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A fixed point theorem in Banach spaces (**)

Un teorema di punto fisso in spazi di Banach

RIASSUNTO. — In questo lavoro, utilizzando un procedimento di approssimazione, si dimostra un teorema di punto fisso per operatori debolmente continui, definiti in spazi di Banach di tipo particolare. Inoltre si applica tale teorema alla risoluzione dell'equazione di Burgers-Hopf: $D_t u - K(u) D_x^2 u + u D_x u = f$, nel caso in cui f appartenga a L^4 e la viscosità $K(u)$ sia variabile.

INTRODUCTION

In a recent paper of mine, (see [5]), extending Canfora's topological degree theory, (see [1]), I obtained some fixed point theorems for bounded weakly closed operators in Hilbert and Banach spaces.

With regard to the Hilbert spaces I showed that if X is a Hilbert space, $T\colon S_r \to X$ a bounded weakly closed operator, defined in the ball S_r with center $0 \in X$ and radius $r > 0$, which satisfies the boundary condition:

$$(1)\quad \begin{cases} \text{there exists } K \in]0,1[\text{ and } a \in X \text{ with } \|a\| = 1 \text{ such that for all} \\ x \in \partial S_r \text{ at least one of the following conditions is satisfied:} \\ \text{i)}\quad (x, T(x)) \leqslant K\|x\|^2, \\ \text{ii)}\quad \begin{cases} (a, x - T(x)) \geqslant 1 - K & \text{if } (a, x) > 0, \\ (a, x - T(x)) \leqslant -(1 - K) & \text{if } (a, x) < 0, \end{cases} \end{cases}$$

and if, furthermore, in the symmetrical points of ∂S_r the same condition (either i) or ii)) is satisfied, then the topological degree $d(I - T; S_r; 0)$ ($I =$ identity) is different from zero and so T has a fixed point in S_r.

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It is then easy to see that, (see [6]), i) can be improved by putting $K = 1$. In this way the Theorem exposed before appears like an extension of a Theorem of M. Shinbrot, (see [4]), according to which all the operators $T\colon S_r \to X$, (where X is a separable Hilbert space), which are weakly continuous [1] and satisfy the condition:

$$(2) \qquad (x, T(x)) \leqslant \|x\|^2 \qquad \text{for each } x \in \partial S_r$$

have a fixed point in S_r.

Concerning, on the other hand Banach spaces, I constructed the topological degree by the same method used in Hilbert spaces, but for bounded weakly closed operators satisfying the condition:

$$(3) \qquad \|T(x)\| \leqslant \|x\| \qquad \text{for all } x \in \partial S_r .$$

It is easy to prove that the degree $d(I - T; S_r; 0)$ of the operators which satisfy (3) is equal to $1 \neq 0$ and so T has a fixed point in S_r.

In this paper, utilizing an approximation method, like that used by M. Shinbrot in [4], I show another fixed point theorem in Banach spaces with particular properties.

To be more precise this theorem asserts that: if X is a reflexive Banach space in which the duality mapping is single-valued and $T\colon S_r \to X$ is a weakly continuous operator such that:

$$(4) \qquad J(x)(T(x)) \leqslant \|x\|^2 \qquad \text{for all } x \in \partial S_r \ (J = \text{duality mapping})$$

then T has a fixed point in S_r.

It is obvious that condition (4) is much better than (3) and has a much wider field of applicability.

Moreover this Theorem is an extension of Shinbrot's Theorem to Banach spaces. In fact, if X is a Hilbert space, the number $J(x)(y)$, $x, y \in X$, coincides exactly with the inner product (x, y).

This paper is divided in two sections.

In the first, after recalling the necessary definitions and theorems, I prove the fixed point theorem enunciated above.

In the second, I use this theorem to study, in a suitable Sobolev space, Burgers-Hopfg's equation:

$$D_t u - K(u) D_x^2 u + u D_x u = f \qquad f \in L^4$$

where the viscosity $K(u)$ is not constant.

For this equation I show an existence theorem in the case when $K(u)$ has a sufficiently small oscillation and the norm of f in L^4 is small enough.

(1) Remark that if X is a reflexive Banach space, the following equivalence holds: T bounded and weakly closed $\Leftrightarrow$ T weakly continuous.

Recently equation (5) has been studied by A. Canfora (see [2]), using the topological degree theory stated above. For (5) A. Canfora establishes an existence theorem, again in the case of variable viscosity (but with an arbitrarily large oscillation), by supposing that f belongs to L^2 and has, in L^2, a small enough norm.

In connexion with this I want to notice that the existence theorem, proved in section 2, is, precisely, a regularity theorem, because I show that, if f is in $L^4 \subset L^2$, the solution u of (5) belongs to the space $H^4_{2,1}(\Omega_\tau)$ obtained closing

$$C^{2,1}_\tau = \{u(x, t) \text{ real}: u, D_x u, D_t u, D^2_x u \in C^0(\bar{\Omega}_\tau);\ u(x, 0) = u(0, t) = u(1, t) = 0\} \qquad 0 \leqslant x \leqslant 1,\ 0 \leqslant t \leqslant \tau$$

with respect to the norm:

$$|||u|||_\tau = \|u\|_{L^4} + \|D_t u\|_{L^4} + \|D_x u\|_{L^4} + \|D^2_x u\|_{L^4}$$

whereas in [2] each solution of (5) obtained in correspondence of $f \in L^2$, belongs to the space $H^2_{2,1}(\Omega_\tau)$ which is the closure of $C^{2,1}_\tau$ with respect to the norm:

$$\|u\|^*_\tau = \|u\|_{L^2} + \|D_t u\|_{L^2} + \|D_x u\|_{L^2} + \|D^2_x u\|_{L^2}.$$

The last remark to be done is that the study of (5) is set in spaces with a sommability exponent equal to 4 only for semplicity, but the proceding used in section 2 could be repeated, replacing L^4 with any L^p ($p \geqslant 2$) and effecting the necessary modifications.

1. - The fixed point theorem

Let X be a reflexive Banach space, with a strictly convex dual X'. From this it follows that the norm on X in Gateaux differentiable and hence, by putting $f(x) = \frac{1}{2}\|x\|^2$, it makes sense to consider the following map $\varphi: X \times \times X \to R$ so defined:

$$\varphi(x, y) = f'(x)(y) = \lim_{t \to 0} \frac{f(x + ty) - f(x)}{t}. \tag{1.1}$$

The following propositions are equivalent: (see [5])

(a) the norm on X is Gateaux differentiable,

(b) $\varphi(x, y)$ is homogeneous in x,

(c) $\varphi(x, y)$ is homogeneous in y,

(d) $\varphi(x, y)$ is a linear in y,

(e) $\varphi(x, y) = J(x)(y)$ ($J =$ duality mapping),

(f) J is single valued

The map φ has also the properties:

p$_1$) $\|x\| = \sqrt{\varphi(x, x)}$,

p$_2$) $|\varphi(x, y)| \leq \|x\| \, \|y\|$ (Cauchy-Schwartz-Buniakovsky inequality),

p$_3$) if X is an inner product space with inner product (x, y), then $(x, y) = \varphi(x, y)$ for all $x, y \in X$.

Now suppose that X has a Schauder basis, that is there exists a sequence $\{e_n\} \subset X$, such that: for each $x \in X$, $x = \sum_{i=1}^{+\infty} \alpha_i e_i$, where α_i are real numbers.

It is not very difficult to find Banach spaces with this property; in fact it is enough to consider the L^p spaces with $p > 1$ (see [8] and [9] for further explanations).

Afterwards let $S_r \subseteq X$ be the closed ball with center $0 \in X$ andr adius $r > 0$, and $T\colon S_r \to X$ a weakly continuous mapping satisfying the condition:

(1.2) $$\varphi(x, T(x)) \leq \|x\|^2 \qquad \text{for all } x \in \partial S_r .$$

In this section I intend to show the following fixed point Theorem:

THEOREM 1.1: *Let X be a reflexive Banach space, endowed with a Schauder basis and with a strictly convex dual X'.*

If $T\colon S_r \subseteq X \to X$ is a weakly continuous operator satisfying (1.2), *then T has a fixed point in S_r.*

Before proving the theorem remark that it is enough to consider the case when T satisfies, instead of (1.2), the more restrictive condition:

(1.3) $$\text{there exists } K \in]0, 1[\text{ such that: } \varphi(x, T(x)) < K\|x\|^2 \text{ for all } x \in \partial S_r .$$

In fact, once one assumes that Theorem 1.1 holds for this type of operators and considers the sequence $\{T_n = n/(n+1)\, T\}$, it is easy to show (regarding this, see [6]) that $\{T_n\}$ is a sequence of weakly continuous mappings which satisfy (1.3) and converges to T uniformly in S_r. Moreover the sequence $\{x_n\} \subseteq S_r$, formed by the fixed points of the operators T_n converges, in thə weak-topology, to a fixed point of T.

PROOF OF THEOREM 1.1: Since X has a Schauder basis there exists a sequence $\{e_n\} \subset X$, such that: $x = \sum_{i=1}^{+\infty} \alpha_i e_i$ for all $x \in X$.

Consider the sequence $\{P_n\}$ of linear continuous operators defined by:

$$P_n(x) = P_n\Big(\sum_{i=1}^{+\infty} \alpha_i e_i\Big) = \sum_{i=1}^{n} \alpha_i e_i \qquad \text{for each } x \in X .$$

Of course, for all $n \in N$, the range of the projector P_n is included in a finite dimensional subspace $X_n \subset X$.

Let Φ_n be the restriction of the mapping $P_n T$ to the set $S_r \cap X_n$; from the continuity of P_n and the weak-continuity of T it follows immediately that $\Phi_n \colon S_r \cap X_n \to X_n$ is a continuous map.

Moreover it holds:

$$(1.4) \qquad \lim_{n \to +\infty} \varphi(x, P_n T(x)) = \varphi(x, T(x)) \quad \text{uniformly in } S_r .$$

To prove (1.4) observe that, if I denotes the identity in X, it turns out that:

$$|\varphi(x, P_n T(x)) - \varphi(x, T(x))| \leq \|x\| \, \|P_n T(x) - T(x)\| \leq$$
$$\leq \|x\| \, \|P_n - I\| \, \|T(x)\| \underset{n}{\to} 0 \qquad \text{uniformly in } S_r .$$

In fact $P_n \to I$ in the space of the continuous linear operators from X to X and $\|T(x)\|$ is bounded, for x belonging to S_r, because if it was not true there would exist a sequence $\{x_n\} \subseteq S_r$ such that $\|T(x_n)\| \to \infty$, in contradiction with the assumption that T is weakly continuous.

From (1.3) and (1.4) it follows that:

(1.5) there exists $\nu \in N$ such that for all $n > \nu$ and for all $x \in \partial S_r$

$$\varphi(x, P_n T(x)) < K\|x\|^2$$

and consequently:

(1.6) for all $n > \nu$ and for all $x \in \partial(S_r \cap X_n)$ $\varphi(x, \Phi_n(x)) < K\|x\|^2$.

(1.6) allows me to say that the topological degree $d(I - \Phi_n; S_r \cap X_n; 0)$ is different from zero and so Φ_n has a fixed point in $S_r \cap X_n$. In fact, putting $\Phi_u^t(x) = x - t\Phi_n(x)$, $t \in [0, 1]$, I obtain:

$$\varphi(x, \Phi_n^t(x)) = \varphi(x, x - t\Phi_n(x)) = \varphi(x, x) - t\varphi(x, \Phi_n(x)) =$$
$$= \|x\|^2 - t\varphi(x, \Phi_n(x)) > \|x\|^2 - tK\|x\|^2 > (1 - K)\|x\|^2 > 0$$

for all $x \in \partial(S_r \cap X_n)$ and this implies that $\Phi_n^t(x)$ is different from zero, for all $x \in \partial(S_r \cap X_n)$ and $t \in [0, 1]$. Therefore, in virtue of the homotopy degree property,

$$d(I - \Phi_n; S_r \cap X_n; 0) = d(\Phi_n^1; S_r \cap X_n; 0) = d(\Phi_n^0; S_r \cap X_n; 0) =$$
$$= d(I; S_r \cap X_n; 0) = 1 \neq 0 .$$

Then let $\{x_n\}_{n > \nu} \subset S_r$ be the sequence of the fixed points of the mappings Φ_n. Since X is a reflexive space and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ which converges, in the weak topology, to a point $x \in S_r$.

On the other hand, from the assumption of weak continuity over T, it follows that $T(x_{n_k}) \rightharpoonup T(x)$ and this, together with the properties of P_{n_k}, gives:

$$|\langle y, P_{n_k} T(x_{n_k})\rangle - \langle y, T(x)\rangle| \leqslant |\langle y, P_{u_k} T(x_{n_k}) - T(x_{n_k})\rangle| +$$
$$+ |\langle y, T(x_{n_k}) - T(x)\rangle| \to 0 \qquad \text{for all } y \in X',$$

that is $P_{n_k} T(x_{n_k}) \to T(x)$.

After all I have:

$$x_{n_k} = \Phi_{n_k}(x_{n_k}) = P_{n_k} T(x_{n_k}) \rightharpoonup T(x) \quad \text{and} \quad x_{n_k} \rightharpoonup x. \tag{1.7}$$

But this clearly implies that X is equal to $T(x)$ as I wanted to prove.

REMARK 1.1: The previous Theorem is valid even if, instead of supposing that X has a Schauder basis, one assumes that X is of a more general type, precisely a π_λ space ($\lambda \geqslant 1$), (see [9]).

That means that $X = \overline{\bigcup_{\alpha \in A} E_\alpha}$ where $\{E_\alpha\}_{\alpha \in A}$ is a set, directed by inclusion, of finite dimensional subspaces of X, such that, for every $\alpha \in A$, there is a projection P_α from X onto E_α with $\|P_\alpha\| \leqslant \lambda$.

In fact it could be proved that, in this case, one can repeat the same proceeding, considering, instead of the sequence $\{P_n\}$, a uniformly bounded net $\{P_\alpha\}_{\alpha \in A}$ of projections which tends strongly to the identity on X and whose existence is guaranteed by being X a π_λ space.

2. - AN APPLICATION

Let Ω_τ be, for all real positive number τ, the open rectangle $]0,1[\times$ $\times]0,\tau[$ and $\mathcal{H}_\tau = H^4_{2,1}(\Omega_\tau)$ the Sobolev space obtained closing

$$C^{2,1}_\tau = \{u(x,t) \text{ real}: u, D_x u, D_t u, D^2_x u \in C^0(\bar{\Omega}_\tau);\ u(x,0) = u(0,t) = u(1,t) = 0\}$$
$$0 \leqslant x \leqslant 1,\ 0 \leqslant t \leqslant \tau$$

with respect to the norm:

$$|||u|||_\tau = \|u\|_{L^4} + \|D_t u\|_{L^4} + \|D_x u\|_{L^4} + \|D^2_x u\|_{L^4} \tag{2.1}$$

where $\|\cdot\|_{L^4} = \|\cdot\|_{L^4(\Omega_\tau)}$ is defined in the usual way.

Remark that the heat operator $L_0 u = D_t u - D^2_x u$ is an algebraical topological isomorphism between $\mathcal{H}_\tau$ and $L^4(\Omega_\tau)$. That allows me to introduce in $\mathcal{H}_\tau$ another norm, equivalent to (2.1):

$$\|u\|_\tau = \left(\int_{\Omega_\tau} (D_t u - D_x u)^4 \, dx\, dt\right)^{\frac{1}{4}} = \|L_0 u\|^{L^4}. \tag{2.2}$$

Moreover, since each function $u \in \mathcal{K}_\tau$ belongs also to $H^2_{2,1}(\Omega_\tau)$ (²), I can utilize the results of [2] and say that all function $u \in \mathcal{K}_\tau$ can be extended to the whole $\bar{\Omega}_\tau$ like a Holder continuous function and the following estimates hold:

$$\text{(2.3)} \qquad \sup_{\substack{(x,t)\\(x',t')}\big\}\in\bar{\Omega}_\tau} \frac{|u(x,t)-u(x',t')|}{|x-x'|^{\beta_1}+|t-t'|^{\beta_2}} \leqslant \gamma\|u\| \qquad \text{for all } u\in\mathcal{K}_\tau$$

$$\text{(2.3)}' \qquad |u(x,t)| \leqslant \delta\|u\|_\tau \qquad \text{for each } (x,t)\in\bar{\Omega}_\tau \text{ and } u\in\mathcal{K}_\tau$$

with $0<\beta_1<\frac{1}{2}$, $0<\beta_2<\frac{1}{4}$ and $\delta=\tau^{\frac{1}{4}}$.

Now consider the operator $A\colon \mathcal{K}_\tau \to L^4(\Omega_\tau)$ defined by:

$$Au = D_t u - K(u) D^2_x u + u D_x u$$

where $K(\cdot)$ is a continuous functional on $C^0(\bar{\Omega}_\tau)$ such that $K(u) = 1 + \nu(u)$, with $\nu(u) \geqslant 0$, for all $u \in C^0(\bar{\Omega}_\tau)$.

I am interested in studying the equation:

$$\text{(2.4)} \qquad Au = f \qquad f\in L^4(\Omega_\tau),\ u\in\mathcal{K}_\tau .$$

In order to solve (2.4) I will use the fixed point theorem of the previous section. To do this I need to transform problem (2.4) in a fixed point problem for a suitable operator T satisfying the properties stated in Theorem 1.1.

For this purpose remark that $\mathcal{K}_\tau$ is a reflexive Banach space with a strictly convex dual and endowed with a Schauder basis.

Moreover, putting

$$\text{(2.5)} \qquad \varphi_\tau(u,v) = \frac{1}{\|u\|^2_\tau}\int_{\Omega_\tau} (L_0 u)^3 L_0 v \, dx\, dt = \varphi_{L^4}(L_0 u, L_3 v) \qquad u, v\in\mathcal{K}_\tau$$

and recalling the expression of the duality map in L^4 ([13]) I can assert that $\varphi_\tau(\cdot,\cdot)\colon \mathcal{K}_\tau\times\mathcal{K}_\tau \to R$ is exactly the duality map $J(u)(v)$ related to the space $\mathcal{K}_\tau$ endowed with the norm (2.2) and so it coincides with the mapping φ used in Theorem 1.1.

The next remark to be done is that, if $L_0^{-1}\colon L^4(\Omega_\tau)\to\mathcal{K}_\tau$ is the inverse operator of L_0 and f is fixed in $L^4(\Omega_\tau)$, then equation (2.4) is equivalent to:

$$\text{(2.6)} \qquad u + L_0^{-1}(Au) = L_0^{-1} f + u \qquad u\in\mathcal{K}^\tau .$$

But (2.6), by putting $T_f u = L_0^{-1} f - L_0^{-1}(Au) + u$, $T_f = T\colon \mathcal{K}_\tau \to \mathcal{K}_\tau$, becames:

$$\text{(2.7)} \qquad u = T(u) \qquad u\in\mathcal{K}_\tau .$$

(²) $H^2_{2,1}(\Omega_\tau)$ is the Sobolev space obtained closing $C^{2,1}_\tau$ with respect to the norm:

$$\|u\|^*_\tau = \|u\|_{L^2} + \|D_t u\|_{L^2} + \|D_x u\|_{L^2} + \|D^2_x u\|_{L^2} .$$

At this point, on the basis of Theorem 1.1, I will be able to solve (2.4), for a given $f \in L^4(\Omega_\tau)$, if I prove that T is weakly continuous and satisfies (1.2), with φ defined by (2.5).

I begin with:

LEMMA 2.1: *T: $\mathcal{H}_\tau \to \mathcal{H}_\tau$ is a weakly continuous operator.*

PROOF: As L_0^{-1} is a linear continuous operator, hence weakly continuous, it suffices to prove that A: $\mathcal{H}_\tau \to L^4(\Omega_\tau)$ is weakly continuous. To do this it is enough to show that A is weakly compact and weakly closed.

Since $\mathcal{H}_\tau$ is a reflexive space, the weak compactness of A follows easily from its boundness.

So it remains to be proved only that A is weakly closed; namely:

$$\text{(2.8)} \qquad \left\{ \begin{array}{l} u_n \underset{\mathcal{H}_\tau}{\rightharpoonup} u \\ Au_n \underset{L^4(\Omega_\tau)}{\rightharpoonup} v \end{array} \right. \Rightarrow Au = v\,.$$

In order to verify (2.8) observe that the sequence $\{u_n\}$, being weakly convergent, is bounded in $\mathcal{H}_\tau$ and this, for (2.3), (2.3)′ and Ascoli-Arzelà's Theorem, implies that there exists a subsequence $\{u_{n_k}\}$, which converges to u in $C^0(\bar{\Omega}_\tau)$.

Then, by the assumption of continuity of the functional $K(\cdot)$ I have that $K(u_{n_k}) \to K(u)$ in R. Moreover, still from the weak convergence of $\{u_n\}$ to u, in $\mathcal{H}_\tau$, it follows that the sequences $\{u_n\}$, $\{D_t u_n\}$, $\{D_x u_n\}$, $\{D_x^2 u_n\}$ converge, in the weak-topology of $L^4(\Omega_\tau)$, respectively to u, $D_t u$, $D_x u$ and $D_x^2 u$.

Now, let w be a function belonging to $L^{4/3}(\Omega_\tau)$; writing $\langle w, \psi\rangle$ instead of $\int_{\Omega_\tau} w\psi$, $\psi \in L^4(\Omega_\tau)$ I have:

$$\begin{aligned} &|\langle w, Au_{n_k}\rangle - \langle w, Au\rangle| = \\ &\qquad = |\langle w, D_t u_{n_k} - K(u_{n_k}) D_x^2 u_{n_k} + u_{n_k} D_x u_{n_k} - D_t u + K(u) D_x^2 u - u D_x u\rangle| \leqslant \\ &\qquad \leqslant |\langle w, D_t u_{n_k} - D_t u\rangle| + |\langle w, [K(u_{n_k}) - K(u)] D_x^2 u_{n_k}\rangle| + \\ &\qquad + |\langle w, K(u)[D_x^2 u_{n_k} - D_x^2 u]\rangle| + |\langle w, [u_{n_k} - u] D_x u_{n_k}\rangle| + \\ &\qquad\qquad + |\langle w, u[D_x u_{n_k} - D_x u]\rangle| \to 0\,. \end{aligned}$$

This proves that $Au_{n_k} \rightharpoonup Au$ in $L^4(\Omega_\tau)$ and hence $Au = v$, as I wanted to show.

At this point remark that, from the equivalence between the norms (2.1) and (2.2), it follows that:

$$\text{(2.9)} \qquad \left| \begin{array}{l} \text{there exists } c > 0 \text{ such that:} \\ \qquad |||u|||_\tau \leqslant c\|u\|_\tau = c\|L_0 u\|_{L^4} \qquad \text{for all } u \in \mathcal{H}_\tau\,. \end{array} \right.$$

Lemma 2.2: *If there exists a constant* $0 \leqslant M < 1/4c^4$ *such that:*

$$\text{(2.10)} \qquad \begin{cases} 1 \leqslant K(u) = 1 + \nu(u) \leqslant 1 + M\,, \\ \|f\| \leqslant \dfrac{(1-4Mc^4)^2}{4c\delta} \qquad (\delta = \tau^{\frac{1}{4}})\,, \end{cases}$$

then:

$$\text{(2.11)} \qquad \left|\; \begin{array}{l} \textit{there exists } r > 0 \textit{ such that: for all } u \in \partial S_r \\ \qquad \varphi_\tau(u, T(u)) \leqslant \|u\|_\tau^2 = \|L_0 u\|_{L^4}^2\,. \end{array}\right.$$

Proof: By (2.5) I have:

$$\varphi_\tau(u, T(u)) = \frac{1}{\|u\|_\tau^2} \int_{\Omega_\tau} (L_0 u)^3 (f - Au + L_0 u)\, dx\, dt =$$

$$= \frac{1}{\|u\|_\tau^2} \left[\int_{\Omega_\tau} (L_0 u)^3 f\, dx\, dt - \int_{\Omega_\tau} (L_0 u)^3 Au\, dx\, dt \right] + \|L_0 u\|_{L^4}^2 =$$

$$= \varphi_{L^4}(L_0 u, f) - \frac{1}{\|u\|_\tau^2} \int_{\Omega_\tau} (L_0 u)^3 Au\, dx\, dt + \|u\|_\tau^2\,.$$

Thus in order to obtain (2.11), it will be enough to prove that:

$$\text{(2.12)} \qquad \left|\; \begin{array}{l} \text{there exists } r > 0 \text{ such that: for all } u \in \partial S_r \\ \qquad \varphi_{L^4}(L_0 u, f) \leqslant \dfrac{1}{\|u\|_\tau^2} \displaystyle\int_{\Omega_\tau} (L_0 u)^3 Au\, dx\, dt = \varphi_{L^4}(L_0 u, Au)\,. \end{array}\right.$$

For this purpose it is convenient to estimate the second part of (2.12):

$$\varphi_{L^4}(L_0 u, Au) = \frac{1}{\|u\|_\tau^2} \int_{\Omega_\tau} \{[(D_t u)^3 - (D_x^2 u)^3 - 3(D_t u)^2 D_x^2 u + 3 D_t u (D_x^2 u)^2] \cdot$$

$$\cdot [D_t u - K(u) D_x^2 u + u D_x u]\}\, dx\, dt = \frac{1}{\|u\|_\tau^2} \int_{\Omega_\tau} \{(D_t u)^4 - K(u) D_x^2 u (D_t u)^3 +$$

$$+ u D_x u (D_t u)^3 - (D_x^2 u)^3 D_t u + K(u)(D_x^2 u)^4 - u D_x u (D_x^2 u)^3 - 3 (D_t u)^3 D_x^2 u +$$

$$+ 3K(u)(D_t u)^2 (D_x^2 u)^2 - 3u D_x u (D_t u)^2 D_x^2 u + 3 (D_t u)^2 (D_x^2 u)^2 +$$

$$- 3K(u) D_t u (D_x^2 u)^3 + 3u D_t u (D_x^2 u)^2 D_x u\}\, dx\, dt\,.$$

Recalling that $K(u) = 1 + \nu(u)$ and $0 \leqslant \nu(u) \leqslant M$, I obtain:

$$\varphi_{L^4}(L_0 u, Au) = \frac{1}{\|L_0 u\|_{L^4}^2} \int_{\Omega_\tau} (D_t u - D_x u)^4\, dx\, dt +$$

$$+ \frac{1}{\|L_0 u\|_{L^4}^2} \int_{\Omega_\tau} u D_x u [D_t u - D_x^2 u]^3\, dx\, dt + \frac{1}{\|L_0 u\|_{L^4}^2} \cdot$$

$$\cdot \int_{\Omega_\tau} \{-\nu(u) D_x^2 u (D_t u)^3 + \nu(u)(D_x^2 u)^4 + 3\nu(u)(D_t u)^2 (D_x^2 u)^2 - 3\nu(u) D_t u (D_x^2 u)^2\} \cdot$$

$$\cdot dx\, dt \geqslant \|L_0 u\|_{L^4}^2 + \varphi_{L^4}(L_0 u, u D_x u) - \frac{\nu(u)}{\|L_0 u\|_{L^4}^2} \int_{\Omega_\tau} (D_t u)^3 D_x^2 u \, dx\, dt -$$

$$- \frac{3\nu(u)}{\|L_0 u\|_{L^4}^2} \int_{\Omega_\tau} (D_x^2 u)^3 D_t u \, dx\, dt \,.$$

On the other hand, by means of (2.9), (2.3)′ and the properties of the duality mapping, stated in the previous section (see Prop. p_2)) I have:

$$\varphi_{L^4}(L_0 u, Au) \geqslant \|L_0 u\|_{L^4}^2 - \|L_0 u\|_{L^4} \|u D_x u\|_{L^4} - \frac{\nu(u)}{\|L_0 u\|_{L^4}^2} \varphi_{L^4}(D_t u, D_x^2 u) \|D_t u\|_{L^4}^2 +$$

$$- \frac{3\nu(u)}{\|L_0 u\|_{L^4}^2} \varphi_{L^4}(D_x^2 u, D_t u) \|D_x^2 u\|_{L^4}^2 \geqslant \|L_0 u\|_{L^4}^2 - c\delta \|L_0 u\|_{L^4}^3 +$$

$$- \frac{\nu(u)}{\|L_0 u\|_{L^4}^2} \|D_t u\|_{L^4}^3 \|D_x^2 u\|_{L^4} - \frac{3\nu(u)}{\|L_0 u\|_{L^4}^2} \|D_x^2 u\|_{L^4}^3 \|D_t u\|_{L^4} \geqslant \|L_0 u\|_{L^4}^2 - c\delta \|L_0 u\|_{L^4}^3 -$$

$$- Mc^4 \|L_0 u\|_{L^4}^2 - 3Mc^4 \|L_0 u\|_{L^4}^2 = (1 - 4Mc^4)\|L_0 u\|_{L^4}^2 - c\delta \|L_0 u\|_{L^4}^3 \,.$$

It is then easy to see that, on the basis of the previous inequality and under the assumptions over M and f, (2.12) is satisfied by all the real numbers $r > 0$ belonging to the interval

$$\left[\frac{1 - 4Mc^4 - \sqrt{(1 - 4Mc^4)^2 - 4c\delta\|f\|}}{2c\delta}, \quad \frac{1 - 4Mc^4 + \sqrt{(1 - 4Mc^4)^2 - 4c\delta\|f\|}}{2c\delta}\right].$$

In fact, as soon as r belongs to this interval and the norm of u is equal to $r > 0$, I get:

$$(2.13) \qquad c\delta \|L_0 u\|_{L^4}^3 - (1 - 4Mc^4)\|L_0 u\|_{L^4}^2 + \|f\|_{L^4} \|L_0 u\|_{L^4} \leqslant 0$$

from which, applying another time property p_2 of the duality map, it follows that:

$$\varphi_{L^4}(L_0 u, f) \leqslant \|f\|_{L^4} \|L_0 u\| \leqslant (1 - 4Mc^4)\|L_0 u\|_{L^4}^2 - c\delta \|L_0 u\|_{L^4}^3 \leqslant \varphi_{L^4}(L_0 u, Au) \,.$$

This proves (2.12).

Finally, on the basis of Lemma 2.1 and 2.2 I can enunciate the following:

THEOREM 2.1: *If $K(u)$ verifies the condition:*

$$1 \leqslant K(u) \leqslant 1 + M \leqslant 1 + \frac{1}{4c^4},$$

then, for each $f \in L^4(\Omega_\tau)$, *with*

$$\|f\|_{L^4} \leqslant \frac{(1-4Mc^4)^2}{4c\delta}$$

there exists a solution $u \in \mathcal{H}_\tau$ *of the problem* (2.4) *with*

$$\|u\|_\tau \leqslant \frac{1-4Mc^4+\sqrt{(1-4Mc^4)^2-4c\delta\|f\|}}{2\delta c}\,.$$

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