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Fixed Points of Lower Semicontinuous Multifunctions and Applications: Alternative and Minimax Theorems (**)

Punti fissi di multifunzioni semicontinue inferiormente ed applicazioni: teoremi di alternativa e di minimax

RIASSUNTO. — In questo lavoro vengono stabiliti un teorema di coincidenza ed un teorema di punto fisso per multifunzioni semicontinue inferiormente su particolari valori delle quali si fa soltanto l'ipotesi che siano insiemi non vuoti. Questi risultati vengono quindi utilizzati per dimostrare nuovi teoremi di alternativa e di minimax.

INTRODUCTION

Two of the most important notions on multifunctions are that of lower and upper semicontinuity, each of which extends the usual concept of continuity for a single-valued function. Let us recall that, given two topological spaces S, Σ, a multifunction Φ from S into Σ is said to be *lower* (resp. *upper*) *semicontinuous* if, for every open set $\Omega \subseteq \Sigma$, the set $\Phi^-(\Omega) = \{s \in S: \Phi(s) \cap \cap \Omega \neq \emptyset\}$ (resp. $\Phi^+(\Omega) = \{s \in S: \Phi(s) \subseteq \Omega\}$) is open in S. In particular, the above notions play a very important role in fixed point theory for multifunctions. In turn, this theory has an extremely wide area of applications including minimax theory, game theory, variational inequalities, differential equations. Perhaps, the most known and used fixed point theorem for multifunctions is the so called Fan-Kakutani theorem (see, for instance, the Theorem on p. 186 of [1]), stated below as Theorem A.

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THEOREM A: *Let X be a non-empty compact convex subset of a Hausdorff locally convex topological vector space. Let F be an upper semicontinuous multifunction from X into X, with non-empty, closed and convex values. Then, there exists $x^* \in X$ such that $x^* \in F(x^*)$.*

Consider now the following simple and meaningful example.

EXAMPLE 1: Let F_1, F_2, F_3 be the multifunctions from $[0, 1]$ into $[0, 1]$ defined by putting:

$$F_1(x) = \begin{cases} \left[\frac{x}{3}, \frac{x}{2}\right] & \text{if } x \in]0, 1], \\ ]0, 1] & \text{if } x = 0, \end{cases}$$

$$F_2(x) = \begin{cases} \{1\} & \text{if } x \in \left[0, \frac{1}{2}\right[, \\ \{0, 1\} & \text{if } x = \frac{1}{2}, \\ \{0\} & \text{if } x \in \left]\frac{1}{2}, 1\right], \end{cases}$$

$$F_3(x) = \begin{cases} \left[\frac{1}{2}, 1\right] & \text{if } x \in \left[0, \frac{1}{2}\right[, \\ [0, 1] & \text{if } x = \frac{1}{2}, \\ \left[0, \frac{1}{2}\right] & \text{if } x \in \left]\frac{1}{2}, 1\right]. \end{cases}$$

Of course, these three multifunctions are upper semicontinuous. Observe that F_1 has no fixed point, although $F_1(x)$ is convex for all $x \in [0, 1]$ and closed for all $x \in]0, 1]$. Likewise, F_2 has no fixed point, although $F_2(x)$ is closed for all $x \in [0, 1]$ and convex for all $x \in [0, 1] \setminus \{\frac{1}{2}\}$. This shows that, in Theorem A, it is essential that each value of F is non-empty, closed and convex. Observe, furthermore, that F_3 has just one fixed point, although each set $F(x)$ has no isolated point. This shows that Theorem A cannot guarantee the existence of many fixed points of F.

The purpose of this paper is to show how the two drawbacks stressed by Example 1 with regard to upper semicontinuous multifunctions, disappear in the case of lower semicontinuous multifunctions. In other words, we establish a fixed point theorem for lower semicontinuous multifunctions that are allowed to have many non-closed or non-convex values (Theorem 2). It is also proved that such multifunctions have infinitely many fixed points, provided each of their values has no isolated point. Before Theorem 2, we prove a kind of coincidence theorem (Theorem 1). We then present some consequences of Theorems 1 and 2. More precisely, these consequences consist in new alternative and minimax theorems.

Results

Before stating our main results, let us introduce some notation. If S is a topological space and $A \subseteq S$, $\dim_S(A) \leq 0$ means that $\dim(T) \leq 0$ for every set $T \subseteq A$ which is closed in S, where $\dim(T)$ denotes the covering dimension of T. If $(\Sigma, \|\cdot\|)$ is a normed space and $B \subseteq \Sigma$, $(\overline{B})_{\|\cdot\|}$ denotes the closure of B in the $\|\cdot\|$-topology. Finally, let Γ, Λ be two non-empty sets. From now on, we call *multifunction from* Γ *into* Λ any function F from Γ into the family of all *non-empty* subsets of Λ. If $\Gamma \subseteq \Lambda$, we put $\mathrm{Fix}(F) = \{y \in \Gamma : y \in F(y)\}$.

Our first result is the following

Theorem 1: *Let* $(U, \|\cdot\|)$, $(V, \|\cdot\|_1)$ *be two Banach spaces and* $X \subseteq U$, $Y \subseteq V$ *two non-empty sets. Let* τ *be a topology on* X, *weaker than the* $\|\cdot\|$*-topology, and* τ_1 *a topology on* Y, *weaker than the* $\|\cdot\|_1$*-topology, such that* (X, τ) *and* (Y, τ_1) *are compact and Hausdorff. Let* C, Z *be two subsets of* X *and* D, W *two subsets of* Y, *with* C, D *countable and* $\dim_{(X,\tau)}(Z) \leq 0$, $\dim_{(Y,\tau_1)}(W) \leq 0$. *Finally, let* F *be a multifunction from* X *into* V *and* G *a multifunction from* Y *into* U *such that:*

(1) F *is* $(\tau, \|\cdot\|_1)$*-lower semicontinuous,* $F(x)$ *is* $\|\cdot\|_1$*-closed for every* $x \in X \setminus C$, $\overline{(F(x))}_{\|\cdot\|_1}$ *is convex for every* $x \in X \setminus Z$ *and* $\overline{(\mathrm{conv}\,(F(X)))}_{\|\cdot\|_1} \subseteq Y$;

(2) G *is* $(\tau_1, \|\cdot\|)$*-lower semicontinuous,* $G(y)$ *is* $\|\cdot\|$*-closed for every* $y \in Y \setminus D$, $\overline{(G(y))}_{\|\cdot\|}$ *is convex for every* $y \in Y \setminus W$ *and* $\overline{(\mathrm{conv}\,(G(Y)))}_{\|\cdot\|} \subseteq X$.

Under such hypotheses, there exists $(x^*, y^*) \in X \times Y$ *such that* $x^* \in G(y^*)$ *and* $y^* \in F(x^*)$.

Proof: By Theorem 7.1 of [2], there exist a $(\tau, \|\cdot\|_1)$-continuous function $f: X \to Y$ and a $(\tau_1, \|\cdot\|)$-continuous function $g: Y \to X$ such that $f(x) \in F(x)$ and $g(y) \in G(y)$ for every $(x, y) \in X \times Y$. Now, put $h(x, y) = (g(y), f(x))$ for every $(x, y) \in X \times Y$. Thus, with obvious meaning of the symbols, h is a $(\tau \times \tau_1, \|\cdot\| \times \|\cdot\|_1)$-continuous function from $X \times Y$ into $X \times Y$. Since the topology $\tau \times \tau_1$ is weaker than the $\|\cdot\| \times \|\cdot\|_1$-topology, h is $(\|\cdot\| \times \|\cdot\|_1, \|\cdot\| \times \|\cdot\|_1)$-continuous. Let $T = \overline{(\mathrm{conv}\,(h(X \times Y)))}_{\|\cdot\| \times \|\cdot\|_1}$. Thus, from our assumptions, it follows that T is a $(\|\cdot\| \times \|\cdot\|_1)$-compact convex subset of $X \times Y$. Then, applying the classical Schauder fixed point theorem to $h|_T$, we find $(x^*, y^*) \in X \times Y$ such that $x^* = g(y^*)$, $y^* = f(x^*)$. Hence, $x^* \in G(y^*)$ and $y^* \in F(x^*)$, that is our conclusion.

Our main fixed point theorem is the following.

Theorem 2: *Let* $(U, \|\cdot\|)$, X, τ, C, Z *be as in Theorem* 1 *and let* F *be a* $(\tau, \|\cdot\|)$*-lower semicontinuous multifunction from* X *into* U *such that* $F(x)$ *is* $\|\cdot\|$*-closed for every* $x \in X \setminus C$, $\overline{(F(x))}_{\|\cdot\|}$ *is convex for every* $x \in X \setminus Z$ *and* $\overline{(\mathrm{conv}\,(F(X)))}_{\|\cdot\|} \subseteq X$.

Then, $\mathrm{Fix}(F) \neq \emptyset$. *If, in addition, for every* $x \in \mathrm{Fix}(F)$, x *is a* $\|\cdot\|$*-accumulation point of* $F(x)$, *then* $\mathrm{Fix}(F)$ *is uncountable.*

PROOF: Our first assertion, that is $\mathrm{Fix}(F) \neq \emptyset$, follows from reasonings similar to that used in the proof of Theorem 1. So, suppose that, for every $x \in \mathrm{Fix}(F)$, x is a $\|\cdot\|$-accumulation point of $F(x)$. Consider the multifunction $\tilde{F}$ from X into X defined by putting:

$$\tilde{F}(x) = \begin{cases} F(x) \setminus \{x\} & \text{if } x \in \mathrm{Fix}(F), \\ F(x) & \text{if } x \in X \setminus \mathrm{Fix}(F). \end{cases}$$

Observe that $\overline{(\tilde{F}(x))}_{\|\cdot\|} = \overline{(F(x))}_{\|\cdot\|}$ for every $x \in X$. From this, it follows that $\tilde{F}$ is $(\tau, \|\cdot\|)$-lower semicontinuous. On the other hand, $\tilde{F}(x)$ is $\|\cdot\|$-closed for every $x \in X \setminus (C \cup \mathrm{Fix}(F))$ and $\overline{(\tilde{F}(x))}_{\|\cdot\|}$ is convex for every $x \in X \setminus Z$. By definition, $\mathrm{Fix}(\tilde{F}) = \emptyset$. Hence, $\mathrm{Fix}(F)$ must be uncountable, since, otherwise, by the first part of the theorem, we would have $\mathrm{Fix}(\tilde{F}) \neq \emptyset$.

REMARK 1: We do not know, in general, whether, in Theorem 2, to guarantee that $\mathrm{Fix}(F)$ is uncountable, it suffices to assume that, for every $x \in \mathrm{Fix}(F)$, $F(x)$ contains more than a point. However, this is true if, for instance, for every $x, y \in X$, with $x \neq y$, there exist a τ-open set $\Omega_1 \subseteq U$ and a $\|\cdot\|$-open set $\Omega_2 \subseteq U$ such that $x \in \Omega_1$, $y \in \Omega_2$, $\Omega_1 \cap \Omega_2 = \emptyset$. In fact, in this case, it is possible to check again that the multifunction $\tilde{F}$ defined in the proof of Theorem 2 is $(\tau, \|\cdot\|)$-lower semicontinuous.

The next proposition will be useful in the sequel.

PROPOSITION 1: *Let* S, Σ *be two topological spaces and* Φ *a multifunction from* S *into* Σ *such that the set* $\{\sigma \in \Sigma : \Phi^-(\sigma)$ *is open*$\}$ *is dense in* Σ *and* $\overline{\Phi(s)} = \overline{\mathrm{int}(\Phi(s))}$ *for every* $s \in S$. *Then,* Φ *is lower semicontinuous.*

PROOF: Let Ω by any open set in Σ and $s_0 \in \Phi^-(\Omega)$. Since $\overline{\Phi(s_0)} = \overline{\mathrm{int}(\Phi(s_0))}$, we have $\mathrm{int}(\Phi(s_0)) \cap \Omega \neq \emptyset$. Since the set $\{\sigma \in \Sigma : \Phi^-(\sigma)$ is open$\}$ is dense in Σ, there exists $\bar{\sigma} \in \mathrm{int}(\Phi(s_0)) \cap \Omega$ such that $\Phi^-(\bar{\sigma})$ is open. Hence, we have $s_0 \in \Phi^-(\bar{\sigma}) \subseteq \Phi^-(\Omega)$, that completes the proof.

Now, we present some applications of Theorems 1 and 2. We begin by proving a minimax theorem. In the sequel, the notions of lower and upper semicontinuity, when referred to real functions, are the usual ones. We also recall that a real function φ, defined on some convex subset of a vector space, is said to be *quasi-convex* (resp. *quasi-concave*), if, for every $t \in \mathbf{R}$, the set $\varphi^{-1}(]-\infty, t[)$ (resp. $\varphi^{-1}(]t, +\infty[)$) is convex.

THEOREM 3: *Let* $(U, \|\cdot\|)$, $(V, \|\cdot\|_1)$, X, Y, τ, τ_1, Z, W *be as in Theorem* 1. *Suppose, in addition, that* X, Y *are convex and norm-closed. Let* f *be a real function on* $X \times Y$ *such that:*

(1) *the function* $f(x, \cdot)$ *is* τ_1*-lower semicontinuous for every* $x \in X$ *and quasi-convex for every* $x \in X \setminus Z$;

(2) *the function* $f(\cdot, y)$ *is* τ*-upper semicontinuous for every* $y \in Y$ *and quasi-convex for every* $y \in Y \setminus W$.

Then, we have $\max_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \max_{x \in X} f(x, y)$.

PROOF: Let $\alpha = \max_{x \in X} \min_{y \in Y} f(x, y)$ and $\beta = \min_{y \in Y} \max_{x \in X} f(x, y)$. Of course, $\alpha \leq \beta$. Assume $\alpha < \beta$. Fix $a, b \in]\alpha, \beta[$ such that $a < b$. For every $\bar{x} \in X$, $\bar{y} \in Y$, put

$$F(\bar{x}) = \overline{(\{y \in Y : f(\bar{x}, y) < a\})}_{\tau_1}, \qquad \Phi(\bar{x}) = \{y \in Y : f(\bar{x}, y) < a\},$$

$$G(\bar{y}) = \overline{(\{x \in X : f(x, \bar{y}) > b\})}_{\tau}, \qquad \Psi(\bar{y}) = \{x \in X : f(x, \bar{y}) > b\}.$$

Plainly, $F(x) \neq \emptyset$, $G(y) \neq \emptyset$ for every $x \in X$, $y \in Y$. Moreover, it is possible to check that the multifunctions F, G satisfy the hypotheses of Theorem 1 and that $F(x) \subset \Phi(x)$, $G(y) \subseteq \Psi(y)$ for every $x \in X$, $y \in Y$. Hence, by that theorem, there exists $(x^*, y^*) \in X \times Y$ such that $x^* \in \Psi(y^*)$ and $y^* \in \Phi(x^*)$, that is $b < f(x^*, y^*) < a$, a contradiction.

Theorem 3, substantially, is an improvement of the classical Sion minimax theorem [3].

Now, we establish three alternative theorems.

THEOREM 4: *Let* $(U, \|\cdot\|)$, X, τ, Z *be as in Theorem 3 and let* f *be a real function on* $X \times X$ *such that*:

(1) *the function* $f(x, \cdot)$ *is* $\|\cdot\|$*-lower semicontinuous for every* $x \in X$ *and quasi-convex for every* $x \in X \setminus Z$;

(2) *the function* $f(\cdot, y)$ *is* τ*-upper semicontinuous for every* $y \in X$.

Then, for any τ*-lower semicontinuous real function* φ *on* X, *at least one of the following assertions holds*:

(i) *There exists* $x_0 \in X$ *such that* $\inf_{y \in X} f(x_0, y) > \varphi(x_0)$.

(ii) *There exists* $x^* \in X$ *such that* $f(x^*, x^*) < \varphi(x^*)$.

PROOF: Let φ be as in the statement. For every $x \in X$, put

$$F(x) = \overline{(\{y \in X : f(x, y) < \varphi(x)\})}_{\|\cdot\|} \quad \text{and} \quad \Phi(x) = \{y \in X : f(x, y) < \varphi(x)\}.$$

Suppose that assertion (i) does not hold. Then, $F(x) \neq \emptyset$ for every $x \in X$. It is seen that the multifunction F satisfies the hypotheses of Theorem 2 and that $F(x) \subset \Phi(x)$ for every $x \in X$. Hence, by that theorem, there exists $x^* \in X$ such that $x^* \in \Phi(x^*)$, and so assertion (ii) holds.

THEOREM 5: *Let* $(U, \|\cdot\|)$, X, τ, Z *be as in Theorem* 3, *with* X *non reducing to a single point. Let* f *be a real function on* $X \times X$ *such that*:

(1) *the function* $f(x, \cdot)$ *is* $\|\cdot\|$*-continuous for every* $x \in X$ *and quasi-convex for every* $x \in X \setminus Z$;

(2) *the set* $\{y \in X: f(\cdot, y)$ *is* τ*-upper semicontinuous*$\}$ *is* $\|\cdot\|$*-dense in* X.

Then, for any τ*-lower semicontinuous real function* φ *on* X, *at least one of the following assertions holds*:

(i) *There exists* $x_0 \in X$ *such that* $\inf_{y \in X} f(x_0, y) > \varphi(x_0)$.

(ii) *The set* $\{x \in X: f(x, x) < \varphi(x)\}$ *is uncountable.*

PROOF: Let φ be as in the statement. Suppose that assertion (i) does not hold. For every $x \in X$, put

$$\Phi(x) = \{y \in X: f(x, y) \le \varphi(x)\}, \qquad H(x) = \{y \in X: f(x, y) < \varphi(x)\}$$

and

$$F(x) = \overline{(H(x))}_{\|\cdot\|}.$$

Thanks to (1), $H(x)$ is $\|\cdot\|$-open in X and $F(x) \subseteq \Phi(x)$. Thanks to (2), the set $\{y \in X: H^{-}(y)$ is τ-open$\}$ is $\|\cdot\|$-dense in X. Then, by Proposition 1, the multifunction H, and so F, is $(\tau, \|\cdot\|)$-lower semicontinuous. Since X is not a singleton, for every $x \in \mathrm{Fix}(F)$, x is a $\|\cdot\|$-accumulation point of $F(x)$. Hence, by Theorem 2, $\mathrm{Fix}(\Phi)$ is uncountable, that is our conclusion.

REMARK 2: It is worth noticing that, in Theorem 4, condition (2) is not replaceable with condition (2) of Theorem 5, even if $Z = \emptyset$. To see this, it suffices to take $X = [0, 1]$, with the usual topology, and $f: X \times X \to \mathbf{R}$ defined as follows:

$$f(x, y) = \begin{cases} 1 & \text{if } (x, y) \in \{0\} \times [0, 1[\,, \\ -1 & \text{if } (x, y) \in (]0, 1] \times \{0\}) \cup \{(0, 1)\}\,, \\ y & \text{if } (x, y) \in\,]0, 1] \times]0, 1]\,. \end{cases}$$

Of course, for every $x \in X$, the function $f(x, \cdot)$ is lower semicontinuous and quasi-convex. Moreover, $\{y \in X: f(\cdot, y)$ is upper semicontinuous$\} = [0, 1[$. However, for every $x \in X$, $f(x, x) > 0$ and there is $y \in X$ such that $f(x, y) < 0$.

The final result is the following.

THEOREM 6: *Let* $(U, \|\cdot\|)$, X, τ *be as in Theorem* 3. *Suppose, in addition, that the algebraic dimension of* X *is greater than one. Let* f *be a real function on* $X \times X$ *such that*:

(1) *for every* $x \in X$, *the function* $f(x, \cdot)$ *is* $\|\cdot\|$*-continuous and, for each* $r \in \mathbf{R}$, *the set* $\{y \in X: f(x, y) = r\}$ *is convex and* $\|\cdot\|$*-nowhere dense in* X;

(2) *the sets* $\{y \in X : f(\cdot, y)$ *is* τ*-upper semicontinuous*$\}$ *and* $\{y \in X : f(\cdot, y)$ *is* τ*-lower semicontinuous*$\}$ *are both* $\|\cdot\|$*-dense in* X.

Then, for any τ*-continuous real function* φ *on* X, *at least one of following assertions holds:*

(i) *There exists* $x_0 \in X$ *such that* $\inf_{y \in X} f(x_0, y) > \varphi(x_0)$.

(ii) *There exists* $x_1 \in X$ *such that* $\sup_{y \in X} f(x_1, y) < \varphi(x_1)$.

(iii) *The set* $\{x \in X : f(x, x) = \varphi(x)\}$ *is uncountable.*

Proof: Let φ be as in the statement. Suppose that assertions (i) and (ii) do not hold. For every $x \in X$, put

$$F(x) = \{y \in X : f(x, y) = \varphi(x)\}.$$

By Theorem 2.2 of [4] and by the proof of Theorem 2.2 of [5] (see also Remark 2.3), the multifunction F is $(\tau, \|\cdot\|)$-lower semicontinuous. Let $\bar{x} \in \mathrm{Fix}(F)$. By our assumptions on X, it follows that $X \setminus \{\bar{x}\}$ is non-empty and $\|\cdot\|$-connected. On the other hand, there are $y', y'' \in X \setminus \{\bar{x}\}$ such that $f(\bar{x}, y') < \varphi(\bar{x})$ and $f(\bar{x}, y'') > \varphi(\bar{x})$. Thus, by the $\|\cdot\|$-continuity of the function $f(\bar{x}, \cdot) - \varphi(\bar{x})$, there is $\bar{y} \in F(\bar{x}) \setminus \{\bar{x}\}$. Taking into account that each $F(x)$ is convex, our conclusion, that is assertion (iii), follows from Theorem 2.

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