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Algebraic conditions on partial differential operators for existence of micro-local fundamental solutions with singularities carried by proper cones (**)

Micro-iperbolicità di operatori differenziali a coefficienti costanti

SUNTO. — Si caratterizzano algebricamente i polinomi differenziali a coefficienti costanti che ammettono soluzioni fondamentali micro-locali con singolarità contenuta in coni propri. A tal fine si raffina il criterio di Kawai [3] per la stima della micro-analiticità di iperfunzioni di Fourier.

0. - We aim to show how elementary arguments lead to a microlocalization of the theorem by Kashiwara [2].

A hyperfunction E is said to be a microlocal fundamental solution at $\xi^0 \in S^{n-1}$ ($= (n-1)$-dimensional real sphere) to the partial differential operator $P = P(D)$ in $\mathbf{R}^n$ with constant coefficients, if there is a neighbourhood $\Delta \ni \xi^0$ in S^{n-1} such that $P(D)E - \delta$ is microanalytic in $\mathbf{R}^n \times \Delta$. When such solutions form a couple $E_\pm$ with singular supports in proper convex closed cones $\pm\Gamma^*$ contained, apart from the vertices, in half spaces $\pm\langle x, \vartheta\rangle > 0$, then we call $P(D)$ $\mathbb{C}$-hyperbolic at ξ^0 to the direction ϑ.

On the other hand we say that the characteristic form $P_m(\zeta)$, $\zeta \in \mathbf{C}^n$ is locally hyperbolic at ξ^0 to ϑ if for some neighbourhood $\Delta \ni \xi^0$ and for some ε the following condition is fulfilled

$$(0.1) \qquad P_m(\xi + \tau\vartheta) = 0\,,\ \xi \in \Delta\,,\ |\tau| < \varepsilon \Rightarrow \operatorname{Im}\tau = 0\,.$$

We relate the former definitions by

THEOREM. *$P(D)$ is $\mathbb{C}$-hyperbolic at ξ^0 (to ϑ) $\Leftrightarrow$ $P_m(\zeta)$ is locally hyperbolic at ξ^0 (to the same ϑ).*

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Regarding Γ^* as the dual cone of an open convex cone $\Gamma \subset \mathbf{R}^n$, we claim that local hyperbolicity to Γ is equivalent to existence of $E_\pm$, in correspondence with any $\Gamma' \subset\subset \Gamma$, with the estimation sing supp $E_\pm \subset \pm \Gamma'^*$. The fact that local hyperbolicity is sufficient is classical and becomes rather elementary if we point out: (*a*) When Δ is small, $P_m(\zeta)$ is locally hyperbolic at any $\xi \in \Delta$ to any $\eta \in \Gamma_\xi$ (where Γ_ξ is the cone defined from the localization $(P_m)_\xi(\zeta)$ at ξ, as the connected component of ϑ in $\{\eta \in \mathbf{R}^n : (P_m)_\xi(\eta) \neq 0\}$). (*b*) The mapping $\xi \to \Gamma_\xi$ is lower semicontinuous; possibly by shrinking Γ_ξ, which will not affect the subsequent estimates of sing supp, let us make it continuous. (*c*) $P(\zeta) \neq 0$ in the regions defined by $\zeta = \xi \pm i\eta$ with $\xi/|\xi| \in \Delta$, $|\xi| > c_1$, $\eta \in \Gamma'_\xi \subset\subset \Gamma_\xi$, $c_2|\xi|^\mu < |\eta| < c_3|\xi|$ for suitable constants c_i and for μ defined from $q = \deg (P_m)_\xi(\zeta)$ as $(q-1)/q$. If we then put

$$E_\pm(x) = (2\pi)^{-n} \int_{D_\pm} \frac{\exp[ix\zeta]}{P(\zeta)} d\zeta$$

where the paths $D_\pm \ni \zeta = \xi \mp i\eta$ are chosen in the previous regions we obtain microlocal fundamental solutions with the desired properties. In fact since we can deform $|\eta|$ within the interval $c_2|\xi|^\mu \mapsto c_3|\xi|$ and $\eta/|\eta|$ within the set $\Gamma'_\xi \cap S^{n-1}$, then letting $\Gamma'_\xi \uparrow \Gamma_\xi$ we conclude that $E_\pm$, mod $A^*(\mathbf{R}^n \times \Delta)$, satisfy the estimates for singular spectrum

$$\text{S.S. } E_\pm \subset \{0\} \times \bar{\Delta} \cup \bigcup_{(P_m(\xi)=0,\ \xi \in \bar{\Delta})} \Gamma^*_\xi \times \{\xi\} . \tag{0.2}$$

If we project (0.2) on the base space and remember that $\xi \mapsto \Gamma^*_\xi$ is (upper semi-) continuous, we attain the conclusion.

At last we point out that when $P_m(\zeta)$ is locally hyperbolic (at any $\xi \in S^{n-1}$) to some θ_ξ continuously depending on ξ, then by covering S^{n-1} by a family $\bigcup_j \Delta_j$ and by noting that the mappings $\xi \mapsto \Gamma^*_\xi$, $\xi \in \Delta_j$, agree on $\Delta_j \cap \Delta_{j'}$, we obtain a «true» fundamental solution, from the microlocal ones, satisfying (0.2) with $\bar{\Delta}$ replaced by S^{n-1}; on the other hand if we have estimates of the latter type involving proper cones Γ^*_ξ continuously depending on ξ, then we conclude from the necessity of local hyperbolicity in the stated theorem that $P_m(\zeta)$ is locally hyperbolic at any $\xi \in S^{n-1}$ to some θ_ξ (with $\xi \mapsto \theta_\xi$ continuous).

1. - Proof of the necessity of local hyperbolicity

We will deal with the sheaf $\mathcal{F}^{-\delta}$ of modified Fourier hyperfunctions of exponential decay with type δ and the corresponding sheaf $\tilde{\mathcal{F}}^{-\delta}$ of analytic functions. Recall that $\mathcal{F}^{-\delta}/\tilde{\mathcal{F}}^{-\delta}$ is flabby and that it agrees with $\mathcal{B}/\mathcal{A}$ on bounded sets of $\mathbf{R}^n$. If then S_R is the sphere of radius R, we know that $E = E_+$

accepts a modification, mod $A(S_R)$, to $\underset{\sim}{E} \in \mathfrak{F}^{-\epsilon}$ which is a section $\mathfrak{F}^{-\epsilon}$ outside $\Gamma^* \cap \tilde{S}_R$. For the new $\underset{\sim}{E}$ we have the identity

$$P(D)\underset{\sim}{E}(x) - \delta(x) = F(x) \tag{1.1}$$

where F satisfies:

i) sing supp $F \subset \tilde{S}_R = K$,

ii) $\pi_x(\text{S.S. } F \cap \{\xi \in \Delta\}) \subset \Gamma^* \cap \partial S_R \subset K'$, ($K'$ = convex hull of $\Gamma^* \cap \partial S_R$).

(Here (x, ξ) is the variable on the sphere boundle $(\mathbf{R}^n \cup S^{n-1}\infty) \times S^{n-1}$, π_x is the first projection, and finally sing supp and S.S. are considered also with respect to the mentioned decay conditions.) By Fourier transform (1.1) is equivalent to

$$P(\zeta)\hat{\underset{\sim}{E}}(\zeta) = 1 + \hat{F}(\zeta) \tag{1.2}$$

in which i) and ii) are reflected into the growth mode of $\hat{F}(\zeta)$ according to the following modification of the theorem of Kawai [3].

Lemma. *Let $F(x)$ be modified real analytic rapidly decreasing outside a compact convex set K (and therefore admitting a Fourier transform $\hat{F}(\zeta)$ which accepts, $\forall \epsilon > 0$, analytic continuation on some $\{\zeta = \xi + i\eta : |\eta| < \delta'(1 + |\xi|)\}$, $\delta' = \delta'_\epsilon$, with infra-exponential growth after multiplication by* $\exp[-H_{K_\epsilon}(\eta)]$, $K_\epsilon = K + \{|x| < \epsilon\}$.
The following are equivalent

i) *$F(x)$ is microanalytic to Δ outside a compact convex set $K' \subset K$,*

ii) *$\forall \epsilon$, $\forall \Delta' \subset\subset \Delta$ there is $\delta' = \delta'_{\epsilon,\Delta'}$ s.t.* $\exp[-H_{K'_\epsilon}(\eta)]\,\hat{F}(\zeta)$ *verifies the infraexponential estimate on $\{\zeta = \xi + i\eta : \xi/|\xi| \in \Delta',\ |\eta| < \delta'(1 + |\xi|)\}$.*

Proof. With an intermediate Δ'', $(\Delta' \subset\subset \Delta'' \subset\subset \Delta)$, let us consider an expression « boundary value » $F(x) = \sum_{j=1}^{N} F_j(x + i\Gamma_j 0)$ with $\Gamma_1^* \cap S^{n-1} \subset \Delta$, $\Gamma_j^* \cap \Delta'' = \emptyset$, $j \neq 1$; besides F_1 is modified analytic exponentially decreasing outside K' and F_j outside K. (To this end it is sufficient to use the curvilinear wave decomposition of $\delta(x)$ multiplied by a factor $\exp[-x^2]$.) Concerning $\hat{F}_j(\zeta)$, $j \neq 1$, let us calculate it, for $\xi/|\xi| \in \Delta'$, as $\hat{F}_j(\zeta) = \int_{D_j} \exp[-iz\zeta]\,F_j(z)\,dz$, where the path D_j is formed by the z in the form $z = x + it(x)\gamma_j$ with

(*a*) $\gamma_j \in \Gamma_j$, so chosen that $\langle \gamma_j, \xi \rangle < -c$, $\forall \xi \in \Delta'$, due to the assumption $\Gamma_j^* \cap \Delta'' = \emptyset$, $\Delta' \subset\subset \Delta''$; (for large Δ', γ_j possibly depends on ξ).

(*b*) $t(x) = c'(1 + |x|)$ with c' so small that the path D_j runs in the domain where $F_j(z)$ is holomorphic.

For $z \in D_j$ we have $|F_j(z)| < C \exp[-\delta|x|]$ and therefore the integral converges absolutely to define a rapidly decreasing holomorphic function in

$$\left\{\zeta = \xi + i\eta : \frac{\xi}{|\xi|} \in \Delta',\ |\eta| < \delta'(1 + |\xi|)\right\}$$

provided that $\delta' < \delta \wedge \varepsilon'$, $\varepsilon' = \varepsilon\varepsilon'$, since then

$$|\exp[-iz\zeta]\, F_j(z)| < C \exp[\delta'(1 + |\xi|)|x| - \varepsilon'(1 + |x|)|\xi| - \delta|x|]\,.$$

By applying the classical rule to $\hat{F}_1(\zeta)$ we can recognize that $\hat{F}_1(\zeta)\exp[-H_{K'_\varepsilon}(\eta)]$ verifies the infraexponential estimate in the whole of $\mathbf{R}^n \times \{|\eta| < \delta'(1 + |\xi|)\}$ for suitable $\delta' = \delta'_\varepsilon$.

End of proof of the theorem. Since for small ε $K'_\varepsilon \subset\subset \{\langle x, \theta\rangle > 0\}$, then for some c_1, c_2, we have $H_{K'_\varepsilon}(\eta) < -c_1|\eta|$ whenever $\eta/|\eta|$ belongs to the c_2-neighbourhood $I = I_{c_2}$ of $-\theta$ in S^{n-1}; and so in view of the lemma $\forall \gamma > 0\ \exists C_\gamma$:

$$(1.3)\qquad |\hat{P}(\zeta)| < C_\gamma \exp[\gamma|\xi| - c_1|\eta|] \quad \text{if} \quad \frac{\xi}{|\xi|} \in \Delta',\ |\eta| < \delta'(1 + |\xi|),\ \frac{\eta}{|\eta|} \in I\,.$$

If then (0.1) were violated there would exist a root $\zeta = \xi - it\theta$ of $P_m(\zeta) = 0$ with $\xi \in \Delta'$, $0 < |t| < \delta'$. (Suppose $t > 0$ for the opposite could be similarly handled by replacing E_+ by E_-). Then there would exist a sequence $\zeta_\nu = \xi_\nu - i\eta_\nu$ of roots of $P(\zeta) = 0$ with $|\zeta_\nu| \to \infty$, $\zeta_\nu/|\zeta_\nu| \to \zeta/|\zeta|$, from which $|\eta_\nu|/|\xi_\nu| \to t$, $\xi_\nu/|\xi_\nu| \to \xi$, $\eta_\nu/|\eta_\nu| \to \theta$. It follows, for large ν, $\xi_\nu/|\xi_\nu| \in \Delta'$, $-\eta_\nu/|\eta_\nu| \in I$, $|\eta_\nu| < \delta'(1 + |\xi_\nu|)$ and thus setting $\zeta = \zeta_\nu$ in (1.2) we would conclude, in view of (1.3),

$$1 < C_\gamma \exp[\gamma|\xi_\nu| - c_1|\eta_\nu|] < C_\gamma \exp\left[|\xi_\nu|\left(\gamma - \frac{tc_1}{2}\right)\right]$$

which is a contradiction when $\gamma < tc_1/2$.

Remark. In view of (c) of § 0, local hyperbolicity characterizes the existence of « good » microlocal fundamental solutions in spaces of generalized distributions $\gamma_0^{(\delta)'}$, $\delta > q/(q-1)$ (and actually in $\mathfrak{D}'$ if we can put $\mu = 0$ in (c) which happens when P is weaker than P_m).

REFERENCES

[1] A. Kaneko, *On the global existence of real analytic solutions of linear partial differential equations on unbounded domain*, preprint.

[2] M. Kashiwara, *On C-hyperbolic partial differential operators with constant coefficients*, *Surikaiseki-Kenkyusho Kokyuroku*, **145** (1972), 168-171.

[3] T. Kawai, *On the theory of Fourier hyperfunctions and its applications to partial differential equations with constant coefficients*, J. Fac. Sci. Univ. Tokyo Sec. IA, **17** (1970), 467-517.

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