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Biorthogonal Systems in Metrizable Locally Convex Spaces (**)

SUMMARY. — Every separable metrizable locally convex space X has two quasi complementary subspaces equipped with bases with brackets, moreover X has a one-norming M-basis. In particular, if X is isomorphic to R^N, there exists an overfilling convex basis.

Sistemi biortogonali in spazi metrizzabili e localmente convessi

RIASSUNTO. — Ogni spazio localmente convesso X, separabile e metrizzabile, ha due sottospazi quasi complementari dotati di basi con parentesi, inoltre X ha una M-base con la proprietà di essere 1-normante. In particolare, se X è isomorfo a R^N, esiste una base «overfilling» in senso convesso.

INTRODUCTION

In the Note X is a metrizable locally convex space, with the quasi-norm

$$(1) \qquad ((x)) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(x)}{1+p_n(x)},$$

where (p_n) is the sequence of the seminorms which define the topology.

The first aim of the Note is to single out the most regular sequence which can represent a separable X, by extending where it is possible the theory of the normed spaces. This is the subject of § 1 and 2.

The most regular known sequence, complete in a separable Banach space B, is the norming M-basis; that is an M-basis (x_n) of B, with (x_n, f_n) biorthogonal, such that (see [6] p. 225 and [3] p. 115; see also [3], Remark of p. 44)

$$(*) \qquad \begin{cases} \|x\| < \sup\{|f(x)|;\, f \in \operatorname{span}(f_n),\ \|f\| < K\}, \\ \text{where } K > 1, \text{ for every } x \text{ of } B. \end{cases}$$

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Firstly we extend this definition, in the framework of the locally convex spaces. Let $(f_n) \subset X^*$ (the dual of X) such that

$$((f_n))_n = \sup\{|f_n(x)|;\, x \in X,\, p_n(x) < 1\} < \infty\,, \qquad \text{for every } n\,;$$

then we set

$$(2) \qquad \begin{cases} (((f_n))) = \sup\limits_n ((f_n))_n\,; \\ ((x))_{(f_n)} = \sum\limits_{n=1}^{\infty} \dfrac{1}{2^n} \dfrac{|f_n(x)|}{1+|f_n(x)|}\,, \qquad \text{for every } x \text{ of } X\,. \end{cases}$$

Then, if (x_n) is complete in X with (x_n, f_n) biorthogonal, we say that (x_n) is *K-norming M-basis* of X, where $1 < K < \infty$, if, for every m and for every x of X,

$$(3) \qquad \begin{cases} p_m(x) < \sup\{|g(x)|;\, g \in \operatorname{span}(f_n),\, ((g))_m < K\}\,, \\ \text{hence } ((x)) < \sup\{((x))_{(g_n)};\, (g_n) \subset \operatorname{span}(f_n),\, (((g_n))) < K\}\,. \end{cases}$$

This seems to be the best extension of (∗): indeed, as in the normed spaces, the norming M-basis has geometric characterizations and is union of two basic with brackets sequences.

It follows that every separable X has a 1-norming M-basis, moreover every linearly independent sequence of X has a basic block sequence.

The second aim of the Note is to let light where the theory of the locally convex spaces goes away from the theory of the normed spaces.

It is known that X has a continuous norm if and only if X has no subspace isomorphic to R^N (the space of the real sequences). If X has a continuous norm every linearly independent sequence of X has an ω-independent subsequence [4]; on the other hand V.M. Kadec stated ([6] p. 858, without proof) that in R^N there exists a linearly independent sequence which has no ω-independent subsequences.

We say that (x_n) is *overfilling convex basis* of X if, for every $(n_k)_{k=1}^{\infty} \subseteq (n)$ and for every $\bar{x}$ of X, there exists $(\bar{a}_k)$ of numbers such that

$$\bar{x} = \sum_{k=1}^{\infty} \bar{a}_k x_{n_k}\,, \quad \text{with} \quad \sum_{k=1}^{\infty} |\bar{a}_k| < 1\,.$$

Then § 3 is essentially a proof of Kadec's statement, since, if there are subspaces isomorphic to R^N, we prove the existence of overfilling convex basic sequences.

In § 1′, 2′ and 3′ there are the proofs of § 1, 2 and 3 respectively.

Standard definitions and notations

If $(x_n) \cup (y_n) \subset X$, we recall that (y_n) is said to be *block sequence* of (x_n) if there exists an increasing sequence (t_n) of natural numbers so that, setting

$t_0 = 0$,

$$y_m \in \operatorname{span}(x_n)_{n=t_{m-1}+1}^{t_m} \qquad \text{for every } m.$$

Moreover (x_n) is said to be *complete* in X if $[x_n]$ $(=\overline{\operatorname{span}}\,(x_n)) = X$. If $Y \subset X$ and $F \subset X^*$ we set

$$Y^\perp = \{f \in X^*; f(x) = 0 \text{ for every } x \text{ of } Y\};$$

$$F_\perp = \{x \in X; f(x) = 0 \text{ for every } f \text{ of } F\};$$

moreover we say that F is *total on* X if $F_\perp = \{0\}$.

Let $(x_n) \subset X$ and $(f_n) \subset X^*$, (x_n, f_n) is said to be *biorthogonal system* if

$$f_m(x_n) = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n, \end{cases} \qquad \text{for every } m \text{ and } n;$$

it is the same as saying that (x_n) is *minimal*, that is $x_m \notin [x_n]_{n \neq m}$ for every m. Let (x_n, f_n) be biorthogonal, (x_n) is said to be

(i) *M-basis* of X if (x_n) is complete in X and $[f_n]$ is total on X;

(ii) *basis with brackets* of X *for* (t_n), where (t_n) is an increasing sequence of natural numbers, if, setting $t_0 = 0$,

$$x = \sum_{m=1}^{\infty} \Big\{ \sum_{n=t_{m-1}+1}^{t_m} f_n(x) x_n \Big\}, \qquad \text{for every } x \text{ of } X;$$

(iii) *basis* of X if in (ii) $t_n = n$ for every n.

Remark that we use « basis » for « Schauder basis » ([5] p. 144 and p. 152).

Moreover (x_n) is said to be *M-basic* (*basic with brackets*) (*basic*) if it is M-basis (basis with brackets) (basis) of $[x_n]$.

Finally, for $Y \subseteq X$ and x of X,

$$\operatorname{dist}(x, Y) = \inf\{\|x + y\|; y \in Y\}.$$

§ 1. - Geometric properties and characterizations of the norming *M*-bases

From the theory of the normed spaces we recall that (see [5] p. 58; see also [3] p. 2)

(x_n) is basis in a normed space $\Leftrightarrow$

$\Leftrightarrow$ *there exists* $1 \leq K < \infty$ *so that, for every* $(a_n)_{n=1}^{m+p}$ *of numbers,*

$$\Big\|\sum_{n=1}^{m} a_n x_n\Big\| \leq K \Big\|\sum_{n=1}^{m+p} a_n x_n\Big\|.$$

Then, if $(x_n) \subset X$ and (t_n) is an increasing sequence of natural numbers, we call

(4)

basis with brackets constant for (t_n) the number: $\inf K$ so that, for every m and for every $x \in [x_k]_{k=1}^{t_m}$

$$((x)) < K \operatorname{dist}(x, [x_n]_{n > t_m}) ;$$

basis constant the number: $\inf K$ so that, for every m and for every $x \in [x_k]_{k=1}^{m}$,

$$((x)) < K \operatorname{dist}(x, [x_n]_{n > m}) .$$

That is a basis constant is a basis with brackets constant for (t_n) with $t_n = n$ for every n.

The basis constant characterizes particular basic sequences, the same for the basic with brackets sequences, indeed:

Proposition 1: *a)* (x_n) *has a finite basis constant* $\Rightarrow (x_n)$ *is basic;*

b) (x_n) *is linearly independent and has a finite basis with brackets constant for* $(t_n) \Rightarrow (x_n)$ *is basic with brackets for* (t_n).

Now we pass to the norming M-bases.

A geometric characterization is:

Proposition 2: *Suppose* (x_n) *minimal and* $1 < K < \infty$, *then:* (x_n) *is K-norming M-basic* $\Leftrightarrow$ *for every m and for every x of* $[x_k]$,

(5) $$p_m(x) < K \sup_n \{\inf \{p_m(x+y) ; y \in [x_k]_{k > n}\}\} .$$

However, in the locally convex case, the basis constant and the norming M-basis do not keep the same properties of the normed spaces. Precisely the basis constant does not characterize the basic sequences (the same for the basic with brackets sequences); while the norming M-basis has some properties stronger than for the basis; indeed

Example : Let

$$(x_n) = \bigcup_{m=1}^{\infty} (x_{mn})_{n=1}^{\infty}$$

be a linearly independent sequence of a linear space.

Then, for every

$$((a_{mn})_{n=1}^{km})_{m=1}^{r}$$

of numbers, set

$$(6)\quad \left|\begin{array}{l} p_1\left(\sum_{m=1}^{r}\sum_{n=1}^{h_m} a_{mn}x_{mn}\right)=\sum_{m=1}^{r}\sum_{n=1}^{h_m}\frac{|a_{mn}|}{2^n}, \\ p_k\left(\sum_{m=1}^{r}\sum_{n=1}^{h_m} a_{mn}x_{mn}\right)=\begin{cases} \left|\sum_{n=1}^{h_{k-1}} a_{k-1,n}\right| + \sum_{n=1}^{h_{k-1}}\frac{|a_{k-1,n}|}{2^{k-1}} & \text{if } 1<k<r+1, \\ 0 & \text{if } k>r+1. \end{cases} \end{array}\right.$$

Let X be the completion of span (x_n) in the topology of (1).
Then it follows that

a) X has a continuous norm;

b) (x_n) is basis of X;

c) (x_n) does not have a finite basis constant;

d) (x_n) is not K-norming M-basis of X, for any finite K.

Remark that the space of Example can already be considered a « good » locally convex space, since there is a continuous norm.

Moreover, extending (4), we call *norming M-basis constant* of (x_n) the number $= \inf K$ with the following property:

there exists an increasing sequence (l_n) of natural numbers so that, for every x of $[x_k]_{k=1}^{m}$ and for every m,

$$(7)\qquad ((x)) < K \operatorname{dist}(x, [x_n]_{n>l_m}).$$

Then it follows that

THEOREM I: *For every K-norming M-basic sequence (x_n) of X there exists an equivalent quasi-norm so that (x_n) acquires a norming M-basis constant* $<3K$.

It is also possibly to extend a known property of the normed spaces ([6] p. 458), precisely:

COROLLARY 1: *Every norming M-basis sequence is union of two basic with brackets sequences.*

§ 2. - EXISTENCE AND STABILITY OF THE NORMING *M*-BASES

Firstly we state the existence of the norming M-bases.

THEOREM II: *Every separable complete X has a 1-norming M-basis.*

Moreover by § 1 it follows that

Corollary 2: *Every linearly independent sequence of X has a basic block sequence.*

This property can also be desumed by [2].

About the stability for sufficiently near sequences the following questions seem to be open.

Problem 1: Let (x_n) be minimal (M-basic) (norming M-basic) (basic with brackets) (basic) in X, does there exist (ε_n) of positive numbers such that, if $(y_n) \subset X$ with $\langle\!\langle y_n - x_n \rangle\!\rangle < \varepsilon_n$ for every n, then (y_n) is minimal (M-basic) (norming M-basic) (basic with brackets) (basic) too?

Next question concerns the stability of the completeness.

Problem 2: Let $(x_n) \subset X$, does there exist (ε_n) of positive numbers such that, if $(y_n) \subset [x_n]$ with $\langle\!\langle y_n - x_n \rangle\!\rangle < \varepsilon_n$ for every n, then (y_n) is complete in $[x_n]$?

We recall that these questions have positive answers in the normed spaces ([5], pp. 84-109; [6], p. 138, Th. 3.2).

§ 3. - On the existence of overfilling convex bases

A characteristic of the overfilling convex bases is that they do not contain any type of regular subsequences. The existence of these sequences is strictly connected to the quasi-norm of (2). Indeed for every $\bar{x}$ of X there exists $(\bar{f}_n)$ of X^* such that

$$\langle\!\langle \bar{x} \rangle\!\rangle_{(\bar{f}_n)} = \langle\!\langle \bar{x} \rangle\!\rangle, \quad \text{with} \quad \langle\!\langle x \rangle\!\rangle_{(\bar{f}_n)} < \langle\!\langle x \rangle\!\rangle \qquad \text{for every } x \text{ of } X.$$

In general these new quasi-norms are only weaker, but not equivalent, to the original quasi-norm; however if this happens we have the apparition of the overfilling convex bases, indeed:

Theorem III: *If X^* contains (f_n) so that $\langle\!\langle x \rangle\!\rangle_{(f_n)}$ is equivalent to the original quasi-norm, it follows that X has an overfilling convex basis.*

Hence R^N has both a basis (the natural basis) and an overfilling convex basis, moreover:

Corollary 3: *X has an overfilling convex basic sequence $\Leftrightarrow$ X has a subspace isomorphic to R^N.*

About the stability for « near » sequences we only point out that:

Remark: *If (x_n) is overfilling convex basis of X, every (y_n) of X with*

$$\lim_{n \to \infty} \langle\!\langle y_n - x_n \rangle\!\rangle = 0\,,$$

is overfilling convex basis with brackets of X.

We do not consider the stability of the overfilling convex basic sequences (x_n) of X, since, as in the normed spaces, if $[x_n]$ has infinite codimension in X, it is easy to see that, for every (ε_n) of positive numbers, there exists a minimal sequence (y_n) of X, with $\|y_n - x_n\| < \varepsilon_n$ for every n.

§ 1' - Proofs of § 1

Proof of Example: The continuous norm is p_1, which does not give the topology, since, for every $k > 1$, p_k is not continuous in the topology of p_1, indeed by (6)

$$\lim_{n\to\infty} p_1(x_{k-1,n}) = 0\,, \quad \text{while} \quad p_k(x_{k-1,n}) = 1 + \frac{1}{2^{k-1}} \qquad \text{for } n > 1\,.$$

Moreover $p_1(x_{m1}) = 1$ for every m, while $\lim_{m\to\infty} p_k(x_{m1}) = 0$ for $k > 1$, hence p_1 is not continuous in the topology of $(p_k)_{k>1}$.

By (6) there exists (f_n) of linear functionals such that

$$(x_n, f_n) \text{ is biorthogonal}\,,$$

$$f_n \text{ is continuous for } p_1 \text{ for every } n\,;$$

hence ([7] p. 216-217) $(f_n) \subset X^*$.

By (1) and (6), for every m, $[x_{mn}]_{n=1}^{\infty}$ is a Banach space isomorphic to l^1, with the norm $p_1(x) + p_{m+1}(x)$, where $(x_{mn})_{n=1}^{\infty}$ is the basis.

Let $\bar{x} \in X$.

Then

$$\bar{x} = \lim_{k\to\infty} v_k\,; \quad \text{where} \quad v_k = \sum_{m=1}^{r_k} v_{km} \qquad \text{for every } k\,;$$

where

$$v_{km} = \sum_{n=1}^{r_{km}} a_{kmn} x_{mn} \qquad \text{for } 1 < m < r_k\,, \text{ for every } k\,.$$

From above and by (1) and (6) it is easy to see that

$$(v_k) \text{ is convergent} \Rightarrow \begin{cases} (v_{km})_{k=1}^{\infty} \text{ converges to } \bar{x}_m \text{ for every } m\,, \\ \sum_{m=1}^{\infty} p_1(\bar{x}_m) \text{ is convergent}\,. \end{cases}$$

Then $\sum_{m=1}^{\infty} \bar{x}_m$ converges, moreover by above $x \in [f_n]_\perp$ implies $p_n(x) = 0$, $\forall n$, hence

$$\bar{x} = \sum_{m=1}^{\infty} \bar{x}_m = \sum_{m=1}^{\infty} \Bigl(\sum_{n=1}^{\infty} f_{mn}(\bar{x}) x_{mn} \Bigr) = \sum_{n=1}^{\infty} f_n(\bar{x}) x_n\,.$$

Therefore (x_n) is basis of X.

Fix $m>1$, for every i by (1) and (6) it follows that

$$((x_{mn}-x_{m,n+i}))=\frac{1}{2}\frac{p_1(x_{mn}-x_{m,n+i})}{1+p_1(x_{mn}-x_{m,n+i})}+\frac{1}{2^{m+1}}\frac{p_{m+1}(x_{mn}-x_{m,n+i})}{1+p_{m+1}(x_{mn}-x_{m,n+i})}=$$

$$=\frac{1}{2}\frac{(2^i+1)/2^{n+i}}{1+(2^i+1)/2^{n+i}}+\frac{1}{2^{m+1}}\frac{1/2^{m-1}}{1+1/2^{m-1}}<\frac{1}{2^n}+\frac{1}{2^{2m}}, \qquad \text{for every } n;$$

hence, for $n>2m$,

$$((x_{mn}-x_{m,n+i}))<\frac{1}{2^{m-4}}\left(\frac{1}{2^{m+1}}\frac{p_{m+1}(x_{mn})}{1+p_{m+1}(x_{mn})}\right)<\frac{((x_{m,n}))}{2^{m-4}}.$$

That is by (4) (x_n) does not have a finite basis constant, moreover by (7) (x_n) is not K-norming M-basic, for any finite K. This completes the proof of Example.

LEMMA 1: *Suppose that there exist K, with $1<K<\infty$, moreover a sequence (ϵ_n) of positive numbers convergent to 0, such that, for every m, $\epsilon_m<1$ and*

$$\left(\left(\sum_{k=m}^{m+r}a_kx_k\right)\right)<K\left(\left(\sum_{k=m}^{m+r+s}a_kx_k\right)\right)+\epsilon_m, \qquad \text{for every } (a_k)_{k=m}^{m+r+s}.$$

Then (x_n) is basic.

PROOF: By hypothesis there exists $(f_n)\subset X^*$ such that (x_n, f_n) is biorthogonal. Let $\bar{x}\in[x_n]$.

Since the f_n are continuous, there exist $(\bar{a}_n)$ of numbers and an increasing sequence (t_n) of natural numbers, so that

$$(8) \qquad \left(\left(\bar{x}-\left\{\sum_{n=1}^{t_m}f_n(\bar{x})x_n+\sum_{n=t_m+1}^{t_{m+1}}\bar{a}_nx_n\right\}\right)\right)<\frac{1}{2^{m+2}}, \qquad \text{for every } m.$$

Set, for every m,

$$(9) \qquad \bar{y}_m=\sum_{n=1}^{t_m}f_n(\bar{x})x_n+\sum_{n=t_m+1}^{t_{m+1}}\bar{a}_nx_n, \qquad \bar{z}_m=\sum_{n=1}^{m}f_n(\bar{x})x_n.$$

We affirm that

$$(10) \qquad ((\bar{x}-\bar{z}_{t_m}))<\frac{1+K}{2^m}+\epsilon_{t_{m-1}+1} \qquad \text{for every } m.$$

Indeed, if $m>1$, by (8), (9) and by hypothesis it follows that

$$((\bar{x}-\bar{z}_{t_m}))<((\bar{x}-\bar{y}_m))+((\bar{y}_m-\bar{z}_{t_m}))<\frac{1}{2^{m+2}}+((\bar{y}_m-\bar{z}_{t_m}))=$$

$$=\frac{1}{2^{m+2}}+\left(\left(\bar{y}_m-\left\{\bar{y}_{m-1}+\sum_{n=t_{m-1}+1}^{t_m}f_n(\bar{x})x_n-\sum_{n=t_{m-1}+1}^{t_m}\bar{a}_nx_n\right\}\right)\right)<$$

$$<\frac{1}{2^{m+2}}+((\bar{y}_m-\bar{x}))+((\bar{y}_{m-1}-\bar{x}))+\Big(\Big(\sum_{n=t_{m-1}+1}^{t_m}(\bar{a}_n-f_n(\bar{x}))x_n\Big)\Big)<$$
$$<\frac{1}{2^m}+\Big(\Big(\sum_{n=t_{m-1}+1}^{t_m}(\bar{a}_n-f_n(\bar{x}))x_n\Big)\Big)<$$
$$<\frac{1}{2^m}+\varepsilon_{t_{m-1}+1}+K\Big(\Big(\sum_{n=t_{m-1}+1}^{t_m}(\bar{a}_n-f_n(\bar{x}))x_n-\sum_{n=t_m+1}^{t_{m+1}}\bar{a}_nx_n\Big)\Big)=$$
$$=\frac{1}{2^m}+\varepsilon_{t_{m-1}+1}+K((\bar{y}_{m-1}-\bar{y}_m))<\frac{1}{2^m}+\varepsilon_{t_{m-1}+1}+K((\bar{x}-\bar{y}_{m-1}))+K((\bar{x}-\bar{y}_m))<$$
$$<\frac{1}{2^m}+\varepsilon_{t_{m-1}+1}+K\Big(\frac{1}{2^{m+1}}+\frac{1}{2^{m+2}}\Big).$$

We affirm now that

$$(11)\qquad ((\bar{x}-\bar{z}_n))<(1+K)\Big\{\frac{1+K}{2^m}+\frac{K}{2^{m+p}}+\varepsilon_{t_{m-1}+1}\Big\}+K\varepsilon_{t_{m+p-1}+1}+\varepsilon_{t_m+1},$$

for $n>t_m$ and for every m.

Indeed by hypothesis and by (9) and (10), if $t_m<n<t_{m+p}$, it follows that

$$((\bar{x}-\bar{z}_n))=\Big(\Big(\bar{x}-\bar{z}_{t_m}-\sum_{k=t_m+1}^{n}f_k(\bar{x})x_k\Big)\Big)<((\bar{x}-\bar{z}_{t_m}))+\Big(\Big(\sum_{k=t_m+1}^{n}f_k(\bar{x})x_k\Big)\Big)<$$
$$<((\bar{x}-\bar{z}_{t_m}))+K\Big(\Big(\sum_{k=t_m+1}^{t_{m+p}}f_k(\bar{x})x_k\Big)\Big)+\varepsilon_{t_m+1}=((\bar{x}-\bar{z}_{t_m}))+K((\bar{z}_{t_{m+p}}-\bar{z}_{t_m}))+\varepsilon_{t_m+1}<$$
$$<((\bar{x}-\bar{z}_{t_m}))+K((\bar{x}-\bar{z}_{t_{m+p}}))+K((\bar{x}-\bar{z}_{t_m}))+\varepsilon_{t_m+1}.$$

Finally by (9) and (11) it follows that

$$\bar{x}=\lim_{n\to\infty}\bar{z}_n=\sum_{n=1}^{\infty}f_n(\bar{x})x_n,$$

which completes the proof of Lemma 1.

Proof of proposition 1: *a*) is a particular case of Lemma 1; the proof of *b*) is similar.

Proof of Prop. 2: Firstly we prove the straight implication: Suppose that (x_n) is K-norming M-basic, with (x_n, f_n) biorthogonal. Fix m and $\varepsilon>0$. Let $\bar{x}\in[x_n]$.

By (2) and (3) there exists g of X^* such that

$$|g(\bar{x})-p_m(\bar{x})|<\varepsilon p_m(\bar{x}),\quad |g(x)|<Kp_m(x)\ \text{ for every } x \text{ of } X,\ g\in\operatorname{span}(f_n)_{n=1}^{t}.$$

Then for every y of $[x_n]_{n>t}$ it follows that

$$Kp_m(\bar{x}+y)>|g(\bar{x}+y)|=|g(\bar{x})|>p_m(\bar{x})(1-\varepsilon);$$

therefore, since ε is arbitrary, we have (5) for $x=\bar{x}$.

Now we prove the inverse implication.

Suppose that (5) is true. We can suppose (x_n) complete in X.

Fix a positive integer m, let $\bar{x} \in X$ and fix $\varepsilon > 0$.

There exist a natural number n_ε and $\tilde{x}$ of X so that

$$\tilde{x} \in [x_n]_{n=1}^{n_\varepsilon}, \qquad p_m(\bar{x} - \tilde{x}) < \varepsilon .$$

By (5) there exists a natural number l_ε such that

$$p_m(\tilde{x}) < (K + \varepsilon) p_m(\tilde{x} + y) \qquad \text{for every } y \in [x_k]_{k > l_\varepsilon} .$$

Hence by the Hahn-Banach theorem there exists b' of X^* such that

$$b'_\perp \supset (x_k)_{k > l_\varepsilon}, \; b'(\tilde{x}) = p_m(\tilde{x}), \; |b'(x)| < (K + \varepsilon) p_m(x) \qquad \text{for every } x \text{ of } X .$$

Set $b = b'(K/(K + \varepsilon))$, it follows that

$$b_\perp \supset (x_k)_{k > l_\varepsilon}, \qquad b(\tilde{x}) = p_m(\tilde{x}) \frac{K}{K + \varepsilon}, \quad |b(x)| < K p_m(x) \quad \text{for every } x \text{ of } X .$$

Therefore, since (x_n) is complete in X, if (x_n, f_n) is biorthogonal we have that

$$b_\perp \supset [x]_{k > l_\varepsilon} = \bigcap_{k=1}^{l_\varepsilon} [f_k]_\perp ,$$

that is $b \in \operatorname{span} (f_k)_{k=1}^{l_\varepsilon}$ (see [7], p. 39, Th. 3).

Moreover

$$|b(\bar{x})| > b(\tilde{x}) - |b(\tilde{x} - \bar{x})| > \frac{K}{K + \varepsilon} p_m(\tilde{x}) - K p_m(\tilde{x} - \bar{x}) >$$

$$> \frac{K}{K + \varepsilon} p_m(\bar{x}) - K\left(1 + \frac{1}{K + \varepsilon}\right) p_m(\tilde{x} - \bar{x}) > \frac{K}{K + \varepsilon} p_m(\bar{x}) - \varepsilon K\left(1 + \frac{1}{K + \varepsilon}\right) .$$

That is, since ε is arbitrary, we have the thesis for m; which completes the proof of Prop. 2.

Now we state a lemma which defines an equivalent quasi-norm, which will have particular properties.

Lemma 2: *Let* $(x_n) \subset X$, *there exists an equivalent sequence* (q_n) *of seminorms such that, for every* x *of span* (x_n), $(q_n(x))$ *is bounded.*

Proof: It is sufficient to set

$$(12) \qquad q_n(x) = \frac{p_n(x)}{1 + \max\{p_n(x_k);\ 1 < k < n\}} , \qquad \text{for every } n .$$

It follows that

$$q_n(x_k) < 1 \qquad \text{for } n > k \text{, for every } k\,.$$

Hence $(q_n(x_k))_{n=1}^{\infty}$ is bounded for every k.

Therefore, for every $(a_k)_{k=1}^{m}$,

$$q_n\Big(\sum_{k=1}^{m} a_k x_k\Big) < \sum_{k=1}^{m} |a_k|\, q_n(x_k) \qquad \text{for every } n\,;$$

whence

$$\Big(q_n\Big(\sum_{k=1}^{m} a_k x_k\Big)\Big)_{n=1}^{\infty} \qquad \text{is bounded}.$$

This completes the proof of Lemma 2.

In what follows, if (q_n) is the sequence of the seminorms of Lemma 2, for every x of X we set

$$(13) \qquad \left|\begin{array}{ll} \langle\!\langle x \rangle\!\rangle_m = \displaystyle\sum_{n=1}^{m} \frac{1}{2^n} \frac{q_n(x)}{1+q_n(x)} & \qquad \text{for every } m\,; \\ \langle\!\langle x \rangle\!\rangle = \displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} \frac{q_n(x)}{1+q_n(x)}\,. & \end{array}\right.$$

Lemma 3: *Fix $\varepsilon > 0$, there exists a sequence (r_m) of natural numbers so that, for every m and for every x of* span $(x_k)_{k=1}^{m}$,

$$(14) \qquad \langle\!\langle x \rangle\!\rangle_{r_m} > (1-\varepsilon)\langle\!\langle x \rangle\!\rangle\,.$$

Proof: By Lemma 2 we can set, for every n,

$$(15) \qquad y_n = x_n / \max\{q_k(x_n);\ 1 < k < \infty\}\,.$$

Since span $(y_n)_{n=1}^{m}$ = span $(x_n)_{n=1}^{m}$ for every m, it is sufficient to prove the thesis for (y_n) instead of (x_n).

Set, for every $(a_n)_{n=1}^{m}$ of numbers,

$$(16) \qquad \Big\|\sum_{n=1}^{m} a_n y_n\Big\| = \sum_{n=1}^{m} |a_n|\,.$$

In the topology of this norm (y_n) is equivalent to the natural basis of l^1; moreover, for every m and for every $(a_k)_{k=1}^{m}$ of numbers, by (15) and (16) it follows that

$$(17) \qquad q_m\Big(\sum_{k=1}^{m} a_k y_k\Big) < \sum_{k=1}^{m} |a_k|\, q_m(y_k) < \sum_{k=1}^{m} |a_k| = \Big\|\sum_{k=1}^{m} a_k y_k\Big\|\,.$$

Fix $\bar{m}$.

If the thesis is not true, there exists a sequence (v_n) of span $(x_k)_{k=1}^{\bar{m}}$ such that

$$(18) \qquad \langle\!\langle v_n \rangle\!\rangle_n < (1-\varepsilon)\langle\!\langle v_n \rangle\!\rangle\,, \qquad \text{for every } n\,.$$

Since span $(x_i)_{i=1}^{\infty}$ is compact in the topology of (16), there exists $\bar{v}$ of span $(x_i)_{i=1}^{\infty}$ so that

$$\lim_{n\to\infty} v_n/\|v_n\| = \bar{v}\,;$$

hence we can suppose that

$$\left\|\frac{v_n}{\|v_n\|} - \bar{v}\right\| < \frac{1}{2^{2n+1}} \qquad \text{for every } n. \tag{19}$$

Fix a natural number k.

Suppose that there exists a number M_k such that (by Lemma 2)

$$M_k q_k(v_n) > \{\max q_i(v_n);\, i > n\}\,, \qquad \text{for every } n. \tag{20}$$

Then it would follow, for every n, by (13),

$$\langle\!\langle v_n \rangle\!\rangle - \langle\!\langle v_n \rangle\!\rangle_n = \sum_{i=n+1}^{\infty} \frac{1}{2^i} \frac{q_i(v_n)}{1+q_i(v_n)} <$$

$$< M_k q_k(v_n) \sum_{i=n+1}^{\infty} \frac{1}{2^i} \frac{1}{1+q_i(v_n)} < M_k q_k(v_n) \sum_{i=n+1}^{\infty} \frac{1}{2^i} = \frac{M_k q_k(v_n)}{2^n}.$$

By (13) and (18) $\lim\limits_{n\to\infty} q_k(v_n) = 0$, hence we can suppose $q_k(v_n) < 1$, therefore for $n > k$, we have that

$$\langle\!\langle v_n \rangle\!\rangle - \langle\!\langle v_n \rangle\!\rangle_n < \frac{M_k}{2^{n-k-1}} \frac{1}{2^k} \frac{q_k(v_n)}{2} < \frac{M_k}{2^{n-k-1}} \frac{1}{2^k} \frac{q_k(v_n)}{1+q_k(v_n)} < \frac{M_k}{2^{n-k-1}} \langle\!\langle v_n \rangle\!\rangle_n .$$

Instead by (18) it follows that

$$\langle\!\langle v_n \rangle\!\rangle - \langle\!\langle v_n \rangle\!\rangle_n > \varepsilon \langle\!\langle v_n \rangle\!\rangle > \varepsilon \langle\!\langle v_n \rangle\!\rangle_n, \qquad \text{for every } n\,;$$

hence (20) is not possible, that is

$$\lim_{n\to\infty} \frac{\max\{q_i(v_n);\, i > n\}}{q_k(v_n)} = +\infty .$$

Therefore there exists a sequence $s(n)$ of natural numbers such that

$$q_k(v_n) < \frac{1}{2^n} q_{k+s(n)}(v_n)\,, \qquad \text{for every } n .$$

Hence by (17) and (19) for every n it would follow that

$$2^n q_k(\bar{v}) < 2^n q_k\left(\frac{v_n}{\|v_n\|}\right) + 2^n q_k\left(\frac{v_n}{\|v_n\|} - \bar{v}\right) <$$

$$< 2^n q_k\left(\frac{v_n}{\|v_n\|}\right) + \frac{1}{2^{n+1}} < q_{k+s(n)}\left(\frac{v_n}{\|v_n\|}\right) + \frac{1}{2^{n+1}} <$$

$$< q_{k+s(n)}(\bar{v}) + q_{k+s(n)}\left(\frac{v_n}{\|v_n\|} - \bar{v}\right) + \frac{1}{2^{n+1}} < q_{k+s(n)}(\bar{v}) + \frac{1}{2^n}.$$

Hence $(q_n(v))$ would not be bounded, contrary to Lemma 2. Therefore (18) is not possible; which completes the proof of Lemma 3.

PROOF OF TH. I: Let (x_n) be a K-norming M-basic sequence of X. Since the norming M-basis constant is an intrinsic property of the sequence, we can suppose (x_n) complete in X.

Let $(f_n) \subset X^*$ so that

$$(21) \qquad (x_n, f_n) \text{ is biorthogonal}.$$

By (2), (3), (12) and (13) it follows that (x_n) is also K-norming M-basis of X with the quasi-norm of (13), hence

> for every $\bar{x}$ of X, $\bar{n}$ and $\varepsilon > 0$, there exists $\bar{g}_{\bar{n}} \in \operatorname{span}(f_n)$ so that $|\bar{g}_{\bar{n}}(x)| < K q_{\bar{n}}(x)$ for every x of X, $|q_{\bar{n}}(\bar{x}) - \bar{g}_{\bar{n}}(\bar{x})| < \varepsilon$.

Now we shall proceed with the quasi-norm of (13).

Fix $\varepsilon > 0$.

Fix $m > 1$ and let r_m be the number of Lemma 2.

Fix n with $1 < n < r_m$.

Note that $\operatorname{span}(x_k)_{k=1}^m$ is compact in the topology of the seminorm q_n; hence by above there exist $(y_{mnk})_{k=1}^{t_{mn}} \subset X$ and $(g_{mnk})_{k=1}^{t_{mn}} \subset X^*$ so that

$$(22) \quad \left|\begin{array}{l} (y_{mnk})_{k=1}^{t_{mn}} \subset \operatorname{span}(x_k)_{k=1}^m, \qquad (g_{mnk})_{k=1}^{t_{mn}} \subset \operatorname{span}(f_k); \\ \text{for every } \bar{x} \text{ of } \operatorname{span}(x_k)_{k=1}^m, \text{ with } q_n(\bar{x}) \neq 0, \text{ there exists } y_{mn\bar{k}} \text{ so that} \\ q_n\left(\dfrac{\bar{x}}{q_n(\bar{x})} - y_{mn\bar{k}}\right) < \varepsilon, \qquad |q_n(y_{mn\bar{k}}) - g_{mn\bar{k}}(y_{mn\bar{k}})| < \varepsilon, \\ |g_{mn\bar{k}}(x)| < K q_n(x) \qquad \text{for every } x \text{ of } X. \end{array}\right.$$

Let l_m be a natural number so that

$$(23) \qquad ((g_{mnk})_{k=1}^{t_{mn}})_{n=1}^{r_m} \subset \operatorname{span}(f_n)_{n=1}^{l_m}.$$

Fix $\bar{x} \in \operatorname{span}(x_k)_{k=1}^m$.

We shall prove that

$$\langle\!\langle \bar{x} \rangle\!\rangle < \frac{3K'}{1-\varepsilon} \operatorname{dist}(\bar{x}, [x_n]_{n>l_m}), \quad \text{with } K' = \left(\frac{1}{K} - \left(1 + \frac{2}{K}\right)\varepsilon\right)^{-1};$$

hence, since ε is arbitrary, by (7) (x_n) will have a norming M-basis constant $< 3K$, for the quasi-norm of (13).

Fix y of $\operatorname{span}(x_k)_{k>l_m}$, it is sufficient to prove that

$$(24) \qquad \frac{q_n(\bar{x}+y)}{1+q_n(\bar{x}+y)} > \frac{1}{3K'} \frac{q_n(\bar{x})}{1+q_n(\bar{x})}, \qquad \text{for } 1 < n < r_m,$$

indeed by (13), (24) and by Lemma 3 it will follow that

$$\langle\!\langle \bar{x} \rangle\!\rangle < \frac{1}{1-\varepsilon} \langle\!\langle \bar{x} \rangle\!\rangle_{r_m} = \frac{1}{1-\varepsilon} \sum_{n=1}^{r_m} \frac{1}{2^n} \frac{q_n(\bar{x})}{1+q_n(\bar{x})} <$$

$$< \frac{3}{1-\varepsilon} K' \sum_{n=1}^{r_m} \frac{1}{2^n} \frac{q_n(\bar{x}+y)}{1+q_n(\bar{x}+y)} = \frac{3K'}{1-\varepsilon} \langle\!\langle \bar{x}+y \rangle\!\rangle_{r_m} < \frac{3K'}{1-\varepsilon} \langle\!\langle \bar{x}+y \rangle\!\rangle .$$

Fix n with $1 \leq n \leq r_m$.

Note that (24) is obvious if $q_n(y) > 2q_n(\bar{x})$, because

$$q_n(\bar{x}+y)(1+q_n(\bar{x})) = q_n(\bar{x}+y) + q_n(\bar{x}+y)q_n(\bar{x}) >$$

$$> |q_n(\bar{x}) - q_n(y)| + q_n(\bar{x}+y)q_n(\bar{x}) = q_n(y) - q_n(\bar{x}) + q_n(\bar{x}+y)q_n(\bar{x}) >$$

$$> q_n(\bar{x}) + q_n(\bar{x}+y)q_n(\bar{x}) = q_n(\bar{x})(1+q_n(\bar{x}+y)) ;$$

hence

$$\frac{q_n(\bar{x}+y)}{1+q_n(\bar{x}+y)} > \frac{q_n(\bar{x})}{1+q_n(\bar{x})} .$$

Therefore suppose $q_n(y) \leq 2q_n(\bar{x})$, we have that

$$\frac{1+q_n(\bar{x})}{1+q_n(\bar{x}+y)} > \frac{1}{3} \tag{25}$$

indeed

$$\frac{1+q_n(\bar{x})}{1+q_n(\bar{x}+y)} > \frac{1+q_n(\bar{x})}{1+q_n(\bar{x})+q_n(y)} > \frac{1+q_n(\bar{x})}{1+3q_n(\bar{x})} > \frac{1+q_n(\bar{x})}{3+3q_n(\bar{x})} = \frac{1}{3} .$$

Moreover by (21), (22) and (23), if $q_n(\bar{x}) \neq 0$, it follows that

$$q_n(\bar{x}+y) = q_n(\bar{x}) \cdot q_n\left(\frac{\bar{x}}{q_n(\bar{x})} + \frac{y}{q_n(\bar{x})}\right) >$$

$$> q_n(\bar{x}) \left\{ q_n\left(y_{mn\bar{x}} + \frac{y}{q_n(\bar{x})}\right) - q_n\left(\frac{\bar{x}}{q_n(\bar{x})} - y_{mn\bar{x}}\right) \right\} > q_n(\bar{x}) \left\{ q_n\left(y_{mn\bar{x}} + \frac{y}{q_n(\bar{x})}\right) - \varepsilon \right\} >$$

$$> q_n(\bar{x}) \left\{ \frac{1}{K} \left| g_{mn\bar{x}}\left(y_{mn\bar{x}} + \frac{y}{q_n(\bar{x})}\right) \right| - \varepsilon \right\} = q_n(\bar{x}) \left\{ \frac{1}{K} |g_{mn\bar{x}}(y_{mn\bar{x}})| - \varepsilon \right\} >$$

$$> q_n(\bar{x}) \left\{ \frac{1}{K} q_n(y_{mn\bar{x}}) - \left(1 + \frac{1}{K}\right) \varepsilon \right\} >$$

$$> q_n(\bar{x}) \left\{ \frac{1}{K} q_n\left(\frac{\bar{x}}{q_n(\bar{x})}\right) - \frac{1}{K} q_n\left(\frac{\bar{x}}{q_n(\bar{x})} - y_{mn\bar{x}}\right) - \left(1 + \frac{1}{K}\right) \varepsilon \right\} >$$

$$> q_n(\bar{x}) \left\{ \frac{1}{K} q_n\left(\frac{\bar{x}}{q_n(\bar{x})}\right) - \left(1 + \frac{2}{K}\right) \varepsilon \right\} = q_n(\bar{x}) \frac{1}{K'} .$$

Therefore, by (25),

$$\frac{q_n(\bar{x}+y)}{1+q_n(\bar{x}+y)} > \frac{1}{K'}\frac{q_n(\bar{x})}{1+q_n(\bar{x}+y)} =$$

$$= \frac{1}{K'}\frac{q_n(\bar{x})}{1+q_n(\bar{x})}\cdot\frac{1+q_n(\bar{x})}{1+q_n(\bar{x}+y)} > \frac{1}{K'}\frac{1}{3}\frac{q_n(\bar{x})}{1+q_n(\bar{x})}.$$

That is (24) is proved; which completes the proof of Th. I.

Proof of Corollary 1: Let (x_n) be a K-norming M-basic sequence of X. By Th. I there exists an equivalent quasi-norm $\langle\!\langle \ldots \rangle\!\rangle$ such that (x_n) has a norming M-basis constant $< 3K$.

Hence by (7), for every $\varepsilon > 0$, there exists an increasing sequence (l_n) of natural numbers, so that for every m we have

$$\langle\!\langle x \rangle\!\rangle < (3K+\varepsilon)\,\mathrm{dist}\,(x, [x_n]_{n>l_m}), \qquad \text{for every } x \text{ of } \mathrm{span}\,(x_n)_{n=1}^{m}.$$

Set

$$t_0 = 0, \quad t_1 = l_1 \quad \text{and} \quad t_n = l_{t_{n-1}} \qquad \text{for } n > 1;$$

$$(y_n) = \bigcup_{m=0}^{\infty} (x_n)_{n=t_{2m}+1}^{t_{2m+1}}, \qquad (z_n) = \bigcup_{m=0}^{\infty} (x_n)_{n=t_{2m+1}+1}^{t_{2m+2}}.$$

Then $(x_n) = (y_n) \cup (z_n)$; moreover by above and by (4), both (y_n) and (z_n) have a basis with brackets constant $< 3K + \varepsilon$.

On the other hand (x_n) is minimal, hence by b) of Prop. 1 both (y_n) and (z_n) are basic with brackets sequences. Finally a basic with brackets sequence is still basic with brackets for the original quasi-norm; which completes the proof of Corollary 1.

§ 2' - Proofs of § 2

Proof of Th. II: By hypothesis there exist (y_n) of X and (b_n) of X^* so that

$$(26) \qquad \begin{cases} (y_n) \text{ is dense in } X; \\ (b_n) = ((b_{mn})_{n=1}^{\infty})_{m=1}^{\infty}, \text{ so that}, \qquad \text{for every } m \text{ and } n, \\ b_{mn}(y_n) = p_m(y_n), \quad |b_{mn}(x)| \leq p_m(x) \qquad \text{for every } x \text{ of } X. \end{cases}$$

Set $x_1 = y_1$, $f_1 = b_1$.

Fix $m \geq 1$.

Suppose to have $(x)_{n=1}^{2m-1}$ of X and $(f_n)_{n=1}^{2m-1}$ of X^* so that

$$(27) \qquad \begin{cases} (y_n)_{n=1}^{m} \subset \mathrm{span}\,(x_n)_{n=1}^{2m-1}, \quad (b_n)_{n=1}^{m} \subset \mathrm{span}\,(f_n)_{n=1}^{2m-1}, \\ (x_n, f_n)_{n=1}^{2m-1} \text{ is biorthogonal}. \end{cases}$$

If $y_{m+1} \in \operatorname{span}(x_n)_{n=1}^{2m-1}$ we consider y_{m+2}, otherwise we set

$$x_{2m} = y_{m+1} - \sum_{n=1}^{2m-1} f_n(y_{m+1}) x_n ;$$

then, since $\bigcap_{n=1}^{2m-1} x_n^{\perp} \subset x_{2m}^{\perp}$ implies $x_{2m} \in \operatorname{span}(x_n)_{n=1}^{2m-1}$ (see [7] p. 39, Th. 3), there exists f_{2m} of X^* so that

$$f_{2m}(x_{2m}) = 1 , \qquad (x_n)_{n=1}^{2m-1} \subset [f_{2m}]_{\perp} .$$

If $b_{m+1} \in \operatorname{span}(f_n)_{n=1}^{2m}$, we consider b_{m+2}, otherwise we set

$$f_{2m+1} = b_{m+1} - \sum_{n=1}^{2m} b_{m+1}(x_n) f_n ;$$

then, since $\bigcap_{n=1}^{2m} [f_n]_{\perp} \subset [f_{2m+1}]_{\perp}$ implies $f_{2m+1} \in \operatorname{span}(f_n)_{n=1}^{2m}$, there exists x_{2m+1} of X so that

$$f_{2m+1}(x_{2m+1}) = 1 , \qquad (f_n)_{n=1}^{2m} \subset x_{2m+1}^{\perp} .$$

That is we have (27) for $m+1$ instead of m. Then, so proceeding, by (26) we have (x_n) of X and (f_n) of X^* such that

$$(28) \qquad \begin{cases} (x_n, f_n) \text{ is biorthogonal} , \\ (b_n) \subset \operatorname{span}(f_n) . \end{cases}$$

Let $\bar{x} \in X$ and fix m.

By (26) there exists a subsequence (n_k) of (n) so that

$$p_m(\bar{x} - y_{n_k}) < \frac{1}{2^{k+1}} p_m(\bar{x}) \qquad \text{for every } k .$$

Hence for every k by (26) it follows that

$$\begin{aligned} 0 < p_m(\bar{x}) - |b_{mn_k}(\bar{x})| &< p_m(\bar{x} - y_{n_k}) + p_m(y_{n_k}) - ||b_{mn_k}(y_{n_k}) - |b_{mn_k}(\bar{x} - y_{n_k})|| = \\ &= p_m(\bar{x} - y_{n_k}) + p_m(y_{n_k}) - |p_m(y_{n_k}) - |b_{mn_k}(\bar{x} - y_{n_k})|| = \\ &= p_m(\bar{x} - y_{n_k}) + |b_{mn_k}(\bar{x} - y_{n_k})| < 2p_m(\bar{x} - y_{n_k}) < \frac{1}{2^k} p_m(\bar{x}) . \end{aligned}$$

Therefore by (3) and (28) (x_n) is 1-norming M-basis of X, which completes the proof of Th. II.

Proof of Corollary 2: Let (x_n) be a linearly independent sequence of X. There exists (y_n) of X so that

$$(y_n) \text{ is dense in } [x_n], \qquad (y_n) \subset \operatorname{span}(x_n), \qquad (x_n) \subseteq (y_n).$$

Proceeding as in the proof of Th. II we find (v_n) of X so that

$$(v_n) \text{ is 1-norming } M\text{-basis of } [x_n], \qquad (y_n) \subset \operatorname{span}(v_n).$$

Since $(x_n) \subseteq (y_n)$ we have $(x_n) \subset \operatorname{span}(v_n)$; therefore, since (x_n) is linearly independent, it is easy to see that there exists (w_n) of X so that

$$(w_n) \text{ is block sequence both of } (x_n) \text{ and of } (v_n).$$

By Th. I there exists an equivalent quasi-norm so that (v_n) has a norming M-basis constant <3; hence (w_n) has a norming M-basis constant <3 in the equivalent quasi-norm.

Therefore, proceeding as in the proof of Corollary 1, (w_n) has a subsequence (w_{n_k}) which has a basis with brackets constant <3; whence, by (4), there exists a subsequence (z_n) of (w_{n_k}) which has a basis constant <3 in the equivalent quasi-norm; hence by *a*) of proposition 1 (z_n) is basic, therefore (z_n) is basic also for the original quasi-norm. Finally (z_n), subsequence of (w_n), is block sequence of (x_n); which completes the proof of Corollary 2.

§ 3'. - Proofs of § 3

Firstly we prove a numerical property.

Lemma 4: *Let* $((N_{mn})_{n=1}^{p})_{m=1}^{p+1} \cup (N_{n,p+1})_{n=1}^{p}$ *be fixed numbers, with the condition*

$$A = \begin{vmatrix} N_{11} & \cdots & N_{1p} \\ \vdots & & \vdots \\ N_{p1} & \cdots & N_{pp} \end{vmatrix} \neq 0, \tag{29}$$

then, if we choose $N_{p+1,p+1}$ *sufficiently big, there exists a unique* $(a_n)_{n=1}^{p+1}$ *of numbers so that*

$$\sum_{n=1}^{p+1} a_n N_{mn} = \begin{cases} 0 & \text{for } 1 < m < p, \\ 1 & \text{for } m = p+1, \end{cases} \qquad \sum_{n=1}^{p+1} |a_n| + \sum_{m=1}^{p} \sum_{n=1}^{p} |a_n N_{mn}| < \frac{1}{2^{p+1}}.$$

PROOF: Consider the system of equations

$$\left|\begin{array}{l} N_{11}a_1 \quad + \dots + N_{1,p+1}a_{p+1} \quad = 0\,, \\ \vdots \qquad\qquad\quad \vdots \\ N_{p,1}a_1 \quad + \dots + N_{p,p+1}a_{p+1} \quad = 0\,, \\ N_{p+1,1}a_1 + \dots + N_{p+1,p+1}a_{p+1} = 1\,, \end{array}\right.$$

where $a_1, \dots, a_{p+1}$ are unknown terms.

Consider the matrix

$$\begin{bmatrix} N_{11} & \cdots & N_{1,p+1} \\ \cdots & & \cdots \\ N_{p+1,1} & \cdots & N_{p+1,p+1} \end{bmatrix}$$

and let A_{mn} be the algebraical complement of the element N_{mn} of the matrix, for $1 < m,\ n < p+1$; moreover let D be the determinant of the matrix, by (29) it follows that

$$D = N_{p+1,p+1}A + B\,, \quad \text{where} \quad B = \sum_{n=1}^{p} N_{p+1,n}A_{p+1,n}\,.$$

Then $D \neq 0$ if $N_{p+1,p+1}$ is sufficiently big; hence the solutions of our system are

$$a_n = \frac{A_{p+1,n}}{N_{p+1,p+1}A + B} \qquad \text{for } 1 < n < p+1\,.$$

Remark that A, B and $A_{p+1,n}$, for $1 < n < p+1$, do not depend on $N_{p+1,p+1}$; hence, if $N_{p+1,p+1}$ is sufficiently big, we can get $a_1, \dots, a_{p+1}$ very small, as we want. This completes the proof of Lemma 4.

PROOF OF TH. III: By hypothesis there exists (g_n) of X^* so that $\|(\dots)\|_{(g_n)}$ gives the topology. Since X is infinite dimensional, we can suppose (g_n) linearly independent.

Then set $f_1 = g_1$ and let $x_1 \in X$ so that $f_1(x_1) = 1$.

Suppose, for $m > 1$, to have $(x_n)_{n=1}^m$ and $(f_n)_{n=1}^m$ so that

(30) $\qquad (x_n, f_n)_{n=1}^m$ is biorthogonal, $\qquad \operatorname{span}(f_n)_{n=1}^m = \operatorname{span}(g_n)_{n=1}^m\,.$

If $\bigcap_{n=1}^m [f_n]_\perp \subset [g_{m+1}]_\perp$ it would follow $g_{m+1} \in \operatorname{span}(f_n)_{n=1}^m$, which is impossible by (30); hence there exists x_{m+1} of $\bigcap_{n=1}^m [f_n]_\perp$ with $g_{m+1}(x_{m+1}) = 1$; then, setting

$$f_{m+1} = g_{m+1} - \sum_{n=1}^{m} g_{m+1}(x_n)\, f_n\,,$$

we have (30) for $m+1$ instead of m.

Therefore there exist (x_n) of X and (f_n) of X^* so that we can suppose

$$(31)\qquad \begin{cases} (x_n, f_n) & \text{is biorthogonal}, \\ ((x)) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|f_n(x)|}{1+|f_n(x)|} & \text{for every } x \in X. \end{cases}$$

We proceed now to construct the overfilling convex basis of X.

Set $y_1 = 2x_1$.

Fix $p > 1$ and suppose to have $(y_n)_{n=1}^{p}$ so that (see proof of Lemma 4)

$$(32)\qquad \left| \begin{array}{l} \text{(i) } y_m \in \operatorname{span}(x_n)_{n=1}^{m}, \text{ with } f_1(y_m) = 2^m, \quad \text{for } 1 < m < p\,; \\ \text{(ii) if } (n_k)_{k=1}^{q} \subset (n)_{n=1}^{p}, \text{ there exists } (a_k)_{k=1}^{q} \text{ of numbers so that} \\ \sum_{k=1}^{q} a_k f_m(y_{n_k}) = \begin{cases} 0 & \text{for } 1 < m < q, \\ 1 & \text{for } m = q, \end{cases} \quad \text{with determinant (of the coefficients } f_m(y_{n_k})) \neq 0, \\ \sum_{k=1}^{q} |a_k| + \sum_{m=1}^{q-1} \sum_{k=1}^{q-1} |a_k f_m(y_{n_k})| < \frac{1}{2^{n_q}}. \end{array} \right.$$

We pass to choose y_{p+1} of $\operatorname{span}(x_n)_{n=1}^{p+1}$:

Set $f_1(y_{p+1}) = 2^{p+1}$. Fix $\bar{n}$ with $1 < \bar{n} < p$.

By Lemma 4 there exists $f_2(y_{p+1})$ so that there exist a_1, a_2 for which

$$\begin{aligned} & a_1 f_1(y_{\bar{n}}) + a_2 f_1(y_{p+1}) = 0, \\ & a_1 f_2(y_{\bar{n}}) + a_2 f_2(y_{p+1}) = 1, \end{aligned} \qquad \text{with} \begin{vmatrix} f_1(y_{\bar{n}}) & f_1(y_{p+1}) \\ f_2(y_{\bar{n}}) & f_2(y_{p+1}) \end{vmatrix} \neq 0,$$

$$|a_1| + |a_2| + |a_1 f_1(y_{\bar{n}})| < \frac{1}{2^{p+1}}.$$

Since this value of $f_2(y_{p+1})$ depends on $\bar{n}$, where $1 < \bar{n} < p$, we shall choose the biggest.

Now we proceed by finite induction.

That is fix s, with $2 < s < p$, and suppose to have $f_m(y_{p+1})$ for $1 < m < s$ so that, for every t with $2 < t < s$, if $(n_k)_{k=1}^{t-1} \subset (n)_{n=1}^{p}$, there exists $(a_k)_{k=1}^{t}$ of numbers so that

$$\sum_{k=1}^{t-1} a_k f_m(y_{n_k}) + a_t f_m(y_{p+1}) = \begin{cases} 0 & \text{for } 1 < m < t, \\ 1 & \text{for } m = t, \end{cases} \qquad \text{with determinant of the coefficients} \neq 0,$$

$$\sum_{k=1}^{t} |a_k| + \sum_{m=1}^{t-1} \sum_{k=1}^{t-1} |a_k f_m(y_{n_k})| < \frac{1}{2^{p+1}}.$$

Then, for every $(n_k)_{k=1}^{s} \subset (n)_{n=1}^{p}$, Lemma 4 gives a value of $f_{s+1}(y_{p+1})$, so that there exists $(a_k)_{k=1}^{s+1}$ for which (see proof of Lemma 4)

$$\sum_{k=1}^{s} a_k f_m(y_{n_k}) + a_{s+1} f_m(y_{p+1}) = \begin{cases} 0 & \text{for } 1 < m < s, \\ 1 & \text{for } m = s+1, \end{cases} \qquad \text{with determinant of the coefficients} \neq 0,$$

$$\sum_{k=1}^{s+1} |a_k| + \sum_{m=1}^{s} \sum_{k=1}^{s} |a_k f_m(y_{n_k})| < \frac{1}{2^{p+1}}.$$

These values of $f_{p+1}(y_{p+1})$ are a finite sequence, since they depend on the subsequences $(n_k)_{k=1}^{q}$ of $(n)_{n=1}^{p}$, then we shall choose the biggest.

So proceeding we get the values $f_m(y_{p+1})$ for $1 < m < p+1$, that is it is sufficient to set

$$y_{p+1} = \sum_{n=1}^{p+1} f_n(y_{p+1}) x_n .$$

Then we have (32) for $p+1$ instead of p, hence the construction of (y_n) is clear.

Now we prove that (y_n) is overfilling convex basis of X.

Let (n'_k) be an infinite subsequence of (n).

Let $\bar{x} \in X$.

Set

$$q_m = \sum_{n=1}^{m} n \qquad \text{for } m > 1 ,\ q_0 = 0 . \tag{33}$$

Choose n_1 of (n'_k) so that $2^{n_1} > 2|f_1(\bar{x})|$, set

$$a_1 = \frac{f_1(\bar{x})}{2^{n_1}} ,$$

by (i) of (32) it follows that

$$f_1(\bar{x} - a_1 y_{n_1}) = 0 , \qquad \text{with } |a_1| < \tfrac{1}{2} .$$

Fix $p > 1$.

Suppose to have $(n_k)_{k=1}^{q_p} \subset (n'_k)$ and $(a_k)_{k=1}^{q_p}$ of numbers, so that

$$\left| \begin{array}{ll} f_m\left(\bar{x} - \sum_{k=1}^{q_p} a_k y_{n_k}\right) = 0 & \text{for } 1 < m < p ; \\ \sum_{k=q_l+1}^{q_{l+1}} |a_k| + \sum_{m=1}^{l} \sum_{k=q_l+1}^{q_{l+1}} |a_k f_m(y_{n_k})| < \frac{1}{2^{l+1}} , & \text{for } 0 < l < p-1 . \end{array} \right. \tag{34}$$

Choose $(n_k)_{k=q_p+1}^{q_{p+1}} \subset (n'_k)$ so that

$$n_{q_p+1} > n_{q_p} , \quad \frac{1}{2^{n_{q_{p+1}}}} < \frac{1}{2^{p+1}\left(1 + \left| f_{p+1}(\bar{x}) - \sum_{k=1}^{q_p} f_{p+1}(y_{n_k}) \right| \right)} . \tag{35}$$

By (32) and (33) there exists $(a'_k)_{k=q_p+1}^{q_{p+1}}$ of numbers so that

$$\left| \begin{array}{l} f_m\left(\sum_{k=q_p+1}^{q_{p+1}} a'_k y_{n_k} \right) = \begin{cases} 0 & \text{for } 1 < m < p , \\ 1 & \text{for } m = p+1 , \end{cases} \\ \sum_{k=q_p+1}^{q_{p+1}} |a'_k| + \sum_{m=1}^{p} \sum_{k=q_p+1}^{q_{p+1}} |a'_k f_m(y_{n_k})| < \frac{1}{2^{n_{q_{p+1}}}} . \end{array} \right. \tag{36}$$

Now set

$$a_i = \left(f_{p+1}(\bar{x}) - \sum_{k=1}^{q_p} f_{p+1}(y_{n_k}) \right) a'_i \qquad \text{for } q_p + 1 \leq i \leq q_{p+1} .$$

Since $n_{q_{p+1}} > n_{q_p+1}$, by (35) and (36) we have (34) for $p+1$ instead of p. So proceeding we construct (n_k) and (a_k).

By (34) it follows that

$$\sum_{k=1}^{\infty} |a_k| = \sum_{p=0}^{\infty} \left(\sum_{k=q_p+1}^{q_{p+1}} |a_k| \right) < \sum_{p=0}^{\infty} \frac{1}{2^{p+1}} = 1 .$$

Moreover we affirm that

$$\bar{x} = \sum_{k=1}^{\infty} a_k y_{n_k} .$$

Let $\varepsilon > 0$.

Let p_ε be a natural number so that

$$\frac{1}{2^{p_\varepsilon - 1}} < \varepsilon . \tag{37}$$

It is sufficient to verify that

$$\left(\left(\bar{x} - \sum_{k=1}^{i} a_k y_{n_k} \right)\right) < \varepsilon \qquad \text{for every } i > q_{p_\varepsilon} . \tag{38}$$

Indeed let p be a natural number so that

$$p > p_\varepsilon , \qquad q_p + 1 \leq i \leq q_{p+1} .$$

By (34) it follows that

$$\begin{cases} f_m\left(\bar{x} - \sum_{k=1}^{q_p} a_k y_{n_k} \right) = 0 & \text{for } 1 \leq m \leq p ; \\ \sum_{m=1}^{p} \sum_{k=q_p+1}^{i} |a_k f_m(y_{n_k})| < \frac{1}{2^{p+1}} . \end{cases}$$

Therefore by (31) and (37) it follows that

$$\left(\left(\bar{x} - \sum_{k=1}^{i} a_k y_{n_k} \right)\right) =$$

$$= \sum_{m=1}^{p} \frac{1}{2^m} \frac{\left| f_m\left(\sum_{k=q_p+1}^{i} a_k y_{n_k} \right) \right|}{1 + \left| f_m\left(\sum_{k=q_p+1}^{i} a_k y_{n_k} \right) \right|} + \sum_{m=p+1}^{\infty} \frac{1}{2^m} \frac{\left| f_m\left(\bar{x} - \sum_{k=1}^{i} a_k y_{n_k} \right) \right|}{1 + \left| f_m\left(\bar{x} - \sum_{k=1}^{i} a_k y_{n_k} \right) \right|} <$$

$$< \frac{1}{2^{p+1}} \sum_{m=1}^{p} \frac{1}{2^m} + \sum_{m=p+1}^{\infty} \frac{1}{2^m} < \frac{1}{2^{p-1}} < \frac{1}{2^{p_\varepsilon - 1}} < \varepsilon .$$

This proves (38) and completes the proof of Th. III.

Proof of Corollary 3: By Th. III it is sufficient to prove the straight implication. If X has an overfilling convex basic sequence, then, by definition of overfilling convex basic and by b) of Th. of [4], X does not admit a continuous norm: hence, by [1], X contains a subspace isomorphic to R^N.

Proof of Remark: Let $\bar{x} \in X$ and let (n_k') be an infinite subsequence of (n). There exists an infinite subsequence (n_k'') of (n_k') such that

$$(39) \qquad ((x_{n_k''} - y_{n_k''})) < ((x_{n_{k-1}''} - y_{n_{k-1}''}))/2, \qquad \text{for every } k > 1.$$

Set $t_0 = 0$.
There exist $(n_k)_{k=t_0+1}^{t_1} \subset (n_k'')$ and $(a_k)_{k=t_0+1}^{t_1}$ of numbers such that

$$\Big(\Big(\bar{x} - \sum_{k=t_0+1}^{t_1} a_k x_{n_k}\Big)\Big) < \frac{1}{2^2}, \quad \sum_{k=t_0+1}^{t_1} |a_k| < \frac{1}{2}, \quad ((x_{n_1} - y_{n_1})) < \frac{1}{2^3}.$$

By (39) it follows that

$$\Big(\Big(\bar{x} - \sum_{k=t_0+1}^{t_1} a_k y_{n_k}\Big)\Big) < \frac{1}{2^2} + \sum_{k=t_0+1}^{t_1} ((a_k(x_{n_k} - y_{n_k}))) <$$
$$< \frac{1}{2^2} + \sum_{k=t_0+1}^{t_1} ((x_{n_k} - y_{n_k})) < \frac{1}{2^2} + \frac{1}{2^2} \sum_{k=t_0+1}^{t_1} \frac{1}{2^k} < \frac{1}{2}.$$

Fix m and suppose to have an increasing sequence $(t_m)_{m=1}^{p}$ of natural numbers, an increasing finite subsequence $(n_k)_{k=1}^{t_p}$ of (n_k') and $(a_k)_{k=1}^{t_p}$ of numbers, such that

$$\left| \begin{array}{l} \Big(\Big(\bar{x} - \sum_{m=1}^{p} \Big(\sum_{k=t_{m-1}+1}^{t_m} a_k y_{n_k}\Big)\Big)\Big) < \frac{1}{2^p}, \\ \sum_{k=t_{m-1}+1}^{t_m} |a_k| < \frac{1}{2^m} \quad \text{for } 1 \leq m \leq p. \end{array} \right.$$

Then there exist an increasing finite sequence $(n_k)_{k=t_p+1}^{t_{p+1}}$ of (n_k') and $(a_k)_{k=t_p+1}^{t_{p+1}}$ of numbers, such that

$$\left| \begin{array}{l} \Big(\Big(\bar{x} - \sum_{m=1}^{p} \Big(\sum_{k=t_{m-1}+1}^{t_m} a_k y_{n_k}\Big) - \sum_{k=t_p+1}^{t_{p+1}} a_k x_{n_k}\Big)\Big) < \frac{1}{2^{p+2}}, \\ \sum_{k=t_p+1}^{t_{p+1}} |a_k| < \frac{1}{2^{p+1}}, \quad n_{t_p+1} > n_{t_p}, \quad ((x_{n_{t_{p+1}}} - y_{n_{t_{p+1}}})) < \frac{1}{2^{p+2}}. \end{array} \right.$$

Hence by (39) it follows that

$$\Big(\Big(\bar{x} - \sum_{m=1}^{p+1} \Big(\sum_{k=t_{m-1}+1}^{t_m} a_k y_{n_k}\Big)\Big)\Big) < \frac{1}{2^{p+2}} + \sum_{k=t_p+1}^{t_{p+1}} ((a_k(x_{n_k} - y_{n_k}))) <$$
$$< \frac{1}{2^{p+2}} + \sum_{k=t_p+1}^{t_{p+1}} ((x_{n_k} - y_{n_k})) < \frac{1}{2^{p+2}} \Big\{1 + \sum_{k=1}^{t_{p+1}-t_p} \frac{1}{2^k}\Big\} < \frac{1}{2^{p+1}}.$$

This completes the proof of Remark.

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