A Rouché's Type Theorem in Several Complex Variables (**)  

Un teorema del tipo di Rouché in più variabili complesse  

Sotto. — Si stabilisce una estensione n-dimensionale del teorema di Rouché. Come applicazione se ne deduce una rapida dimostrazione del teorema di Bézout.  

Let $\Omega$ be a relatively compact open domain of $\mathbb{C}^n$ (or of any n-dimensional Stein manifold) with connected $C^1$ boundary $\partial \Omega = \Gamma$. Let $f_1, \ldots, f_n, g_1, \ldots, g_n$ be complex-valued functions holomorphic on $\Omega$ and $C^1$ on $\partial \Omega$, and consider the systems of equations  

(1) \[ f_\alpha(z) = 0, \ldots, f_n(z) = 0, \]  

(2) \[ g_\alpha(z) = 0, \ldots, g_n(z) = 0. \]  

In this paper we shall prove the following theorem:  

**THEOREM 1:** Set  

\[ \Gamma_\alpha = \{ z \in \Gamma: |f_\alpha(z) - g_\alpha(z)| < |g_\alpha(z)| \}, \quad \alpha = 1, \ldots, n, \]  

and assume that  

\[ \Gamma = \bigcup_{\alpha=1}^n \Gamma_\alpha. \]  

Then the systems (1) and (2) have finite numbers $N_1$ and $N_2$ of solutions all contained in $\Omega$ (each solution being counted with its multiplicity) and $N_1 = N_2$.  

Moreover, if $\bigcap_{\alpha=1}^n \Gamma_\alpha = \emptyset$, then $0 = N_1 = N_2$.  

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This theorem is an extension to n complex variables of the classical Rouché’s theorem for holomorphic functions of one complex variable (cf. Ahlfors [1], p. 153).

A simpler extension is given by the following assertion:

**Theorem 2**: Assume that

\[ \sum_{a=1}^{n} |f_a(z) - g_a(z)|^2 < \sum_{a=1}^{n} |g_a(z)|^2 \]

for every \( z \in \Omega \). Then the systems (1) and (2) have finite numbers \( N_1 \) and \( N_2 \) of solutions all contained in \( \Omega \) and \( N_1 = N_2 \).

The proof of the latter is quite immediate (cf. n. 2, footnote (1)).

As an application of Theorem 1 we will show that it allows one to give a simple proof of Bézout’s theorem, parallel to the proof of the fundamental theorem of Algebra as a corollary of Rouché’s theorem for one complex variable (cf. Alexandroff-Hopf [2], p. 469). On the contrary Theorem 2 is not sufficient for this purpose (cf. n. 3, Remark 3).

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1. **Proof of Theorem 1**

Set \( V_a = \{ \zeta \in \bar{\Omega} : f_a(\zeta) = 0 \} \), \( W_a = \{ \zeta \in \bar{\Omega} : g_a(\zeta) = 0 \} \), \( a = 1, \ldots, n \). Then \( I_a \cap I \setminus V_a \), \( I_a \cap I \setminus W_a \), and hence \( I = \bigcup_{a=1}^{n} I_a \cap I \setminus \bigcap_{a=1}^{n} V_a \), \( I \setminus I_a \cap I \setminus W_a \), which means that \( \bigcap_{a=1}^{n} V_a \) and \( \bigcap_{a=1}^{n} W_a \) do not intersect \( I = \partial \Omega \). It follows that \( \bigcap_{a=1}^{n} V_a \), \( \bigcap_{a=1}^{n} W_a \) are compact analytic subvarieties of \( \Omega \) and, as such, finite subsets of \( \Omega \) (cf. Gunning-Rossi [6], p. 106). This shows that the solutions of the systems (1) and (2) are at most finitely many and contained in \( \Omega \). Hence also \( N_1 \) and \( N_2 \), the sums of the multiplicities of these solutions, are finite numbers.

We recall that the multiplicity \( \mu \) of a solution \( x^0 \in \Omega \) of the system (1) can be understood, in the most elementary way, as follows: \( \mu = 1 \) if and only if \( x^0 \) is a simple solution, that is, the differentials \( df_1, \ldots, df_n \) are independent at \( x^0 \); in general, for almost all \( \alpha \in \mathbb{C}^n \) sufficiently close to the origin, the system \( f_1(\zeta) = a_1, \ldots, f_n(\zeta) = a_n \) has precisely \( \mu \) simple solutions within a small neighbourhood of \( x^0 \) (cf. Milnor [8], Appendix B).

Next, consider the Cauchy kernel

\[ \omega = (2\pi i)^{-n} \frac{\partial x_1}{\zeta_1} \wedge \ldots \wedge \frac{\partial x_n}{\zeta_n} \]

and the maps \( f = (f_1, \ldots, f_n) \), \( g = (g_1, \ldots, g_n) : \bar{\Omega} \to \mathbb{C}^n \). We shall derive Theorem 1 from the following lemma, which is indeed the main point in the proof and can deserve some interest for itself (cf. n. 2, Remark 1).
Lemma: There exist $C^1$ singular $n$-cycles with integral coefficients $\gamma^1_n, \gamma^2_n$ of $\Omega$ such that

\[
\text{Supp} (\gamma^1_n) \subset \bigcap_{a=1}^n V_a, \quad N_1 = \int_{\gamma^1_n} f^*\omega,
\]

\[
\text{Supp} (\gamma^2_n) \subset \bigcap_{a=1}^n W_a, \quad N_2 = \int_{\gamma^2_n} g^*\omega.
\]

Moreover, if $\bigcap_{a=1}^n \Gamma_a \neq \emptyset$, we may assume

\[
\gamma^1_n = \gamma^2_n = \gamma_n, \quad \text{Supp} (\gamma_n) \subset \bigcap_{a=1}^n \Gamma_a;
\]

while, if $\bigcap_{a=1}^n \Gamma_a = \emptyset$, then

\[
\gamma^1_n \sim 0 \quad \text{in} \quad \bigcap_{a=1}^n V_a, \quad \gamma^2_n \sim 0 \quad \text{in} \quad \bigcap_{a=1}^n W_a.
\]

The proof of this lemma will be given later. Plainly, it already disposes of the case when $\bigcap_{a=1}^n \Gamma_a = \emptyset$. Otherwise, it enables one to extend the usual proof of Rouché's theorem for one complex variable, based on the principle of the argument.

As a matter of fact, let us consider the $C^1$ maps

\[
F_b = \left( s_1, \ldots, s_{n-1}, \frac{\delta_b}{\delta_b}, f_b, f_{b+1}, \ldots, f_n \right) : \bigcap_{a=1}^n \Gamma_a \to \mathbb{C}^n,
\]

$b = 1, \ldots, n$. Then we have on $\bigcap_{a=1}^n \Gamma_a$:

\[
f^*\omega - g^*\omega = \sum_{k=1}^n F_b^k \omega,
\]

so that we are led to show that

\[
\int_{\gamma_n} F_b^k \omega = \int_{F_b(\gamma_n)} \omega = 0, \quad b = 1, \ldots, n.
\]

Setting

\[
A_b = \{ z \in \mathbb{C}^n : z_1 \neq 0, \ldots, z_{n-1} \neq 0, |z_n - 1| < 1, z_{n+1} \neq 0, \ldots, z_n \neq 0 \},
\]

$b = 1, \ldots, n$, it is clear that $F_b(\bigcap_{a=1}^n \Gamma_a) \subset A_b$, and hence

\[
\text{Supp} (F_b(\gamma_n)) = F_b(\text{Supp} (\gamma_n)) \subset A_b.
\]
Moreover the singular homology group with integral coefficients $H_n(A_b)$ is zero, since $A_b$ has the homotopy type of a $(n - 1)$-dimensional torus. It follows that

$$F_{sb}(y_a) \sim 0 \text{ in } C^n \setminus \{\xi \in C^n : \xi_1 \ldots \xi_n = 0\},$$

$b = 1, \ldots, n$, which concludes the proof of Theorem 1.

2. Proof of the Lemma

We shall use the following facts:

i) Given an exact sequence

$$0 \to K_0 \to K_1 \to \ldots \to K_n \to 0$$

of chain complexes (cochain complexes), there exists a canonical homomorphism of graded modules

$$\epsilon: H(K_n) \to H(K_0)$$

of degree $-(n - 1)$ (degree $n - 1$). Moreover $\epsilon$ is natural, in the sense that, if

$$
\begin{array}{cccccc}
0 & \to & K_0 & \to & K_1 & \to & \ldots & \to & K_n & \to & 0 \\
& \downarrow & \phi_1 & & \downarrow & \phi_2 & & \cdots & \downarrow & \phi_n & & \\
0 & \to & K'_0 & \to & K'_1 & \to & \ldots & \to & K'_n & \to & 0
\end{array}
$$

is a commutative diagram of chain maps (cochain maps) with exact rows, then the diagram

$$
\begin{array}{ccc}
H(K_n) & \to & H(K_0) \\
\phi_{n*} & & \phi_{0*} \\
\uparrow & & \uparrow \\
H(K'_n) & \to & H(K'_0)
\end{array}
$$

is also commutative.

This is a straightforward generalization of the parallel well-known fact concerning a short exact sequence. As in that case, $\epsilon$ will be termed connecting homomorphism.

ii) Consider the Martinelli kernel

$$k = (-1)^{n(n-1)/2} \frac{(n-1)!}{(2\pi i)^n} \frac{d\xi_1 \wedge \ldots \wedge d\xi_n}{(\xi_1^2 + \ldots + |\xi_n|^2)^n} \sum_{k=1}^n (-1)^{n-1-k} \frac{d\xi_1 \wedge \ldots \wedge d\xi_{k} \wedge \ldots \wedge d\xi_n}{d\xi_k},$$

and assume that $\Omega$ is oriented by the form $(i/2)^n \sum_{k=1}^n d\xi_1 \wedge \ldots \wedge d\xi_k$ and $\Gamma$ as the
boundary of \( \Omega \). Then
\[
N_1 = \int_{\tilde{f}} \kappa, \quad N_2 = \int_{\tilde{g}} \kappa
\]
(cf. Severi [10], p. 192) (1).

iii) Consider the generalized Mayer-Vietoris exact sequence (cf. Bott-Tu [3]) for complex-valued \( C^\infty \) differential forms relative to the open covering of \( \mathbb{C}^n \setminus \{O\} \) made by the sets \( U_a = \{ z \in \mathbb{C}^n : \phi_a^- = 0 \}, \quad a = 1, \ldots, n \). If
\[
e^*: H^*_C \left( \bigcap_{a \in A} U_a \right) \to H^*_{FB} (\mathbb{C}^n \setminus \{O\})
\]
is the induced connecting homomorphism, then
\[
e^*([\omega]) = [\kappa].
\]

This fact is essentially known: it is indeed a straightforward consequence of a result found in Harvey [7] (Theorem 2.2) (2).

That being stated, let
\[
e^*_k : H_k (\Gamma) \to H_{k-(n-1)} (\Gamma \setminus \bigcup_{a=1}^n V_a),
\]
\[
e^*_1 : H^*_C (\Gamma \setminus \bigcup_{a=1}^n V_a) \to H^*_{FB} (\Gamma \setminus \bigcup_{a=1}^n V_a),
\]
be the connecting homomorphisms induced by the Mayer-Vietoris sequences relative to the open covering \( \{ \Gamma \setminus V_1, \ldots, \Gamma \setminus V_n \} \) of \( \Gamma \), respectively for \( C^1 \) singular chains with integral coefficients and for complex-valued regular differential forms (cf. Whitney [12]) (2). Denote by \( e_\Gamma \) the canonical generator of \( H_{n-1} (\Gamma) \) and choose a representative \( \gamma_k^* \) of the class \( e^*_k (e_\Gamma) \in H_k (\Gamma \setminus \bigcup_{a=1}^n V_a) \).

Since plainly \( e^*_k \) and \( e^*_k \) are dual with respect to the Kronecker index \( \langle \cdot, \cdot \rangle \),

(1) It follows that
\[
N_1 = \int_{\tilde{f}} \kappa, \quad N_2 = \int_{\tilde{g}} \kappa.
\]

Now, suppose that \( f \) and \( g \) satisfy
\[
dist(f(z), g(z)) < dist(g(z), O),
\]
for every \( z \in \Gamma \). The \((2n-1)\)-cycles \( f_\Gamma (\Gamma), g_\Gamma (\Gamma) \) are then homotopic in \( \mathbb{C}^n \setminus \{O\} \) (cf. Alexandroff-Hopf [2], p. 459), and hence the above integrals are equal. This proves Theorem 2.

(2) Cf. also Griffiths-Harris [5], Chapter 5.

(2) Since \( \Gamma \) is assumed to be only \( C^1 \), we may not consider \( C^\infty \) forms on \( \Gamma \).
it follows from i), ii) and iii) that

\[
\int f^* \omega = \langle [f^* \omega], c^*_a(\sigma_\rho) \rangle = \langle c^*_a([f^* \omega]), \sigma_\rho \rangle = c^*_a([f^* k]), \sigma_\rho \rangle = \int f^* k = N_1.
\]

In the same way one shows that

\[
\int g^* \omega = N_2,
\]

where \( c^*_a \) is a representative of the class \( c^*_a(\sigma_\rho) \in H_a\left( \bigcap_{x=1}^{n} \mathcal{W}_x \right) \) (with obvious meaning of \( c^*_a \)).

Next, consider the connecting homomorphism

\[
e_\omega : H_a(\mathcal{I}) \rightarrow H_{a-(n-1)} \left( \bigcap_{x=1}^{n} \mathcal{I}_x \right)
\]

induced by the Mayer-Vietoritis sequence for \( C^1 \) singular chains with integral coefficients relative to the open covering \( \{ \mathcal{I}_x \}_{x=1}^{n} \) of \( \mathcal{I} \). Then the inclusions \( \mathcal{I}_x \subset \mathcal{I} \setminus \mathcal{V}_x, \mathcal{I}_x \subset \mathcal{I} \setminus \mathcal{W}_x, x = 1, ..., n \) induce commutative diagrams

\[
\begin{array}{ccc}
H_a(\mathcal{I}) & \xrightarrow{e_\omega} & H_{a-(n-1)} \left( \bigcap_{x=1}^{n} \mathcal{I}_x \right) \\
\downarrow & & \downarrow \\
H_a(\mathcal{I} \setminus \bigcup_{x=1}^{n} \mathcal{V}_x) & \xrightarrow{e_\omega} & H_{a-(n-1)} \left( \bigcap_{x=1}^{n} \mathcal{I}_x \setminus \bigcup_{x=1}^{n} \mathcal{W}_x \right)
\end{array}
\]

It follows that, if \( \bigcap_{x=1}^{n} \mathcal{I}_x = \emptyset \), any representative \( \gamma_\omega \) of the class \( e_\omega(\sigma_\rho) \in H_a \left( \bigcap_{x=1}^{n} \mathcal{I}_x \right) \) represents also the classes \( e_\omega^1(\sigma_\rho) \) and \( e_\omega^2(\sigma_\rho) \), so that we may take \( \gamma_\omega^1 = \gamma_\omega^2 = \gamma_\omega \). On the contrary, if \( \bigcap_{x=1}^{n} \mathcal{I}_x = \emptyset \), then the homomorphisms \( e_\omega^1, e_\omega^2 \) are both zero, and consequently \( \gamma_\omega^1 \sim 0 \) in \( \mathcal{I} \setminus \bigcup_{x=1}^{n} \mathcal{V}_x \), \( \gamma_\omega^2 \sim 0 \) in \( \mathcal{I} \setminus \bigcup_{x=1}^{n} \mathcal{W}_x \).

The proof of the lemma is then completed.

**Remark 1:** The lemma just proved can be generalized to the case that \( f^* \omega, g^* \omega \) are replaced by two \( n \)-forms \( \varphi, \psi \) such that \( \varphi \) is \( C^1 \) on \( \mathcal{I} \setminus \bigcup_{x=1}^{n} \mathcal{V}_x \) and holomorphic on \( \mathcal{I} \setminus \bigcup_{x=1}^{n} \mathcal{W}_x \), \( \psi \) is \( C^1 \) on \( \mathcal{I} \setminus \bigcup_{x=1}^{n} \mathcal{W}_x \) and holomorphic on \( \mathcal{I} \setminus \bigcup_{x=1}^{n} \mathcal{W}_x \), and \( N_1, N_2 \) are replaced by \( (2\pi i)^n \sum_{x=1}^{n} \operatorname{Res}_{\mathcal{V}_x} \varphi, (2\pi i)^n \sum_{x=1}^{n} \operatorname{Res}_{\mathcal{W}_x} \psi \), respectively.

The proof is essentially the same, using Dolbeault's representatives \( \eta_{\varphi}, \eta_{\psi} \) in place of \( f^* k, g^* k \) (cf. Griffiths-Harris [5], pp. 649-656).
3. - Bézout’s theorem

Now, let $f_1, \ldots, f_n : \mathbb{C}^n \to \mathbb{C}$ be polynomials of respective degrees $v_1, \ldots, v_n$ all positive. Denote by $g_1, \ldots, g_n$ the homogeneous parts of highest degrees of $f_1, \ldots, f_n$, respectively. Then we will prove:

**Theorem 3:** Assume that the homogeneous system $g_1(\xi) = 0, \ldots, g_n(\xi) = 0$ has only the trivial solution $O = (0, \ldots, 0)$. Then the system $f_1(\xi) = 0, \ldots, f_n(\xi) = 0$ has precisely $v_1 \ldots v_n$ solutions, each one counted with its multiplicity.

This is the affine rephrasing of Bézout’s theorem for $n$ divisors of $\mathbb{P}^n(\mathbb{C})$ meeting in a finite set of points, after removing from $\mathbb{P}^n(\mathbb{C})$ a hyperplane $L$ that does not contain any of these points ($L$ may be assumed to be the hyperplane $\infty$ of $\mathbb{C}^n$).

We are going to show that Theorem 1 may be applied to the polynomial systems $f_1(\xi) = 0, \ldots, f_n(\xi) = 0$ and $g_1(\xi) = 0, \ldots, g_n(\xi) = 0$ assuming as $\Omega$ an open ball with center the origin and radius $r$ large enough. After that we’ll need only to prove that $O$ has multiplicity $\mu = v_1 \ldots v_n$ as the unique solution of the homogeneous system $g_1(\xi) = 0, \ldots, g_n(\xi) = 0$.

For every positive real number $r$, let $S(r)$ be the $(2n-1)$-sphere of $\mathbb{C}^n$ with center $O$ and radius $r$. Choose a positive real number $\varepsilon$ such that

$$S(1) \cap \{ \xi \in \mathbb{C}^n : |g_1(\xi)| < \varepsilon, \ldots, |g_n(\xi)| < \varepsilon \} = \emptyset$$

and set

$$S_a(1) = \{ \xi \in S(1) : |g_a(\xi)| > \varepsilon \} ,$$

$a = 1, \ldots, n$. Then $S(1) = \bigcup_{a=1}^n S_a(1)$. For every $r > 1$, let $S_a(r)$ be the transform of $S_a(1)$ by the homothety $\xi \mapsto r\xi$. Then, by the homogeneity of the $g_a$’s,

$$S_a(r) = \{ \xi \in S(r) : |g_a(\xi)| > r^a \varepsilon \} ,$$

$a = 1, \ldots, n$, and of course $S(r) = \bigcup_{a=1}^n S_a(r)$. Next, if we set

$$M_a(r) = \max_{\xi \in S_a(r)} |f_a - g_a| , \quad a = 1, \ldots, n ,$$

since $\deg (f_a - g_a) < v_a - 1$, it is clear that $M_a(r)/r^a$ approaches zero for large $r$. Hence we can find some $r_1 > 1$ such that $M_a(r) < r^a \varepsilon$ for $r > r_1$, $a = 1, \ldots, n$.

It follows that for $r > r_1$ and $\xi \in S_a(r)$ we have

$$|f_a(\xi) - g_a(\xi)| < r^a \varepsilon < |g_a(\xi)| ,$$

$a = 1, \ldots, n$, which yields the desired result.
Now, to show that \( \mu = \tau_1 \ldots \tau_n \), assume that \( \xi_1, \ldots, \xi_n \) are the general homogeneous polynomials of degrees \( \tau_1, \ldots, \tau_n \) with undetermined complex coefficients \( a_j \). Then, as a consequence of Hilbert’s Nullstellensatz, there exists a set \( \{D(a_j)\} \) of homogeneous polynomials in those coefficients, such that the system \( \xi_j(\tau) = 0, \ldots, \xi_n(\tau) = 0 \) has only the trivial solution if and only if the corresponding \( a_j \)’s do not satisfy at least one of the equations \( D(a_j) = 0 \) (cf. van der Waerden [11], vol. 2, p. 158) (\( \dagger \)). It follows that \( \mu \) is a integer-valued function of the \( a_j \)’s defined on a connected open subset of \( \mathbb{C}^N \), where \( N \) is the number of the \( a_j \)’s. This function is continuous, as being equal to the integral \( \int g(\mathbf{k}) (\text{cf. n. 2, ii}), \) and therefore is constant. Then taking \( \xi_j = \zeta^\alpha_j, \alpha = 1, \ldots, n \) \( \dagger \) obviously gives \( \mu = \tau_1 \ldots \tau_n \) (\( \dagger \)).

**Remark 2:** In the case \( n = 2 \) a proof of Bézout’s theorem based on the Cauchy kernel has first been given by Caccioppoli [4]. Though the argument used there is rather intricated and not suited for an easy extension to general \( n \), we have followed here the conceptual lines of Caccioppoli’s proof.

**Remark 3:** Theorem 2 is not suitable to prove that the polynomial systems \( f_1(\tau) = 0, \ldots, f_n(\tau) = 0 \) and \( g_1(\tau) = 0, \ldots, g_n(\tau) = 0 \) have the same number of solutions in a large domain containing \( O \). For example, if \( n = 2, f_1 = \zeta_1^2 + \zeta_2, f_2 = 1 + \zeta_2, \) then

\[
|\xi_1(\tau)|^2 + |\xi_2(\tau)|^2 - |f_1(\tau) - \xi_1(\tau)|^2 - |f_2(\tau) - \xi_2(\tau)|^2 = |\xi_1|^4 - 1,
\]

so that the condition of Theorem 2 is not fulfilled on the boundary of any domain containing \( O \) and the solutions \((1, -1), (-1, -1)\) of the system \( f_1(\tau) = 0, f_2(\tau) = 0 \).

(\( \dagger \)) It is known that the set \( \{D(a_j)\} \) can be replaced by a single \( \langle \text{resultant} \rangle R(a_j) \) (cf. Segre [9], p. 26), but the above elementary result is all we need.

(\( \dagger \)) Another way to derive this equality is comparing the residues \( \text{Res}_\alpha (g^*(\alpha)) = (2\pi i)^n \mu \) and \( \text{Res}_\alpha (\xi_\alpha) = (2\pi i)^n \) by means of the \( \langle \text{transformation law} \rangle \) (cf. Griffiths-Harris [5], p. 637), taking into account the identities \( v_\alpha \xi_\alpha = \sum \beta (\xi_{\beta}, \xi_\alpha) \xi_\beta, \alpha = 1, \ldots, n \).

**References**


