mathscr{B}_{x_n}$ (= Fourier hyperfunctions with hyperfunction parameter) which has x_n as an analytic parameter (due to the flabbiness of $\mathscr{C}$ and $\mathscr{Q}_{x'}\mathscr{B}_{x_n}$) and which coincides with f in $\mathscr{C}$ near x^*.

We can prove as in Lemma 2.4 that $F(D)(f)_{x^*}$ is well defined and thus we can obtain by means of (2.13) an operator on $\mathscr{C}|_{\mathbf{R}^n \times \varrho^{-1}(\Gamma)}$. In the following we relate the two former definitions.

Lemma 2.5. *If F is of type* (2.12) *then the operators defined by* (2.13) *and* (2.11) *agree in* $\mathscr{C}|_{\mathbf{R}^n \times \varrho^{-1}(\Gamma)}$.

Proof. To handle this situation we need to be very careful in the choice of the representative $\tilde{f}$ of $(f)_{x^*}$; we proceed as follows. For some neighbourhood $I = \Omega \times \Delta \ni x^*$, $(\Omega \ni x,\ \Delta \ni \xi)$, contained in $\mathbf{R}^n \times \varrho^{-1}(\Gamma)$ first let us take $\tilde{f}' \in \mathscr{B}$ whose S.S. is the closure in $\mathbf{R}^n \times S^{n-1}$ of $\operatorname{supp} f|_I$ the image of $\tilde{f}'$ in $\mathscr{C}$ being f in I. Since $\tilde{f}'$ is analytic outside Ω, then it accepts a modification mod $\mathscr{A}(\Omega)$ to $\tilde{f} \in \underset{\sim}{\mathscr{Q}}^{-\delta}$ which is a section of the sheaf $\underset{\sim}{\mathscr{P}}^{-\delta}$ of modified real analytic functions which decrease with exponential rate δ. This is a consequence of the flabbiness of $\underset{\sim}{\mathscr{Q}}^{-\delta}/\underset{\sim}{\mathscr{P}}^{-\delta}$ and the fact that the sheaves $\underset{\sim}{\mathscr{P}}^{-\delta}$ and $\mathscr{A}$ agree in $\mathbf{R}^n$. Thus in view of the formula

$$\xi_n^j \hat{\tilde{f}}(\xi) = \widehat{D_n^j \tilde{f}}(\xi),$$

which can be easily proved by duality, we obtain

$$(2\pi)^{-1}\int \exp [ix_n\xi_n]\,\xi_n^{l}\hat{f}(\xi)\,d\xi_n = D_n^{l}\hat{f}(\xi', x_n)\,,$$

where all entries make sense because

(*a*) $\hat{f}$ is analytic in $\{\zeta: |\eta| < \delta(1+|\xi|)\}$ for some δ,

(*b*) $\tilde{f}$ has x_n as an analytic parameter.

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