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CLAUDIO CITRINI - CLELIA MARCHIONNA (*)

Support Domains for a Quasi-Linear String Vibrating Against a Wall: A Unilateral Free Boundary Problem (**) (***)

Domini di appoggio per una corda quasi lineare vibrante contro una parete: un problema unilaterale di frontiera libera

SUNTO. — Si considera il problema unilaterale $y_{\xi\eta} \geq f(\xi, \eta, y, y_\xi, y_\eta)$, $y \geq 0$ in un rettangolo caratteristico, e si danno condizioni sufficienti affinchè la soluzione presenti il fenomeno dell'appoggio, cioè sia $y \equiv 0$ in un opportuno dominio. Si studia a tale scopo, nel caso non lineare, un problema unilaterale di frontiera libera, recentemente introdotto da Amerio nell'ipotesi che la forza esterna f sia indipendente dall'incognita, ottenendo teoremi di esistenza, unicità e regolarità.

1. - INTRODUCTION: THE Π^* PROBLEM

The problem of the motion of a string vibrating in presence of a rigid wall has been studied in many papers. Particular difficulties arise in the case where the string is subject to an external force, directed *towards the wall*, or if the wall is *convex*.

In a recent work [3] L. Amerio considered the case of the linear equation

$$y_{\xi\eta} = f(\xi, \eta) \tag{1.1}$$

(where $\xi = (x + t)/\sqrt{2}$, $\eta = (-x + t)/\sqrt{2}$ are the characteristic coordinates and $f(\xi, \eta)$ is *arbitrary*), introducing the notions of *support line and domain.* This

(*) Dipartimento di Matematica del Politecnico di Milano. Gli autori aderiscono al G.N.A.F.A. del C.N.R.

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allows to formulate *new extension laws* (with respect to [1]), in order to extend the solution *beyond* the first *line of influence of the wall*, until a *second line of influence*, and so on.

In the present paper we generalize some of the results in [3] to the non linear case where the external force f depends also on y and its first derivatives, supposing either that f is directed *towards the wall* (§§ 2 and 3), or that it *changes sign in a neighborhood of the origin* (§ 4).

More precisely, given a rectangle $\bar{R} = [0, \bar{a}] \times [0, \bar{b}]$, we look for a C^1-function $y = y(\xi, \eta)$, solving (at least locally) the following *unilateral problem*:

$$(1.2)\qquad y_{\xi\eta} \geq f(\xi, \eta, y(\xi, \eta), y_\xi(\xi, \eta), y_\eta(\xi, \eta)) \qquad \text{in } \mathcal{D}'(\mathring{\bar{R}}),$$

$$(1.3)\qquad y(\xi, \eta) \geq 0,$$

$$(1.4)\qquad y \cdot [y_{\xi\eta} - f] = 0,$$

$$(1.5)\qquad y(\xi, 0) = A(\xi), \qquad y(0, \eta) = B(\eta) \qquad (\text{with } A(0) = B(0) = 0).$$

In order to obtain local solutions of (1.2)-(1.5) we solve the following *free boundary problem*, denoted by Π^* (see Amerio [3], § 2):

Π^* PROBLEM: *Find a rectangle* $R = [0, a] \times [0, b] \subseteq \bar{R}$, *and a line* Γ:

$$(1.6)\qquad \xi = \gamma(\eta), \qquad \gamma(0) = 0,$$

with $\gamma \in C^0([0, b'])$, $b' < b$, *strictly increasing, such that the equation*

$$(1.7)\qquad y_{\xi\eta} = f(\xi, \eta, y, y_\xi, y_\eta)$$

has a C^1*-solution* $y(\xi, \eta)$ *in the domain*

$$(1.8)\qquad Z = \{\gamma(\eta) < \xi < a,\ 0 < \eta < b'\}$$

satisfying the conditions

$$(1.9)\qquad y(\xi, 0) = A(\xi) \qquad \text{in } [0, a],$$

$$(1.10)\qquad y(\gamma(\eta), \eta) = y_\xi(\gamma(\eta), \eta) = 0 \qquad \text{in } [0, b'],$$

$$(1.11)\qquad y(\xi, \eta) > 0 \qquad \text{in } Z.$$

For a general framework, where the Π^* problem is studied in connection with the mixed initial-boundary value problem for the vibrating string equation with unilateral constraints, we refer to the paper [3] of Amerio.

Let us now introduce some simplifying notations. For each function $u \in C^1(R)$, we denote by $\mathbf{u}$ the vector (u, u_ξ, u_η). Let $f = f(\xi, \eta, z, p, q)$ be a C^1-function: f_* will represent the vector (f_z, f_p, f_q) and $f_* \cdot \mathbf{u}$ the scalar product $f_z u + f_p u_\xi + f_q u_\eta$.

If y is a solution of Π^*, then the formula:

$$y(\xi,\eta) = A(\xi) - A(\gamma(\eta)) + \int_{\gamma(\eta)}^{\xi} d\alpha \int_0^{\eta} f(\alpha,\beta,y(\alpha,\beta))\, d\beta \tag{1.12}$$

holds true in Z. Setting

$$G(\xi,\eta) = A'(\xi) + \int_0^{\eta} f(\xi,\beta,y(\xi,\beta))\, d\beta \qquad \text{in } R, \tag{1.13}$$

we have

$$y_\xi(\xi,\eta) = G(\xi,\eta) \qquad \text{in } Z, \tag{1.14}$$

$$y(\xi,\eta) = \int_{\gamma(\eta)}^{\xi} G(\alpha,\eta)\, d\alpha \qquad \text{in } Z. \tag{1.15}$$

The equation $G(\xi,\eta)=0$ implicitly defines the line Γ:

$$G(\gamma(\eta),\eta) = 0 \qquad \text{in } [0,b']. \tag{1.16}$$

We shall give in §§ 3 and 4 existence, uniqueness and regularity theorems for the solution of the Π^* problem; in § 5 we deduce sufficient conditions for the existence of support domains, and formulate the corresponding extension laws.

2. - Admissible functions

In the present section we consider the Π^* problem *assuming* $f<0$, and we obtain some preliminary results. More precisely, we shall introduce an application τ from a suitable set A of admissible functions into itself. The fixed point of such an application, which we shall obtain in § 3 by means of the contraction mapping theorem, will give the solution of the Π^* problem.

We shall suppose that the *boundary value* $A(\xi)$ and the function $f(\xi,\eta,z,p,q)$ *satisfy the following hypotheses* (see Amerio [3], § 5, Th. 2):

$$\left|\begin{array}{ll} A\in C^2([0,\bar a]), & \\ A(0)=A'(0)=0, & \\ A''(\xi)>a_2>0 & \text{in } [0,\bar a], \end{array}\right. \tag{2.1}$$

so that there exist A_0, A_1, A_2 such that:

$$\left|\begin{array}{ll} a_2<A''(\xi)<A_2 & \text{in } [0,\bar a], \\ 0<A'(\xi)<A_1 & \text{in } [0,\bar a], \\ 0<A(\xi)<A_0 & \text{in } [0,\bar a]. \end{array}\right. \tag{2.2}$$

Note that the condition $A(0) = A'(0) = 0$ is necessary in order that there exists a support domain ([3], § 3). Let us set

$$S = [0, A_0] \times [-A_1, A_1] \times [-M_1, M_1], \; \bar{E} = \bar{R} \times S, \; (M_1 > 0) \tag{2.3}$$

and suppose

$$f \in C^1(\bar{E}), \qquad f(\xi, \eta, z, p, q) < 0 \qquad \text{in } \bar{E}. \tag{2.4}$$

We assume therefore that the external force is directed towards the wall. In the proof of the existence and uniqueness theorem we shall also assume:

(2.5) f_ξ, f_z Lipschitz continuous with respect to z in $\bar{E}$.

Let now a, b, M, M_2 be positive constants, and R be a fixed rectangle $[0, a] \times [0, b] \subset \bar{R}$. The set of *admissible functions* will then be defined by:

$$A = \{z = z(\xi, \eta) | z, z_\xi \in C^1(R), z(\xi, 0) = A(\xi), \forall \xi \in [0, a], \\ 0 < z(\xi, \eta) < A_0, |z_\xi(\xi, \eta)| < A_1, |z_\eta(\xi, \eta)| < M_1, \\ 0 < z_{\xi\xi}(\xi, \eta) < M_2, -M < z_{\xi\eta}(\xi, \eta) < 0\}. \tag{2.6}$$

Remark: If $z \in A$, we have $z_{\eta\xi} = z_{\xi\eta}$ everywhere in R (see Hobson [7], vol. I, pag. 497).

The set A is obviously closed in the Banach space

$$X = \{z = z(\xi, \eta) | z, z_\xi \in C^1(R)\}, \tag{2.7}$$

endowed with the norm (equivalent to the usual one):

$$\|z\| = \|z\|_1 + \|z_{\xi\xi}\|_0 + \sigma \|z_{\xi\eta}\|_0, \tag{2.8}$$

where

$$\|u\|_0 = \operatorname*{Sup}_{(\xi,\eta) \in R} \{\exp[-\varrho\eta] |u(\xi, \eta)|\}, \tag{2.9}$$

$$\|u\|_1 = \|u\|_0 + \|u_\xi\|_0 + \|u_\eta\|_0, \tag{2.10}$$

with ϱ, σ positive constants to be chosen later.

Let now $z \in A$. Consider the function

$$G(\xi, \eta) = A'(\xi) + \int_0^\eta f(\xi, \beta, z(\xi, \beta)) \, d\beta, \tag{2.11}$$

which, under the hypotheses (2.1) and (2.4), belongs to $C^1(R)$. Choosing R as in the following *Lemma* 1, the equation

$$G(\xi, \eta) = 0 \tag{2.12}$$

implicitly defines in R (see Amerio [3]) a line (depending on z)

$$(2.13)\qquad \Gamma:\ \xi=\gamma(\eta)\,,\qquad \gamma\in C^1([0,b'])\,,\ b'<b\,,$$

with

$$(2.14)\qquad \gamma(0)=0\,,\qquad \gamma(\eta)\ \text{strictly increasing in}\ [0,b']\,;$$

if $b'<b$ it is $\gamma(b')=a$, if $b'=b$ it is $\gamma(b')<a$ (fig. 1).

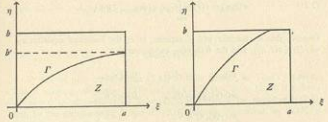

Fig. 1.

The line Γ divides the rectangle R in two parts; setting

$$(2.15)\qquad Z=\{(\xi,\eta)\in R\,|\,\gamma(\eta)<\xi<a,\ 0<\eta<b'\}\,,$$

we have

$$(2.16)\qquad G(\xi,\eta)>0\ \ \text{in}\ \mathring{Z}\,,\qquad G(\xi,\eta)<0\ \ \text{in}\ (R\setminus Z)\,.$$

Let us define the application τ by means of

$$(2.17)\qquad w(\xi,\eta)=\tau(z)=\begin{cases}\displaystyle\int_{\gamma(\eta)}^{\xi}G(x,\eta)\,dx\,, & 0<\eta<b'\,,\\[2mm] \displaystyle\int_{0}^{\xi}G(x,\eta)\,dx\,, & b'<\eta<b\,.\end{cases}$$

It follows

$$(2.18)\qquad w_\xi(\xi,\eta)=G(\xi,\eta)\qquad \text{in}\ R\,,$$

$$(2.19)\qquad w_\eta(\xi,\eta)=\begin{cases}\displaystyle\int_{\gamma(\eta)}^{\xi}G_\eta(x,\eta)\,dx\,, & 0<\eta<b'\ \text{(by (2.12))}\,,\\[2mm] \displaystyle\int_{0}^{\xi}G_\eta(x,\eta)\,dx\,, & b'<\eta<b\,,\end{cases}$$

from which one obtains immediately that $\tau: A \to X$, both G and γ being of class C^1. If $b' < b$, we shall extend the definition of γ to the whole of $[0, b]$, by setting:

$$g(\eta) = \begin{cases} \gamma(\eta) & \text{for } 0 < \eta < b' , \\ a & \text{for } b' < \eta < b . \end{cases} \tag{2.20}$$

Hence we can rewrite (2.17) as

$$w(\xi, \eta) = \tau(z) = \int_{g(\eta)}^{\xi} G(\alpha, \eta)\, d\alpha , \qquad 0 < \eta < b . \tag{2.21}$$

Observe that the function $w(\xi, \eta)$ satisfies, in R, the linearized equation $w_{\xi\eta} = f(\xi, \eta, z(\xi, \eta))$, and the following conditions:

$$\begin{aligned} w(\xi, 0) &= A(\xi) , && 0 < \xi < a , \\ w(\gamma(\eta), \eta) &= 0 , && 0 < \eta < b' , \\ w_\xi(\gamma(\eta), \eta) &= 0 , && 0 < \eta < b' , \\ w(a, \eta) &= 0 , && \text{if } b' < \eta < b . \end{aligned}$$

The following lemma then holds.

LEMMA 1: *Under the hypotheses* (2.1) *and* (2.4) *it is possible to choose the constants a, b, M_2 and M so that*:

$$\tau: A \to A . \tag{2.22}$$

PROOF: Let us set

$$M = \max_{\bar{E}} | f(\xi, \eta, \zeta, p, q) | , \tag{2.23}$$

and let L_1, L_2, L_4, L_5 be the Lipschitz constants of f with respect to ξ, ζ, p, q (in the given order) in $\bar{E}$. Let us choose arbitrarily $M_2 > A_2$ and ϵ, with $0 < \epsilon < a_2$, and set

$$K = L_1 + L_2 A_1 + L_4 M_2 + L_5 M , \tag{2.24}$$

$$a = \min \{\bar{a}, M_1/M\} , \tag{2.25}$$

$$b = \min \{\bar{b}, (a_2 - \epsilon)/K, (M_2 - A_2)/K, A_0/aM, A_1/M\} . \tag{2.26}$$

Substituting in (2.6) the above constants, we shall prove that (2.22) holds, *a*) Let us consider firstly, $\forall \xi \in A$, the derivative $w_{\xi\xi}$. From (2.11), (2.18) we obtain at once

$$(2.27)\qquad w_{\xi\xi}(\xi,\eta) = G_\xi(\xi,\eta) =$$
$$= A''(\xi) + \int_0^\eta [f_\xi(\xi,\beta,z(\xi,\beta)) + f_z(\xi,\beta,z(\xi,\beta))\cdot z_\xi(\xi,\beta)]\,d\beta\,,$$

from which, by (2.2) and (2.24):

$$(2.28)\qquad a_2 - Kb < w_{\xi\xi}(\xi,\eta) < A_2 + Kb \qquad \text{in } R\,.$$

Hence by (2.26) we have

$$(2.29)\qquad 0 < c < w_{\xi\xi}(\xi,\eta) < M_2\,.$$

b) Let us now observe that (2.12) actually defines, by Dini's theorem, a line Γ enjoying properties (2.13) and (2.14). In fact, from (2.29) it follows $G_\xi(\xi,\eta) > 0$ in R, while $G_\eta(\xi,\eta) = f(\xi,\eta,z(\xi,\eta)) < 0$ by hypothesis (2.4). Then, by the definitions of Γ and Z, (2.16) follows. Moreover, by (2.18), it is $w_\xi(\xi,\eta) = G(\xi,\eta)$; hence

$$(2.30)\qquad 0 < w_\xi(\xi,\eta) = A'(\xi) + \int_0^\eta f(\xi,\beta,z(\xi,\beta))\,d\beta < A_1 \qquad \text{in } Z\,,$$

while in $R \setminus Z$, by (2.26):

$$(2.31)\qquad 0 > w_\xi(\xi,\eta) > -Mb > -A_1\,.$$

Consequently

$$(2.32)\qquad |w_\xi(\xi,\eta)| < A_1 \qquad \text{in } R\,.$$

c) From (2.11), (2.19) and (2.25) we have at once, $\forall \xi \in A$:

$$(2.33)\qquad |w_\eta(\xi,\eta)| < Ma < M_1\,,$$

$$(2.34)\qquad -M < w_{\eta\eta}(\xi,\eta) = f(\xi,\eta,z(\xi,\eta)) < 0\,.$$

d) Lastly we obtain, from (2.21):

$$(2.35)\qquad w(\xi,\eta) = A(\xi) - A(g(\eta)) + \int_{g(\eta)}^{\xi} d\alpha \int_0^\eta f(\alpha,\beta,z(\alpha,\beta))\,d\beta\,.$$

We have $w(g(\eta),\eta) = 0$ in $[0,b]$, and $w_\xi(\xi,\eta) > 0$ in $\mathring{Z}$ by (2.30): hence

$w(\xi, \eta) > 0$ in Z. Moreover $A(\xi)$ increases by (2.2), and $f < 0$ in R: hence

$$0 < w(\xi, \eta) < A_0 \qquad \text{in } Z. \tag{2.36}$$

On the other hand it is $w_2(\xi, \eta) < 0$ in $R \setminus Z$; hence we have again $w(\xi, \eta) > 0$, and furthermore, by (2.26):

$$0 < w(\xi, \eta) < \int_0^{a(\eta)} d\alpha \int_0^{\eta} |f(\alpha, \beta, z(\alpha, \beta))| \, d\beta < \tag{2.37}$$
$$< Mab < Ma(A_0/Ma) = A_0 \qquad \text{in } R \setminus Z.$$

Finally

$$0 < w(\xi, \eta) < A_0 \qquad \text{in } R. \tag{2.38}$$

From (2.29), (2.32), (2.33), (2.34) and (2.38) the thesis follows.

3. - Solution of the Π^* problem

Let ζ_1 and $\zeta_2 \in A$, $g_1(\eta)$ and $g_2(\eta)$ be the corresponding lines, defined by

$$g_i(\eta) = \begin{cases} \gamma_i(\eta) & 0 < \eta < b_i', \\ a & b_i' < \eta < b, \end{cases} \qquad (i = 1, 2), \tag{3.1}$$

(see (2.20)), and suppose $b_2' < b_1'$. The following *continuous dependence theorem* then holds:

Theorem 1: *If the constants c, L_3, L_4, L_5 are chosen according to Lemma* 1, *we have*:

$$|\Delta g(\eta)| < (1/c) \int_0^{\eta} \{L_3 |\Delta\zeta(g_2(\eta), \beta)| + \tag{3.2}$$
$$+ L_4 |\Delta\zeta_3(g_2(\eta), \beta)| + L_5 |\Delta\zeta_4(g_2(\eta), \beta)|\} \, d\beta$$

(where $\Delta g = g_1 - g_2$, $\Delta\zeta = \zeta_1 - \zeta_2$).

For the sake of simplicity we shall use in the sequel the following notations:

$$|\Delta z(\xi, \eta)| = |\Delta\zeta(\xi, \eta)| + |\Delta\zeta_3(\xi, \eta)| + |\Delta\zeta_4(\xi, \eta)|, \tag{3.3}$$

$$F_i(\xi, \eta) = f(\xi, \eta, z_i(\xi, \eta)). \tag{3.4}$$

If now

$$L = \max\{L_3, L_4, L_5\}, \tag{3.5}$$

$$\|\Delta g\|_0 = \max_{\eta \in (0, b)} \{\exp[-\varrho\eta] |\Delta g(\eta)|\}, \tag{3.6}$$

we obtain at once the

Corollary 1:

$$\|\Delta g\|_0 < (L/c\varrho)\|\Delta \zeta\|_1, \tag{3.7}$$

the latter norm being defined in (2.10).

It is indeed

$$\exp[-\varrho\eta]\,|\Delta g(\eta)| < (L/c)\int_0^\eta \exp[-\varrho(\eta-\beta)]\exp[-\varrho\beta]\,|\Delta z(g_2(\eta),\beta)|\,d\beta <$$

$$< (L/c)\|\Delta\zeta\|_1 \int_0^\eta \exp[-\varrho(\eta-\beta)]\,d\beta < (L/c\varrho)\|\Delta\zeta\|_1,$$

from which (3.7) follows.

Let us now prove Theorem 1. We firstly consider the interval $[0, b_2']$. It is

$$G_i(\gamma_i(\eta),\eta) = 0 \qquad \text{for } 0 \leq \eta \leq b_2'. \tag{3.8}$$

By Lagrange theorem we obtain:

$$0 = G_1(\gamma_1(\eta),\eta) - G_1(\gamma_2(\eta),\eta) + G_1(\gamma_2(\eta),\eta) - G_2(\gamma_2(\eta),\eta) = \tag{3.9}$$

$$= G_{1\varepsilon}(\delta(\eta),\eta)\cdot\Delta\gamma(\eta) + \int_0^\eta |F_1(\gamma_2(\eta),\beta) - F_2(\gamma_2(\eta),\beta)|\,d\beta,$$

where $\delta(\eta) \in (\gamma_1(\eta), \gamma_2(\eta))$. By the Lipschitz continuity of f, and by (2.29) we have then

$$|\Delta\gamma(\eta)| < (1/c)\int_0^\eta \{L_3|\Delta\zeta(\gamma_2(\eta),\beta)| + L_4|\Delta\zeta_t(\gamma_2(\eta),\beta)| + \tag{3.10}$$

$$+ L_5|\Delta\zeta_n(\gamma_2(\eta),\beta)|\}\,d\beta \qquad \text{in } [0, b_2'].$$

In $(b_2', b]$ it is $g_2(\eta) = a$, while $g_1(\eta)$ is non decreasing. Hence

$$|\Delta g(\eta)| < |\Delta\gamma(b_2')| < \tag{3.11}$$

$$< (1/c)\int_0^{b_2'} \{L_3|\Delta\zeta(a,\beta)| + L_4|\Delta\zeta_t(a,\beta)| + L_5|\Delta\zeta_n(a,\beta)\}\,d\beta <$$

$$< (1/c)\int_0^\eta \{L_3|\Delta\zeta(g_2(\eta),\beta)| + L_4|\Delta\zeta_t(g_2(\eta),\beta)| + L_5|\Delta\zeta_n(g_2(\eta),\beta)|\}\,d\beta.$$

(3.10) together with (3.11) give the thesis.

We are now ready to prove the existence and uniqueness theorems, which represent the main result of the work.

THEOREM 2 (Existence): *Let R be the rectangle $[0, a] \times [0, b]$, where a and b satisfy* (2.25), (2.26) *and*

$$a < 1/L\,, \tag{3.12}$$

with L given in (3.5). *Let moreover $E = R \times S$, where S is defined in* (2.3). *Under the hypotheses* (2.1), (2.4) *and* (2.5), *the Π^* problem has in R at least one solution.*

PROOF: It suffices to show that, if a and b are as above, then $w = \tau(\zeta)$ is a contraction of $\mathcal{A}$ into itself in the norm (2.8), provided that the constants ϱ and σ are suitably chosen.

a) Let us firstly consider the difference $\Delta w = w_1 - w_2$. We have:

$$\begin{aligned} |\Delta w(\xi, \eta)| &= \Big| A(g_2(\eta)) - A(g_1(\eta)) + \\ &\quad + \int_{g_1(\eta)}^{g_2(\eta)} d\alpha \int_0^\eta F_1(\alpha, \beta)\, d\beta + \int_{g_2(\eta)}^{\xi} d\alpha \int_0^\eta \Delta F(\alpha, \beta)\, d\beta \Big| < \\ &< (A_1 + Mb)|\Delta g(\eta)| + L \int_0^\xi d\alpha \int_0^\eta |\Delta z(\alpha, \beta)|\, d\beta\,. \end{aligned} \tag{3.13}$$

Let us now multiply both sides of (3.13) by $\exp[-\varrho\eta]$, and calculate the norm (2.9) of Δw. We have:

$$\begin{aligned} \exp[-\varrho\eta]\,|\Delta w(\xi, \eta)| &< (A_1 + Mb) \exp[-\varrho\eta]\,|\Delta g(\eta)| + \\ &\quad + L \int_0^\eta \exp[-\varrho(\eta - \beta)]\, d\beta \int_0^\xi \exp[-\varrho\beta]\,|\Delta z(\alpha, \beta)|\, d\alpha < \\ &< (A_1 + Mb)\|\Delta g\|_0 + La\|\Delta\zeta\|_1 \int_0^\eta \exp[-\varrho(\eta - \beta)]\, d\beta\,. \end{aligned}$$

By (3.7) we have then

$$\|\Delta w\|_0 < (1/\varrho)[(A_1 + Mb)L/c + La]\,\|\Delta\zeta\|_1\,. \tag{3.14}$$

b) The evaluation of the norm (2.9) of Δw_ξ is immediate:

$$|\Delta w_\xi(\xi, \eta)| = \Big| \int_0^\eta \Delta F(\xi, \beta)\, d\beta \Big| < L \int_0^\eta |\Delta z(\xi, \beta)|\, d\beta\,, \tag{3.15}$$

from which, as before:

$$\|\Delta w_\xi\|_0 < (L/\varrho)\|\Delta\zeta_1\|\,. \tag{3.16}$$

c) The estimate of Δw_ξ, on the other hand, takes the following form:

$$(3.17)\qquad |\Delta w_\eta(\xi,\eta)| = \left| \int_{x_1(\eta)}^{\xi} F_1(x,\eta)\,dx - \int_{x_2(\eta)}^{\xi} F_2(x,\eta)\,dx \right| <$$

$$< \left| \int_{x_1(\eta)}^{x_2(\eta)} F_1(x,\eta)\,dx \right| + \left| \int_{x_2(\eta)}^{\xi} \Delta F(x,\eta)\,dx \right| < M|\Delta g(\eta)| + L \left| \int_{x_2(\eta)}^{\xi} |\Delta z(x,\eta)|\,dx \right| .$$

Multiplying both sides by $\exp[-\varrho\eta]$ we obtain, as usual:

$$(3.18)\qquad |\Delta w_\eta|_\varrho < (ML/c\varrho)\,|\Delta z|_1 + La\,|\Delta z|_1 .$$

Note that the hypothesis (3.12) on a plays an essential role only in this estimate.

d) By hypothesis (2.5) the derivatives of $f(\xi,\eta,z,p,q)$ with respect to ξ, z, p and q satisfy a Lipschitz condition in $E = R \times S$ with respect to the variables z, p and q. Let H be the greatest of the 12 Lipschitz constants in (2.5). We obtain then the following estimates, $\forall z_i \in A$:

$$(3.19)\qquad |f_\xi(\xi,\eta,z_1(\xi,\eta)) - f_\xi(\xi,\eta,z_2(\xi,\eta))| < H|\Delta z(\xi,\eta)| ;$$

$$(3.20)\qquad |f_z(\xi,\eta,z_1(\xi,\eta))\,z_{1\xi}(\xi,\eta) - f_z(\xi,\eta,z_2(\xi,\eta))\,z_{2\xi}(\xi,\eta)| <$$

$$< |\{f_z(\xi,\eta,z_1(\xi,\eta)) - f_z(\xi,\eta,z_2(\xi,\eta))\}\,z_{1\xi}(\xi,\eta)| +$$

$$+ |f_z(\xi,\eta,z_2(\xi,\eta))\{z_{1\xi}(\xi,\eta) - z_{2\xi}(\xi,\eta)\}| < H|\Delta z(\xi,\eta)|\cdot A_1 + L|\Delta z_\xi(\xi,\eta)| .$$

In an analogous way we obtain:

$$(3.21)\qquad |f_p(\xi,\eta,z_1(\xi,\eta))\,z_{1\xi\xi}(\xi,\eta) - f_p(\xi,\eta,z_2(\xi,\eta))\,z_{2\xi\xi}(\xi,\eta)| <$$

$$< H|\Delta z(\xi,\eta)|\cdot M_2 + L|\Delta z_{\xi\xi}(\xi,\eta)| ,$$

$$(3.22)\qquad |f_q(\xi,\eta,z_1(\xi,\eta))\,z_{1\xi\eta}(\xi,\eta) - f_q(\xi,\eta,z_2(\xi,\eta))\,z_{2\xi\eta}(\xi,\eta)| <$$

$$< H|\Delta z(\xi,\eta)|M + L|\Delta z_{\xi\eta}(\xi,\eta)| .$$

Observe that, by (2.27),

$$\Delta w_{\xi\xi} = \int_0^\eta \{ f_\xi(\xi,\beta,z_1(\xi,\beta)) - f_\xi(\xi,\beta,z_2(\xi,\beta)) +$$

$$+ f_z(\xi,\beta,z_1(\xi,\beta))\,z_{1\xi}(\xi,\beta) - f_z(\xi,\beta,z_2(\xi,\beta))\,z_{2\xi}(\xi,\beta) +$$

$$+ f_p(\xi,\beta,z_1(\xi,\beta))\,z_{1\xi\xi}(\xi,\beta) - f_p(\xi,\beta,z_2(\xi,\beta))\,z_{2\xi\xi}(\xi,\beta) +$$

$$+ f_q(\xi,\beta,z_1(\xi,\beta))\,z_{1\xi\eta}(\xi,\beta) - f_q(\xi,\beta,z_2(\xi,\beta))\,z_{2\xi\eta}(\xi,\beta)\}\,d\beta .$$

Adding (3.19)-(3.22) and integrating from 0 to η:

$$(3.23)\qquad |\Delta w_{\xi\xi}(\xi,\eta)| = \left|\int_0^\eta \{F_{1\xi}(\xi,\beta) - F_{2\xi}(\xi,\beta)\}\, d\beta\right| <$$

$$< H(1 + A_1 + M_2 + M)\int_0^\eta |\Delta z(\xi,\beta)|\, d\beta +$$

$$+ L\int_0^\eta \{|\Delta z_\xi(\xi,\beta)| + |\Delta z_{\xi\xi}(\xi,\beta)| + |\Delta z_{\xi\eta}(\xi,\beta)|\}\, d\beta .$$

We can now multiply by $\exp[-\varrho\eta]$ and evaluate the norm:

$$(3.24)\qquad \|\Delta w_{\xi\xi}\|_0 < (H/\varrho)(1 + A_1 + M_2 + M)\|\Delta z\|_1 +$$

$$+ (L/\varrho)(\|\Delta z\|_1 + \|\Delta z_{\xi\xi}\|_0 + \|\Delta z_{\xi\eta}\|_0) .$$

e) The last term to take into account is

$$(3.25)\qquad |\Delta w_{\xi\eta}(\xi,\eta)| = |F_1(\xi,\eta) - F_2(\xi,\eta)| < L|\Delta z(\xi,\eta)| .$$

Multiplying by $\sigma\exp[-\varrho\eta]$ and calculating the norm we have

$$(3.26)\qquad \sigma\|\Delta w_{\xi\eta}\|_0 < \sigma L\|\Delta z\|_1 .$$

f) Let us finally add (3.14), (3.16), (3.18), (3.24) and (3.26). We have by (2.8):

$$(3.27)\qquad \|\Delta w\| <$$

$$< (1/\varrho)\{L[((A_1 + Mb)/c) + a + 1 + (M/c) + 1] + H(1 + A_1 + M_2 + M)\}\cdot$$

$$\cdot\|\Delta z\|_1 + L(a + \sigma)\|\Delta z\|_1 + (L/\varrho)(\|\Delta z_{\xi\xi}\|_0 + \|\Delta z_{\xi\eta}\|_0) =$$

$$= [(C/\varrho) + L(a + \sigma)]\|\Delta z\|_1 + (L/\varrho)\|\Delta z_{\xi\xi}\|_0 + (L/\varrho\sigma)\cdot\sigma\|\Delta z_{\xi\eta}\|_0 < K\|\Delta z\| ,$$

where C denotes the constant in { } and

$$(3.28)\qquad K = \max\{(C/\varrho) + L(a + \sigma),\ L/\varrho\sigma\} .$$

By (3.12) it is possible to choose σ so that $L(a + \sigma) < 1$; for ϱ large enough we shall finally obtain $K < 1$. Hence the contraction mapping theorem applies, and *the existence and uniqueness of a fixed point* $y_1 \in A$ *is proved*. Let Γ_1: $\xi = \gamma_1(\eta)$ be the line which is implicitly defined by

$$(3.29)\qquad G_1(\xi,\eta) = A'(\xi) + \int_0^\eta f(\xi,\beta,\mathbf{y}_1(\xi,\beta))\, d\beta = 0 ,$$

and let Z_1 be the domain given by setting Γ_1 in (2.15). It is immediate to verify that the pair (y_1, γ_1) gives a solution of the Π^* problem in R.

The uniqueness of the solution of the Π^* problem does not follow directly from the previous theorem. This solution, in fact, as defined in § 1, $\in C^1(Z)$, whereas $w(\xi, \eta)$ belongs to a smaller class. Thus we have to prove a *regularity* theorem, from which the uniqueness will follow at once, at least in a suitable rectangle R.

LEMMA 2: *Let $y(\xi, \eta)$ and Γ give a solution of the Π^* problem in a rectangle $\bar{R}$, where* (2.1), (2.4) *and* (2.5) *hold* (see fig. 2). *There exists then a rectangle $R \subset \bar{R}$ such that, letting $Z = R \cap \bar{Z}$, $y_\xi(\xi, \eta)$ and Γ are of class C^1 in Z.*

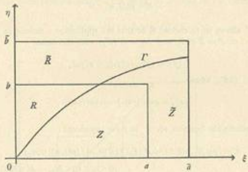

Fig. 2.

PROOF: After restricting, if needed, $\bar{R}$ so that $y(\xi, \eta) \in S$, $\forall (\xi, \eta) \in \bar{Z}$, what is possible because $y(0, 0) = 0$, we have:

$$y_\xi(\xi, \eta) = G(\xi, \eta) = A'(\xi) + \int_0^\eta f(\xi, \beta, y(\xi, \beta))\, d\beta \,. \tag{3.30}$$

Thus we observe that, beyond y, y_ξ and y_η, also $y_{\xi\eta} = G_\eta(\xi, \eta) = f(\xi, \eta, y(\xi, \eta))$ is continuous in $\bar{Z}$. Let us consider the application δ: $C^0(\bar{Z}) \to C^0(\bar{Z})$ given by setting $v = \delta p$ where

$$v(\xi, \eta) = A'(\xi) + \int_0^\eta f(\xi, \beta, y(\xi, \beta), p(\xi, \beta), y_\eta(\xi, \beta))\, d\beta \,. \tag{3.31}$$

A classical argument shows that δ has a unique fixed point in $C^0(\bar{Z})$, which agrees, by (3.30), with $y_\xi(\xi, \eta)$. Let now construct the set A as in (2.6), observing that the inequalities

$$(3.32) \qquad \begin{cases} \max\limits_{\bar{Z}} |y(\xi, \eta)| < A_0 , \\ \max\limits_{\bar{Z}} |y_\xi(\xi, \eta)| < A_1 , \\ \max\limits_{\bar{Z}} |y_\eta(\xi, \eta)| < M_1 , \end{cases}$$

hold; the remaining constants are chosen as in Lemma 1. We have, in particular:

$$(3.33) \qquad -M < y_{\xi\eta}(\xi, \eta) < 0 .$$

Lemma 1 allows us to choose R so that the application τ defined in (2.17) is $A \to A$. Let $Z = \bar{Z} \cap R$, and $X_1 = C^1(Z)$, with the norm

$$(3.34) \qquad |p| = |p|_0 + |p_\xi|_0 + \sigma |p_\eta|_0$$

similar to (2.8), where

$$(3.35) \qquad |u|_0 = \max_{Z} \{\exp[-\varrho\eta] |u(\xi, \eta)|\} .$$

A new admissible function set A' is now introduced:

$$(3.36) \qquad A' = \{p \in X_1 | p(\xi, 0) = A'(\xi), \ \forall \xi \in [0, a]; \ |p(\xi, \eta)| < A_1 ; \\ 0 < p_\xi(\xi, \eta) < M_2 ; -M < p_\eta(\xi, \eta) < 0\} .$$

The same argument of steps *a*), *b*), *c*) of Lemma 1 (with p and v instead of z_ξ and w_ξ respectively) shows that δ: $A' \to A'$, and

$$(3.37) \qquad v_\xi(\xi, \eta) > c > 0 .$$

We can now proceed as in steps *b*), *d*), *e*) of Theorem 1, taking into account that in (3.31) p appears only in the fourth argument of f, while the third and the fifth one are fixed. We obtain thus the following simpler estimates:

$$(3.38) \qquad |\Delta v|_0 < (L/\varrho) |\Delta p|_0 ,$$

$$(3.39) \qquad |\Delta v_\xi|_0 < (H/\varrho)(1 + A_1 + M_2 + M) |\Delta p|_0 + (L/\varrho) |\Delta p_\xi|_0 ,$$

$$(3.40) \qquad \sigma |\Delta v_\eta|_0 < L\sigma |\Delta p|_0 .$$

Adding (3.38)-(3.40) we deduce that, for suitable ϱ and σ, δ is a contraction in the norm (3.34) too. Note that (3.12) is no longer required. Hence the fixed point $y_2 \in A' \subset C^1(Z)$, and we have, by (3.37):

$$y_{2\xi}(\xi, \eta) = G_\xi(\xi, \eta) > \varepsilon > 0 \qquad \text{in } Z. \tag{3.41}$$

It is immediate now to verify that the line Γ: $\xi = \gamma(\eta)$ (that is $G(\xi, \eta) = 0$) is of class C^1, and that

$$\gamma'(\eta) = -G_\eta(\gamma(\eta), \eta)/G_\xi(\gamma(\eta), \eta) > 0 \qquad \text{in } [0, b']. \tag{3.42}$$

Let us introduce the rectangle $R' = [0, a] \times [0, b']$, and divide Z into $Z' \cup Z''$ as in fig. 3. Note that the solution y is given, in Z', also by the formula

$$y(\xi, \eta) = -\int_T f(\alpha, \beta, y(\alpha, \beta))\, dT, \tag{3.43}$$

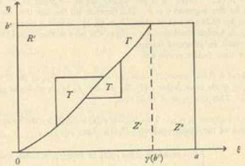

Fig. 3.

because it is a solution of a Cauchy problem too, with data vanishing on Γ. In Z'' a simple Darboux problem must be solved. If $(\xi, \eta) \in R' \setminus Z'$, the same formula (3.43) gives an integral equation which allows us to extend the function $y(\xi, \eta)$ to the whole of R'. By known results (see Cinquini Cibrario, Cinquini [4], pag. 161) we deduce the

THEOREM 3 (Regularity): *Every solution y of a Π^* problem, under the hypotheses* (2.1), (2.4) *and* (2.5), *can be extended to a function $\bar{y} \in C^1(R')$, satisfying in R' a Cauchy problem on Γ, with vanishing data.*

The function

$$G(\xi,\eta)=A'(\xi)+\int_{0}^{\eta} f(\xi,\beta,\bar{y}(\xi,\beta))\,d\beta \tag{3.44}$$

is now defined on the whole of R', where it is of class C^1, and (3.43) may be rewritten as

$$\bar{y}(\xi,\eta)=-\int_{\xi}^{\gamma(\eta)} d\alpha \int_{\varphi(\alpha)}^{\eta} f(\alpha,\beta,\bar{y}(\alpha,\beta))\,d\beta = \tag{3.45}$$

$$=\int_{\gamma(\eta)}^{\xi}\left\{A'(\alpha)+\int_{0}^{\eta} f(\alpha,\beta,\bar{y}(\alpha,\beta))\,d\beta\right\}d\alpha=\int_{\gamma(\eta)}^{\xi} G(\alpha,\eta)\,d\alpha\,,$$

where $\eta=\varphi(\xi)$ is the inverse function of $\xi=\gamma(\eta)$.

Remark: If $b'<b$ the fixed point of τ, given in Theorem 2, does not belong to $C^2(R)$. Indeed $y_{\eta\eta}$ is not continuous at the point (a,b'), and consequently on the segment $\eta=b'$. This depends on the fact that the function $g(\eta)$ given by (2.20) is not of class C^1 in $[0,b]$: indeed g represents Γ only in $[0,b']$. A further restriction on b (as in the next section) would obviously eliminate such an innatural feature.

Let us now finally prove the

Theorem 4 (Uniqueness): *Given two solutions y_1 and y_2 of the Π^* problem, corresponding to the same datum $A(\xi)$, let R_1' and R_2' be the rectangles obtained in Theorem 3. Then $y_1\equiv y_2$ in $R'=R_1'\cap R_2'$.*

Proof: Suppose $\bar{y}_1\not\equiv\bar{y}_2$. This implies $\gamma_1(\eta)\not\equiv\gamma_2(\eta)$, by the uniqueness of the solution of the Cauchy problem (3.43). Let

$$\begin{cases}\bar{\eta}=\max\{\eta\,|\,\gamma_1(\beta)=\gamma_2(\beta)\ \text{in}\ [0,\eta]\}>0\,,\\ \bar{\xi}=\gamma_1(\bar{\eta})=\gamma_2(\bar{\eta})\,.\end{cases} \tag{3.46}$$

In the rectangle $R_\varphi=\{(\xi,\eta)\in R'\,|\,\xi>\bar{\xi},\ \eta<\bar{\eta}\}$ we have again $\bar{y}_1=\bar{y}_2$, because they are both solutions of the same Darboux problem. In particular:

$$\bar{y}_1(\xi,\bar{\eta})=\bar{y}_2(\xi,\bar{\eta})=\bar{A}(\xi) \qquad \text{for } \bar{\xi}<\xi<a'\,, \tag{3.47}$$

and

$$\bar{A}'(\xi)>\varepsilon>0 \qquad \text{for } \bar{\xi}<\xi \tag{3.48}$$

by (3.41). Setting

$$\bar{R}=\{(\xi,\eta)\in R'\,|\,\xi>\bar{\xi},\ \eta>\bar{\eta}\}\,. \tag{3.49}$$

let

$$\overline{A}_0 = \max_{\bar{R}} \max \{\bar{y}_1(\xi, \eta), \bar{y}_2(\xi, \eta)\},$$

$$\overline{A}_1 = \max_{\bar{R}} \max \{\bar{y}_{1\xi}(\xi, \eta), \bar{y}_{2\xi}(\xi, \eta)\},$$

$$\overline{M}_1 = \max_{\bar{R}} \max \{\bar{y}_{1\eta}(\xi, \eta), \bar{y}_{2\eta}(\xi, \eta)\},$$

$$\overline{M}_2 > \max_{\bar{R}} \max \{\bar{y}_{1\xi\xi}(\xi, \eta), \bar{y}_{2\xi\xi}(\xi, \eta)\}.$$

From these constants we can construct, as in Lemma 1, a set $\overline{A}$ of admissible functions, and a corresponding rectangle R. But $\bar{y}_1$ and $\bar{y}_1$ (restricted to R) are both fixed points of the transformation τ, so that they coincide on R, against the hypothesis (3.46).

4. - The case $f(0, 0, 0, 0, 0) = 0$

Let $A(\xi)$ *satisfy* (2.1) ($\Rightarrow$ (2.2)). Set

(4.1) $\quad B_0 = 2A_0, \qquad B_1 = 2A_1, \qquad M_1 > 0;$

(4.2) $\quad S' = [0, B_0] \times [-B_1, B_1] \times [-M_1, M_1], \qquad \bar{E}' = \bar{R} \times S';$

and assume:

(4.3) $\quad f \in C^1(\bar{E}), \qquad f(0, 0, 0, 0, 0) = 0,$

(4.4) $\quad f_\zeta, f_\eta, f_z$ Lipschitz continuous in $\bar{E}$ with respect to z.

We shall define a new set of admissible functions

(4.5)
$$\mathfrak{B} = \{z = z(\xi, \eta) \mid z \in C^2(R),\ z(\xi, 0) = A(\xi),\ \forall \xi \in [0, a],$$
$$z_\eta(0, 0) = z_{\xi\eta}(0, 0) = z_{\eta\eta}(0, 0) = 0,\ 0 < z(\xi, \eta) < B_0,$$
$$|z_\xi(\xi, \eta)| < B_1,\ |z_\eta(\xi, \eta)| < M_1,\ 0 < z_{\xi\xi}(\xi, \eta) < M_2,$$
$$|z_{\xi\eta}(\xi, \eta)| < M,\ |z_{\eta\eta}(\xi, \eta)| < M_2\}.$$

$\mathfrak{B}$ is a closed subset of X and C^2. We give the norm in C^2 (which is consistent with the norm in X)

(4.6)
$$\|u\|_2 = |u|_1 + |u_{\xi\xi}|_0 + \sigma |u_{\xi\eta}|_0 + \sigma |u_{\eta\eta}|_0 = \|u\| + \sigma |u_{\eta\eta}|_0,$$

where the norms in the right hand side are given in (2.8), (2.9), (2.10). We suppose also that

$$(4.7) \qquad F(\xi,\eta) = f(\xi,\eta,z(\xi,\eta))$$

is strictly decreasing in η $\forall \xi \in [0,a]$ and $\forall \zeta \in \mathfrak{B}$.

It is obvious that this hypothesis is satisfied, at least for a rectangle R small enough, if

$$(4.8) \qquad f_q(0,0,0,0,0) < 0$$

and f_q satisfies a Lipschitz condition with respect to all its variables.

We have in fact, $\forall \zeta \in \mathfrak{B}$

$$\frac{\partial}{\partial\eta} f(\xi,\eta,\zeta(\xi,\eta),\zeta_\xi(\xi,\eta),\zeta_\eta(\xi,\eta))|_{(\xi,\eta)=(0,0)} = f_q(0,0,0,0,0) = d < 0.$$

Let K the greatest of the Lipschitz constants of f_q: by a classical argument we obtain

$$\left|\frac{\partial}{\partial\eta} f(\xi,\eta,\zeta(\xi,\eta),\zeta_\xi(\xi,\eta),\zeta_\eta(\xi,\eta)) - f_q(0,0,0,0,0)\right| < \\ < K(a+b+B_0+B_1+M_1) + L(M_1+M+M_2)$$

and therefore

$$(4.9) \qquad \frac{\partial}{\partial\eta} f(\xi,\eta,\zeta(\xi,\eta),\zeta_\xi(\xi,\eta),\zeta_\eta(\xi,\eta)) < \\ < d + K(a+b+B_0+B_1+M_1) + L(M_1+M+M_2).$$

But $\zeta(0,0) = \zeta_\xi(0,0) = \zeta_\eta(0,0) = \zeta_{\xi\eta}(0,0) = \zeta_{\eta\eta}(0,0) = 0$ and we can choose R as small as we want; so it is always possible to reduce B_0, B_1, M_1, M, M_2 so that the right hand side of (4.9) becomes negative.

Hypothesis (4.7) allows us to define the application τ given in (2.17). Now it is possible to prove the following Lemma (similar to Lemma 1):

LEMMA 3: *Under the hypotheses* (2.1), (4.3), (4.4), (4.7) *it is possible to choose* a, b, M_2, M, M_3 *such that*

$$\tau : \mathfrak{B} \to \mathfrak{B}.$$

PROOF: Let $M = \operatorname{Max}_{\mathfrak{B}} |f(\xi,\eta,z,p,q)|$, and L_1, L_2, L_3, L_4, L_5 be the Lipschitz constants of f with respect to the five variables. We choose $M_2 > A_2$, $0 < \varepsilon < a_2$, $M_3 > M^2/\varepsilon$ and we set

$$(4.10) \qquad K_1 = L_1 + L_3 B_1 + L_4 M_2 + L_5 M,$$

$$(4.11) \qquad K_2 = L_2 + L_3 M_1 + L_4 M + L_5 M_2,$$

(4.12) $\quad a = \min\{\bar{a}, M_1/M, (M_2 - M^2/c)/K_2\}$,

(4.13) $\quad b = \min\{\bar{b}, (a_2 - c)/K_1, (M_2 - a_2)/K_1, A_0/aM, A_1/M, ac/M\}$.

We obtain in R (as in Lemma 1) $c < w_{\xi\xi}(\xi, \eta) < M_2$. In fact it suffices to put K_1 instead of K in (2.28). Let us now consider $G(\xi, \eta)$. We have, by (4.7):

$$G(0, \eta) = \int_0^\eta f(0, \beta, z(0, \beta))\, d\beta < 0, \qquad 0 < \eta < b,$$

and

$$G(\xi, 0) = A'(\xi) > 0, \qquad 0 < \xi < a,$$

$$G_\xi(\xi, 0) > c > 0 \qquad \text{as in } \S 2.$$

The equation $G(\xi, \eta) = 0$ defines implicitly a line Γ: $\xi = \gamma(\eta)$, with

$$\gamma \in C^1([0, b']), \qquad \gamma'(\eta) = -f(\gamma(\eta), \eta, z(\gamma(\eta), \eta))/G_\xi(\gamma(\eta), \eta).$$

We note that $\gamma'(\eta) > 0$, $\forall \eta > 0$. We want to show that $\gamma'(\bar{\eta}) < 0$ is inconsistent with the definition of Γ; in fact under this hypothesis we have

$$f(\gamma(\bar{\eta}), \bar{\eta}, z(\gamma(\bar{\eta}), \bar{\eta})) > 0$$

and therefore, by (4.7),

$$f(\gamma(\bar{\eta}), \eta, z(\gamma(\bar{\eta}), \eta)) > 0 \qquad \forall \eta < \bar{\eta}$$

and

$$G(\gamma(\bar{\eta}), \bar{\eta}) = A'(\gamma(\bar{\eta})) + \int_0^{\bar{\eta}} f(\gamma(\bar{\eta}), \beta, z(\gamma(\bar{\eta}), \beta))\, d\beta > 0,$$

which is absurd. We have moreover

(4.14) $$0 < \gamma'(\eta) < M/c$$

from which it follows, by the last of (4.13),

(4.15) $$Mb/c < a.$$

We obtain so $b = b'$, $\gamma \in C^1([0, b])$ and (see (2.19)) $w_\xi \in C^1(R)$. We can verify, as in Lemma 1, that

(4.16) $$|w_\xi(\xi, \eta)| < A'(\xi) + \int_0^\eta |f(\xi, \beta, z(\xi, \beta))|\, d\beta < A_1 + bM < B_1$$

and, with a straightforward computation,

$$|w_\eta(\xi,\eta)| < M_1\,,$$

$$|w_{\xi\eta}(\xi,\eta)| < M\,,$$

$$0 < w(\xi,\eta) < B_0\,.$$

Lastly we prove the estimate of $w_{\eta\eta}$, which is given by the formula (see (2.19))

$$w_{\eta\eta}(\xi,\eta) = \frac{\partial}{\partial\eta}\int_{\gamma(\eta)}^{\xi} f(\alpha,\eta,z(\alpha,\eta))\,d\alpha = \tag{4.17}$$

$$= \int_{\gamma(\eta)}^{\xi} (f_\eta + f_z z_\eta + f_p z_{\xi\eta} + f_q z_{\eta\eta})\,d\alpha - f\big(\gamma(\eta),\eta,z(\gamma(\eta),\eta)\big)\gamma'(\eta)\,.$$

We have thus

$$|w_{\eta\eta}(\xi,\eta)| < aK_2 + M^2/c < M_2\,. \tag{4.18}$$

We can state now the following

COROLLARY 2: *Let* $z_1, z_2 \in \mathfrak{B}$ *and* $g_1(\eta)$, $g_2(\eta)$ *the corresponding lines defined in* (2.20). *Under the hypotheses* (2.1), (4.3), (4.4), (4.7) *we have* $|\Delta g|_0 < L/c\varrho\,|\Delta z_1|$.

It corresponds closely to Corollary 1, and can be proved identically.

We shall now repeat, with few modifications, the argument of § 3. Let us now prove

THEOREM 5 (Existence): *Let* $R = [0,a]\times[0,b]$ *and assume that* a, b *are given by* (4.12), (4.13); *let moreover* a *satisfy* (3.12), *where* $L = \text{Max}\,\{L_2, L_4, L_5\}$. *Under the hypotheses* (2.1), (4.3), (4.4), (4.7), *the* II^* *problem has at least one solution in* R.

PROOF: τ maps $\mathfrak{B}$ into itself as a contraction in the norm (4.6), for ϱ and σ suitably chosen. We observe that (3.14), (3.16), (3.18) and (3.26) are true in $\mathfrak{B}$. Let $E' = R\times S'$, with S' defined in (4.2), and let H be the greatest of the Lipschitz constants of all the first derivatives of $f(\xi,\eta,z,p,q)$ with respect to (z,p,q) in E'. We have, as in (3.24),

$$|\Delta w_{\xi\xi}|_0 < H(1 + B_1 + M_2 + M)/\varrho\cdot|\Delta z|_1 + L/\varrho\cdot|\Delta z|_1 + \tag{4.19}$$

$$+ L/\varrho\,|\Delta z_{\xi\xi}|_0 + L/\varrho\cdot|\Delta z_{\eta\eta}|_0\,.$$

We have to prove now the estimate of $|\Delta w_{\eta\eta}|_0$; $\Delta w_{\eta\eta}$ is given by

$$(4.20)\qquad \Delta w_{\eta\eta}(\xi,\eta)=\frac{\partial}{\partial\eta}\int_{\gamma_2(\eta)}^{\xi}\Delta F(\alpha,\eta)\,d\alpha+\frac{\partial}{\partial\eta}\int_{\gamma_1(\eta)}^{\gamma_2(\eta)}F_1(\alpha,\eta)\,d\alpha=$$
$$=\int_{\gamma_2(\eta)}^{\xi}\Delta F_\eta(\alpha,\eta)\,d\alpha-\Delta F(\gamma_2(\eta),\eta)\gamma_2'(\eta)+\int_{\gamma_1(\eta)}^{\gamma_2(\eta)}F_{1\eta}(\alpha,\eta)\,d\alpha+$$
$$+F_1(\gamma_2(\eta),\eta)\gamma_2'(\eta)-F_1(\gamma_1(\eta),\eta)\gamma_1'(\eta).$$

The estimate of ΔF_η is analogous to the previous one for ΔF_ξ in (3.19)-(3.22), bearing in mind that in R it is:

$$|F_{1\xi}(\xi,\eta)|<L_1+L(A_1+M_2+M),$$
$$|F_{1\eta}(\xi,\eta)|<L_2+L(M_1+M_3+M),$$

and that $|\gamma_i'(\eta)|<M/\varepsilon$ by (4.14). It follows then that

$$(4.21)\qquad |\Delta w_{\eta\eta}(\xi,\eta)|<\int_0^a\{H(1+M_1+M+M_3)|\Delta z(\alpha,\eta)|+$$
$$+L[|\Delta\zeta_\eta(\alpha,\eta)|+|\Delta\zeta_{\xi\eta}(\alpha,\eta)|+|\Delta\zeta_{\eta\eta}(\alpha,\eta)|]\}\,d\alpha+LM/\varepsilon|\Delta z(\gamma_2(\eta),\eta)|+$$
$$+\{L_2+L(M_1+M+M_3)+M[L_1+L(A_1+M_2+M)]/\varepsilon\}|\Delta\gamma(\eta)|.$$

Denoting by D the last constant in $\{\ \}$, and keeping the norms, we obtain thus:

$$(4.22)\qquad \sigma|\Delta w_{\eta\eta}|_0<\sigma a(H(1+M_1+M+M_3)+L)\|\Delta\zeta\|_1+$$
$$+La\sigma|\Delta\zeta_{\xi\eta}|_0+La\sigma|\Delta\zeta_{\eta\eta}|_0+LM\sigma/\varepsilon\|\Delta\zeta\|_1+DL\sigma/\varepsilon\varrho\cdot\|\Delta\zeta\|_1.$$

We add (3.14), (3.16), (3.18), (3.26), (4.19) and (4.22) and we obtain, for σ small and ϱ large enough:

$$(4.23)\qquad \|\Delta w\|_2<C\|\Delta\zeta\|_2,\qquad \text{with } C<1,$$

and the thesis follows. In order to prove uniqueness and regularity of the solution of the Π^* problem (at least in a suitable rectangle R) we must prove some theorems analogous of Lemma 2, also under the different hypotheses made on f. We obtain, in fact:

LEMMA 4: *Let y, Γ be a solution of the Π^* problem in $\bar{R}$, where (2.1), (4.3), (4.4), (4.7) hold; there exists then a rectangle $R\subset\bar{R}$ such that, letting $Z=\bar{Z}\cap R$, $y_\xi(\xi,\eta)$ and Γ are of class C^1 in Z and $\gamma'(\eta)>0$, for $\eta>0$.*

Then we can extend uniquely y in R' using (3.45), that is:

$$\tilde{y}(\xi,\eta)=\int_{\gamma(\eta)}^{\xi} G(\alpha,\eta)\,d\alpha \,. \tag{4.24}$$

Γ is however tangent to the characteristic $\xi=0$ in $(0,0)$, so the classical results on the solution of a Cauchy problem are not sufficient to guarantee that $\tilde{y}\in C^2(R')$. We must prove therefore the property directly, by means of the techniques used in Lemma 2 for the continuity of $y_{\xi\xi}$.

THEOREM 6: *Every solution y of Π^*, under the hypotheses* (2.1), (4.3), (4.4), (4.7), *can be extended to a function $\tilde{y}\in C^2(R')$, defined in* (4.24), *satisfying in R' a Cauchy problem with vanishing data.*

PROOF: We already verified that $y_{\xi\xi}\in C^0(Z)$; the same proof works in order to get that $y_{\xi\xi}\in C^0(R')$. We show also that $y_{\xi\eta}\in C^0(R')$.

Let

$$B_0=\operatorname*{Max}_{R'}|\tilde{y}(\xi,\eta)|\,,\qquad B_1=\operatorname*{Max}_{R'}|\tilde{y}_\xi(\xi,\eta)|\,,\qquad M_1=\operatorname*{Max}_{R'}|\tilde{y}_\eta(\xi,\eta)|$$

and S' given by (4.2), $M=\operatorname*{Max}_{R'\times S'}|f(\xi,\eta,z,p,q)|$.

$$\mathcal{B}'=\{q\in C^1(R')\,|\,|q(\xi,\eta)|<M_1,\ |q_\xi(\xi,\eta)|<M,\ |q_\eta(\xi,\eta)|<M_2\}\,. \tag{4.25}$$

We introduce in $C^1(R')$ the norm

$$|\!|\!|q|\!|\!|'=|q|'+\sigma|q_\xi|'+\sigma|q_\eta|' \tag{4.26}$$

with

$$|u|'=\operatorname*{Max}_{R'}\{\exp[-\varrho|\xi-\gamma(\eta)|]\,|u(\xi,\eta)|\}\,. \tag{4.27}$$

Let $\omega:\mathcal{B}'\to C^1(R')$ be the map defined by $\omega u=v$, with

$$v(\xi,\eta)=\int_{\gamma(\eta)}^{\xi} f(\alpha,\eta,\tilde{y}(\alpha,\eta),\tilde{y}_\xi(\alpha,\eta),u(\alpha,\eta))\,d\alpha\,. \tag{4.28}$$

Since R, R' are constructed by Lemma 3, we have immediately

$$\omega:\mathcal{B}'\to\mathcal{B}'\,. \tag{4.29}$$

We obtain also, using (3.18), (4.19), (4.20), and noting that only the last argument is varying,

$$(4.30)\qquad |\Delta v(\xi,\eta)| < L_3\left|\int_{\gamma(\eta)}^{\xi}|\Delta u(\alpha,\eta)\,d\alpha\right|,$$

$$(4.31)\qquad |\Delta v_\xi(\xi,\eta)| < L_3|\Delta u(\xi,\eta)|,$$

$$(4.32)\qquad |\Delta v_\eta(\xi,\eta)| < H(1+M_1+M+M_2)\int_{\gamma(\eta)}^{\xi}|\Delta u(\alpha,\eta)|\,d\alpha +$$
$$+ L\int_{\gamma(\eta)}^{\xi}|\Delta u_\eta(\alpha,\eta)|\,d\alpha + L_3|\Delta u(\gamma(\eta),\eta)|\,|\gamma'(\eta)|,$$

then, multiplying by the exponential weight all the last three formulas and multiplying also (4.31), (4.32) by σ, we have, by addition:

$$(4.33)\qquad \|\Delta v\|' < (L_3 + \sigma H(1+M_1+M+M_2))/\varrho\cdot\|\Delta u\|' +$$
$$+ L_3(1+M/\varepsilon)\sigma\|\Delta u\|' + L_3\sigma/\varrho\,\|\Delta u_\eta\|'.$$

Therefore ω is a contraction too, for a suitable choice of ϱ, σ, and we prove that y_η, unique fixed point of ω, belongs to C^1. We have lastly the uniqueness theorem

Theorem 7: *Let y_1 and y_2 be solutions of the II^* problem. There exists then a rectangle R' such that $y_1(\xi,\eta) = y_2(\xi,\eta)$ in R'.*

The proof is the same of Theorem 4.

5. - Support domains

a) Let us now consider the problem (1.2)-(1.5), with

$$(5.1)\qquad A(\xi)\in C^2([0,\bar a]),\qquad B(\eta)\in C^2([0,\bar b]),\qquad A(0)=B(0),$$

$$(5.2)\qquad f\in C^{1,3}(\bar R\times S),$$

where S is a parallelepiped in $R_+\times R^2$ as in (2.3). A necessary condition for the existence of a support domain is

$$(5.3)\qquad A(0) = A'(0) = B(0) = B'(0) = 0.$$

While it is easy to verify (see [3]) that, if (5.3) does not hold, the solution of Darboux problem with data $A(\xi)$ and $B(\eta)$ satisfies, at least locally, the unilateral condition (1.3), and is strictly positive in an open rectangle $(0,a')\times$

$\times (0, b')$. Suppose therefore that (5.3), (2.4), (2.5) hold, and moreover that

$$A'(0) > 0 , \qquad B'(0) > 0 . \tag{5.4}$$

It is then possible to solve the Π_1^* problem with respect to the datum $A(\xi)$, and the analogous Π_2^* one relative to $B(\eta)$, and obtain two lines

$$\begin{cases} \Gamma_1 \colon \xi = \gamma_1(\eta) \Leftrightarrow \eta = \varphi_1(\xi) , \\ \Gamma_2 \colon \eta = \gamma_2(\xi) . \end{cases} \tag{5.5}$$

If there exists an interval $[0, \bar{a}]$ such that

$$\varphi_1(\xi) < \gamma_2(\xi) \qquad \text{for } 0 < \xi < \bar{a} , \tag{5.6}$$

then the function

$$y(\xi, \eta) = \begin{cases} y_1(\xi, \eta) & \text{in } Z_1 = \{0 < \xi < \bar{a}, 0 < \eta < \varphi_1(\xi)\} , \\ 0 & \text{in } \Delta \ = \{0 < \xi < \bar{a}, \varphi_1(\xi) < \eta < \gamma_2(\xi)\} , \\ y_2(\xi, \eta) & \text{in } Z_2 = \{0 < \xi < \bar{a}, \gamma_2(\xi) < \eta < b_1\} , \end{cases} \tag{5.7}$$

satisfies the unilateral problem, and admits Δ as support domain. (5.7) gives obviously an *extension law* for the unilateral problem. Since

$$\gamma_1'(0) = -f(0, 0, 0, 0, 0)/A'(0) , \qquad \gamma_2'(0) = -f(0, 0, 0, 0, 0)/B'(0) , \tag{5.8}$$

the condition

$$A'(0) B'(0) < f^2(0, 0, 0, 0, 0) \tag{5.9}$$

implies (5.6). If, on the other hand,

$$A'(0) B'(0) > f^2(0, 0, 0, 0, 0) \tag{5.10}$$

the solution of the free Darboux problem is again strictly positive in $(0, a') \times \times (0, b')$, and no support domain exists. Moreover, if, instead of (5.4),

$$A'(0) > 0 , \qquad B(\eta) = 0 \tag{5.11}$$

holds, we get the simpler solution

$$y(\xi, \eta) = \begin{cases} y_1(\xi, \eta) & \text{in } Z_1 , \\ 0 & \text{in } \Delta = R \setminus Z_1 , \end{cases} \tag{5.12}$$

(an analogous case holds if $A(\xi) = 0$).

b) Let us now turn to the case considered in § 4. It is $f = 0$ at the origin. The hypothesis (5.4) again implies existence and uniqueness of the solution of the Π_1^* and Π_2^* problems; (5.6), however, can no longer hold. In fact, Γ_1 is necessarily tangent, at the origin, to the η-axis, and Γ_2 to the ξ-axis, so that no support domain exists. Suppose, on the other hand, that (5.11) holds. Then (5.12) gives a solution of the support problem. The case $B'(0) = 0$, $B(\eta) \neq 0$ for the quasi-linear equation is still open; for the linear one see Amerio [3].

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