Support Domains
for a Quasi-Linear String Vibrating Against a Wall:
A Unilateral Free Boundary Problem (***) (***)

Domini di appoggio per una corda quasi lineare
vibrante contro una parete:
un problema unilaterale di frontiera libera

1. - INTRODUCTION: THE II* PROBLEM

The problem of the motion of a string vibrating in presence of a rigid wall has been studied in many papers. Particular difficulties arise in the case where the string is subject to an external force, directed towards the wall, or if the wall is convex.

In a recent work [3] L. Amerio considered the case of the linear equation

\[ y_{xx} = f(\xi, \eta) \]

(1.1)

(where \( \xi = (x + t)/\sqrt{2} \), \( \eta = (-x + t)/\sqrt{2} \) are the characteristic coordinates and \( f(\xi, \eta) \) is arbitrary), introducing the notions of support line and domain. This

(**) Lavoro parzialmente finanziato con fondi M.P.I.
(***) Memoria presentata il 24 luglio 1984 da Luigi Amerio, uno dei XL.
allows to formulate new extension laws (with respect to [1]), in order to extend
the solution beyond the first line of influence of the wall, until a second line of influence,
and so on.

In the present paper we generalize some of the results in [3] to the non
linear case where the external force $f$ depends also on $y$ and its first deriva-
tives, supposing either that $f$ is directed towards the wall (§§ 2 and 3), or that
it changes sign in a neighborhood of the origin (§ 4).

More precisely, given a rectangle $R = [0, a] \times [0, b]$, we look for a $C^1$-
function $y = y(\xi, \eta)$, solving (at least locally) the following unilateral problem:

\begin{align*}
(1.2) \quad & y_{\eta\eta} + f(\xi, \eta, y(\xi, \eta), y_\xi(\xi, \eta), y_\eta(\xi, \eta)) \quad \text{in } D(\tilde{R}), \\
(1.3) \quad & y(\xi, \eta) > 0, \\
(1.4) \quad & y[y_\eta - f] = 0, \\
(1.5) \quad & y(\xi, 0) = A(\xi), \quad y(0, \eta) = B(\eta) \quad \text{(with } A(0) = B(0) = 0). 
\end{align*}

In order to obtain local solutions of (1.2)-(1.5) we solve the following free
boundary problem, denoted by $\Pi^*$ (see Amerio [3], § 2):

**$\Pi^*$ problem:** Find a rectangle $R = [0, a] \times [0, b] \subset \tilde{R}$, and a line $\Gamma$:

(1.6) \quad $\xi = \gamma(\eta), \quad \eta(0) = 0$,

with $\gamma \in C^1([0, b'])$, $b' < b$, strictly increasing, such that the equation

(1.7) \quad $y_{\eta\eta} = f(\xi, \eta, y, y_\xi, y_\eta)$

has a $C^1$-solution $y(\xi, \eta)$ in the domain

(1.8) \quad $Z = \{y(\eta) < \xi < a, 0 < \eta < b'\}$

satisfying the conditions

\begin{align*}
(1.9) \quad & y(\xi, 0) = A(\xi) \quad \text{in } [0, a], \\
(1.10) \quad & y(\gamma(\eta), \eta) = y_\xi(y(\eta), \eta) = 0 \quad \text{in } [0, b'], \\
(1.11) \quad & y(\xi, \eta) > 0 \quad \text{in } Z.
\end{align*}

For a general framework, where the $\Pi^*$ problem is studied in connection
with the mixed initial-boundary value problem for the vibrating string equa-
tion with unilateral constraints, we refer to the paper [3] of Amerio.

Let us now introduce some simplifying notations. For each function
$u \in C^1(R)$, we denote by $u$ the vector $(u, u_\xi, u_\eta)$. Let $f = f(\xi, \eta, y, y_\xi, y_\eta)$ be a $C^1$-function; $f_\ast$ will represent the vector $(f_\xi, f_\eta, f_\eta)$ and $\cdot u$ the scalar product $f_\ast u + f_\eta u_\xi + f_\xi u_\eta$. 
If \( y \) is a solution of \( \Pi^* \), then the formula:

\[
y(\xi, \eta) = A(\xi) - A(y(\eta)) + \int_{y(\eta)}^{\xi} \int_{\beta}^{\xi} f(x, \beta, y(x, \beta)) \, d\beta
\]

holds true in \( Z \). Setting

\[
G(\xi, \eta) = A'(\xi) + \int_{\eta}^{\xi} f(\xi, \beta, y(\xi, \beta)) \, d\beta \quad \text{in } R,
\]

we have

\[
y_y(\xi, \eta) = G(\xi, \eta) \quad \text{in } Z,
\]

\[
y(\xi, \eta) = \int_{\eta}^{\xi} G(x, \eta) \, dx \quad \text{in } Z.
\]

The equation \( G(\xi, \eta) = 0 \) implicitly defines the line \( T' \):

\[
G(y(\eta), \eta) = 0 \quad \text{in } [0, \eta'].
\]

We shall give in §§ 3 and 4 existence, uniqueness and regularity theorems for the solution of the \( \Pi^* \) problem; in § 5 we deduce sufficient conditions for the existence of support domains, and formulate the corresponding extension laws.

2. - Admissible functions

In the present section we consider the \( \Pi^* \) problem assuming \( f < 0 \), and we obtain some preliminary results. More precisely, we shall introduce an application \( r \) from a suitable set \( A \) of admissible functions into itself. The fixed point of such an application, which we shall obtain in § 3 by means of the contraction mapping theorem, will give the solution of the \( \Pi^* \) problem.

We shall suppose that the boundary value \( A(\xi) \) and the function \( f(\xi, \eta, r, p, q) \) satisfy the following hypotheses (see Amerio [3], § 5, Th. 2):

\[
A \in C^2([0, \tilde{\sigma}])
\]

\[
A(0) = A'(0) = 0,
\]

\[
A'(\xi) > a_2 > 0 \quad \text{in } [0, \tilde{\sigma}],
\]

so that there exist \( A_0, A_1, A_2 \) such that:

\[
a_2 < A'(\xi) < A_2 \quad \text{in } [0, \tilde{\sigma}],
\]

\[
0 < A'(\xi) < A_1 \quad \text{in } [0, \tilde{\sigma}],
\]

\[
0 < A(\xi) < A_0 \quad \text{in } [0, \tilde{\sigma}].
\]
Note that the condition $A(0) = A'(0) = 0$ is necessary in order that there exists a support domain ([3], § 3). Let us set

\begin{equation}
S = [0, A_0] \times [-A_1, A_1] \times [-M_1, M_1], \quad \mathcal{E} = \mathbb{R} \times S, \quad (M_1 > 0)
\end{equation}

and suppose

\begin{equation}
f \in C^1(\mathcal{E}), \quad f(\xi, \eta, z, \rho, \vartheta) < 0 \quad \text{in } \mathcal{E}.
\end{equation}

We assume therefore that the external force is directed towards the wall. In the proof of the existence and uniqueness theorem we shall also assume:

\begin{equation}
f_1, f_2 \text{ Lipschitz continuous with respect to } z \text{ in } \mathcal{E}.
\end{equation}

Let now $a, b, M, M_2$ be positive constants, and $R$ be a fixed rectangle $[0, a] \times [0, b] \subset \mathbb{R}$. The set of admissible functions will then be defined by:

\begin{equation}
A = \{ z = \varphi(\xi, \eta) | \varphi, \varphi \in C^1(R), z(\xi, 0) = A(\xi), \forall \xi \in [0, a],
0 < \varphi(\xi, \eta) < A_0, |\varphi(\xi, \eta)| < A_1, |\varphi(\xi, \eta)| < M_1,
0 < \varphi(\xi, \eta) < M_2, -M < \varphi(\xi, \eta) < 0 \}.
\end{equation}

**Remark:** If $\varphi \in A$, we have $\varphi_{\xi \eta} = \varphi_{\eta \xi}$ everywhere in $R$ (see Hobson [7], vol. I, pag. 497).

The set $A$ is obviously closed in the Banach space

\begin{equation}
X = \{ z = \varphi(\xi, \eta) | \varphi, \varphi \in C^1(R) \},
\end{equation}

endowed with the norm (equivalent to the usual one):

\begin{equation}
\| z \| = |z|_1 + |z|_0 + \sigma |z|_0,
\end{equation}

where

\begin{equation}
\| u \|_0 = \sup \{ \exp \{-q \eta \| u(\xi, \eta) \| \} \},
\end{equation}

\begin{equation}
\| u \|_1 = |u|_0 + |u|_0 + |u|_0,
\end{equation}

with $q, \sigma$ positive constants to be chosen later.

Let now $\varphi \in A$. Consider the function

\begin{equation}
G(\xi, \eta) = A'(-\xi) + \int_{\frac{\pi}{2}}^{\varphi(\xi, \eta)} \frac{f(\xi, \beta, z(\xi, \beta))}{\beta} \, d\beta,
\end{equation}

which, under the hypotheses (2.1) and (2.4), belongs to $C^1(R)$. Choosing $R$ as in the following Lemma 1, the equation

\begin{equation}
G(\xi, \eta) = 0
\end{equation}
implicitly defines in $\mathcal{R}$ (see Amerio [3]) a line (depending on $\gamma$)

\begin{equation}
I': \xi = \gamma(\eta), \quad \gamma \in C^1([0, b')], \quad b' < b,
\end{equation}

with

\begin{equation}
\gamma(0) = 0, \quad \gamma(\eta) \text{ strictly increasing in } [0, b'];
\end{equation}

if $b' < b$ it is $\gamma(b') = a$, if $b' = b$ it is $\gamma(b') < a$ (fig. 1).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{Fig. 1.}
\end{figure}

The line $I'$ divides the rectangle $\mathcal{R}$ in two parts; setting

\begin{equation}
Z = \{(\xi, \eta) \in \mathcal{R} | \gamma(\eta) < \xi < a, \quad 0 < \eta < b'\},
\end{equation}

we have

\begin{equation}
G(\xi, \eta) > 0 \text{ in } \hat{\mathcal{R}}, \quad G(\xi, \eta) < 0 \text{ in } (\mathcal{R} \setminus \mathcal{Z}).
\end{equation}

Let us define the application $\tau$ by means of

\begin{equation}
\tau(\xi) = \int_{\tau(\eta)}^{\xi} G(\xi, \eta) \, dx, \quad 0 < \eta < b',
\end{equation}

\begin{equation}
\tau(\xi) = \int_{\xi}^{\tau(\eta)} G(\xi, \eta) \, dx, \quad b' < \eta < b.
\end{equation}

It follows

\begin{equation}
w(\xi, \eta) = G(\xi, \eta) \quad \text{in } \mathcal{R},
\end{equation}

\begin{equation}
w(\xi, \eta) = \int_{\tau(\eta)}^{\xi} G(\xi, \eta) \, dx, \quad 0 < \eta < b' \text{ (by (2.12))},
\end{equation}

\begin{equation}
w(\xi, \eta) = \int_{\eta}^{\xi} G(\xi, \eta) \, dx, \quad b' < \eta < b,
\end{equation}
from which one obtains immediately that \( \tau : \mathcal{A} \to X \), both \( G \) and \( \gamma \) being of class \( C^1 \). If \( b' \leq b \), we shall extend the definition of \( \gamma \) to the whole of \([0, b]\), by setting:

\[
\gamma(\eta) = \begin{cases} 
\gamma(\eta) & \text{for } 0 < \eta < b', \\
\eta & \text{for } b' < \eta < b.
\end{cases}
\]

Hence we can rewrite (2.17) as

\[
u(\xi, \eta) = \tau(\gamma) = \int_{a(\eta)}^{\xi} G(a, \eta) \, da,
\]

\(0 < \eta < b.\)

Observe that the function \( u(\xi, \eta) \) satisfies, in \( R \), the linearized equation \( u_{\mathcal{B}} = f(\xi, \eta, z(\xi, \eta)) \), and the following conditions:

\[
u(\xi, 0) = A(\xi), \quad 0 < \xi < a,
\]

\[u(\gamma(\eta), \eta) = 0, \quad 0 < \eta < b',\]

\[u_\xi(\gamma(\eta), \eta) = 0, \quad 0 < \eta < b',\]

\[u(a, \eta) = 0, \quad \text{if } b' < \eta < b.\]

The following lemma then holds.

**Lemma 1:** Under the hypotheses (2.1) and (2.4) it is possible to choose the constants \( a, b, M_2 \) and \( M \) so that:

\[
\tau : \mathcal{A} \to \mathcal{A}.
\]

**Proof:** Let us set

\[
M = \max_{\xi, \eta} |f(\xi, \eta, z, p, q)|,
\]

and let \( L_1, L_2, L_4, L_5 \) be the Lipschitz constants of \( f \) with respect to \( \xi, z, p, q \) (in the given order) in \( \overline{B} \). Let us choose arbitrarily \( M_2 > A_2 \) and \( \epsilon \), with \( 0 < \epsilon < a_2 \), and set

\[
K = L_1 + L_2 A_1 + L_4 M_2 + L_5 M,
\]

\[
a = \min \left( a, \frac{M_2}{M} \right),
\]

\[
b = \min \left( b, \left( a_2 - \epsilon \right) / K, \left( M_2 - A_2 \right) / K, A_4 / a M, A_5 / M \right).
\]

Substituting in (2.6) the above constants, we shall prove that (2.22) holds,
a) Let us consider firstly, \( \forall \xi \in A \), the derivative \( w_{\xi\eta} \). From (2.11), (2.18) we obtain at once

\[
(2.27) \quad w_{\xi\eta}(\xi, \eta) = G_{\xi}(\xi, \eta) = A'(\xi) + \int_0^1 [f_1(\xi, \beta, \tau(\xi, \beta)) + f_2(\xi, \beta, \tau(\xi, \beta)) \cdot \tau_2(\xi, \beta)] \, d\beta,
\]

from which, by (2.2) and (2.24):

\[
(2.28) \quad a_k - K \beta < w_{\xi\eta}(\xi, \eta) < A_4 + K \beta \quad \text{in } R.
\]

Hence by (2.26) we have

\[
(2.29) \quad 0 < \epsilon < w_{\eta\eta}(\xi, \eta) < M_4.
\]

b) Let us now observe that (2.12) actually defines, by Dini's theorem, a line \( \Gamma \) enjoying properties (2.13) and (2.14). In fact, from (2.29) it follows \( G_{\xi}(\xi, \eta) > 0 \) in \( R \), while \( G_{\eta}(\xi, \eta) = f(\xi, \eta, z(\xi, \eta)) < 0 \) by hypothesis (2.4). Then, by the definitions of \( \Gamma \) and \( Z \), (2.16) follows. Moreover, by (2.18), it is \( w_{\eta}(\xi, \eta) = G(\xi, \eta) \); hence

\[
(2.30) \quad 0 < w_{\eta}(\xi, \eta) = A'(\xi) + \int_0^1 f(\xi, \beta, z(\xi, \beta)) \, d\beta < A_4 \quad \text{in } Z,
\]

while in \( R \setminus Z \), by (2.26):

\[
(2.31) \quad 0 > w_{\eta}(\xi, \eta) > - A_4.
\]

Consequently

\[
(2.32) \quad |w_{\eta}(\xi, \eta)| < A_4 \quad \text{in } R.
\]

c) From (2.11), (2.19) and (2.25) we have at once, \( \forall \xi \in A \):

\[
(2.33) \quad |w_{\eta}(\xi, \eta)| < M_3 < M_1,
\]

\[
(2.34) \quad - M < w_{\eta\eta}(\xi, \eta) = f(\xi, \eta, z(\xi, \eta)) < 0.
\]

d) Lastly we obtain, from (2.21):

\[
(2.35) \quad w(\xi, \eta) = A(\xi) - A(g(\eta)) + \int_{s(\xi)}^\eta f(x, \beta, z(x, \beta)) \, d\beta.
\]

We have \( w(g(\eta), \eta) = 0 \) in \([0, b]\), and \( w(\xi, \eta) > 0 \) in \( \bar{Z} \) by (2.30); hence
\[ w(\xi, \eta) > 0 \text{ in } Z. \text{ Moreover } A(\xi) \text{ increases by (2.2), and } f < 0 \text{ in } R: \text{ hence} \]

\[ 0 < w(\xi, \eta) < A_0 \quad \text{in } Z. \]

On the other hand it is \( w_0(\xi, \eta) < 0 \text{ in } R \setminus Z; \text{ hence we have again } w(\xi, \eta) > 0, \]
and furthermore, by (2.26):

\[ 0 < w(\xi, \eta) < \int_0^{\frac{\pi}{4}} \left| \int_0^{\beta} f(x, \beta, z(x, \beta)) \, d\beta \right| d\beta < \]

\[ < Mab < Ma (A_0/Ma) = A_0 \quad \text{in } R \setminus Z. \]

Finally

\[ 0 < w(\xi, \eta) < A_0 \quad \text{in } R. \]

From (2.29), (2.32), (2.33), (2.34) and (2.38) the thesis follows.

3. - Solution of the II* Problem

Let \( z_1 \text{ and } z_2 \in A, g_1(\eta) \text{ and } g_2(\eta) \) be the corresponding lines, defined by

\[ \begin{align*}
  g_i(\eta) & = \begin{cases} 
  \gamma_i(\eta) & 0 < \eta < b_i' \\
  a & b_i' < \eta < b_i ,
  \end{cases} 
  \end{align*} \quad (i = 1, 2),
\]

(see (2.20)), and suppose \( b_2' < b_1' \). The following continuous dependence theorem then holds:

**Theorem 1**: If the constants \( c, L_3, L_4, L_5 \) are chosen according to Lemma 1, we have:

\[ |Ag(\eta)| < (1/c) \left[ \int_0^{\beta} \left| L_3 |\Delta z_i(g_i(\eta), \beta)\right| + 
\right. 
\]

\[ + L_4 |\Delta z_4(g_4(\eta), \beta)| + L_3 |\Delta z_3(g_3(\eta), \beta)| \, d\beta \]

(\text{where } Ag = g_1 - g_2, \Delta z = z_1 - z_2).

For the sake of simplicity we shall use in the sequel the following notations:

\[ |\Delta z(\xi, \eta)| = |\Delta z_1(\xi, \eta)| + |\Delta z_2(\xi, \eta)| + |\Delta z_3(\xi, \eta)| , \]

\[ F_1(\xi, \eta) = f(\xi, \eta, z_1(\xi, \eta)). \]

If now

\[ L = \max \{ L_3, L_4, L_5 \} , \]

\[ \|Ag\|_0 = \max_{\eta \in \partial D} \{ \exp \left( -c\eta \right) |Ag(\eta)| \} , \]
we obtain at once the

**Corollary 1:**

(3.7) \[ \|A_g\|_0 < (L/\epsilon) \|A_z\|_1, \]

the latter norm being defined in (2.10).

It is indeed

\[ \exp \left[ -\eta \|A_g\| \right] < (L/\epsilon) \int_0^\eta \exp \left[ -\eta (\eta - \beta) \right] \|A_z(\eta, \beta)\| d\beta < \]

\[ < (L/\epsilon) \int_0^\eta \exp \left[ -\eta (\eta - \beta) \right] d\beta < (L/\epsilon) \|A_z\|_1, \]

from which (3.7) follows.

Let us now prove Theorem 1. We firstly consider the interval \([0, b_1^*]\).

It is

(3.8) \[ G_F(\gamma_1(\eta), \eta) = 0 \quad \text{for} \quad 0 < \eta < b_1^*. \]

By Lagrange theorem we obtain:

(3.9) \[ 0 = G_1(\gamma_1(\eta), \eta) - G_1(\gamma_2(\eta), \eta) + G_1(\gamma_2(\eta), \eta) - G_2(\gamma_2(\eta), \eta) = \]

\[ = G_{12}(\delta(\eta), \eta) \cdot A\gamma(\eta) + \int_0^\eta \left[ F_1(\gamma_2(\eta), \beta) - F_2(\gamma_2(\eta), \beta) \right] d\beta, \]

where \(\delta(\eta) \in (\gamma_1(\eta), \gamma_2(\eta))\). By the Lipschitz continuity of \(f\), and by (2.29) we have then

(3.10) \[ \|A\gamma(\eta)\| < (1/\epsilon) \int_0^\eta \{ L_{\|A\gamma(\eta, \beta)\|} + L_{\|A\gamma(\eta, \beta)\|} + L_{\|A\gamma(\eta, \beta)\|} \} d\beta \quad \text{in} \quad [0, b_1^*]. \]

In \((b_1^*, b)\) it is \(g_2(\eta) = a\), while \(g_1(\eta)\) is non decreasing. Hence

(3.11) \[ \|A\gamma(b_1^*)\| < (1/\epsilon) \int_0^{b_1^*} \{ L_{\|A\gamma(a, \beta)\|} + L_{\|A\gamma(a, \beta)\|} + L_{\|A\gamma(a, \beta)\|} \} d\beta < \]

\[ < (1/\epsilon) \int_0^{b_1^*} \{ L_{\|A\gamma(a, \beta)\|} + L_{\|A\gamma(a, \beta)\|} + L_{\|A\gamma(a, \beta)\|} \} d\beta. \]

(3.10) together with (3.11) give the thesis.
We are now ready to prove the existence and uniqueness theorems, which represent the main result of the work.

**Theorem 2 (Existence):** Let $R$ be the rectangle $[0, a] \times [0, b]$, where $a$ and $b$ satisfy (2.25), (2.26) and

$$a < 1/L,$$

with $L$ given in (3.5). Let moreover $E = R \times S$, where $S$ is defined in (2.3). Under the hypotheses (2.1), (2.4) and (2.5), the II* problem has in $R$ at least one solution.

**Proof:** It suffices to show that, if $a$ and $b$ are as above, then $w = \tau(\xi)$ is a contraction of $A$ into itself in the norm (2.8), provided that the constants $\rho$ and $\sigma$ are suitably chosen.

$a)$ Let us firstly consider the difference $\Delta w = w_1 - w_2$. We have:

$$|\Delta w(\xi, \eta)| = |A(\zeta(\eta)) - A(\zeta(\eta))| +$$

$$+ \int_{\zeta(\eta)}^{\zeta(\eta)} \left[ F_1(x, \beta) d\beta \right] + \int_{\zeta(\eta)}^{\zeta(\eta)} \left[ F_2(x, \beta) d\beta \right] <$$

$$< (A_1 + Mb)|A g(\eta)| + L \int_{\zeta(\eta)}^{\zeta(\eta)} |\Delta z(x, \beta)| d\beta.$$

Let us now multiply both sides of (3.13) by $\exp[-\rho \eta]$, and calculate the norm (2.9) of $\Delta w$. We have:

$$\exp[-\rho \eta]|\Delta w(\xi, \eta)| < (A_1 + Mb) \exp[-\rho \eta]|A g(\eta)| +$$

$$+ L \int_{\zeta(\eta)}^{\zeta(\eta)} \exp[-\rho (\eta - \beta)] d\beta \exp[-\rho \beta] |\Delta z(x, \beta)| dx <$$

$$< (A_1 + Mb)|A g| + L \int_{\zeta(\eta)}^{\zeta(\eta)} \exp[-\rho (\eta - \beta)] d\beta.$$

By (3.7) we have then

$$|\Delta w| < (1/\rho)((A_1 + Mb)L + L_\sigma) |\Delta z|_1.$$

$b)$ The evaluation of the norm (2.9) of $\Delta w$ is immediate:

$$|\Delta w(\xi, \eta)| = \left| \int_{\eta}^{\eta} F_1(x, \beta) d\beta \right| < L \int_{\eta}^{\eta} |\Delta z(\xi, \beta)| d\beta,$$

from which, as before:

$$|\Delta w| < (L/\rho) |\Delta z|_1.$$
The estimate of $\Delta w_\epsilon$, on the other hand, takes the following form:

$$\begin{align*}
|\Delta w_\epsilon(\xi, \eta)| &= \left| \int_{\nu_k(\epsilon)} F_1(x, \eta) \, dx - \int_{\nu_k(\epsilon)} F_2(x, \eta) \, dx \right| \\
&< \left| \int_{\nu_k(\epsilon)} F_1(x, \eta) \, dx \right| + \left| \int_{\nu_k(\epsilon)} \Delta F(x, \eta) \, dx \right| < M|\Delta g(\eta)| + L \left| \int_{\nu_k(\epsilon)} \Delta z(x, \eta) \, dx \right|.
\end{align*}$$

Multiplying both sides by $\exp[-\rho \eta]$ we obtain, as usual:

$$|\Delta w_\epsilon| \leq (ML|\eta|)\{A_\tau\} + L\epsilon\frac{\epsilon}{A_\tau}.$$

Note that the hypothesis (3.12) on $a$ plays an essential role only in this estimate.

By hypothesis (2.5) the derivatives of $f(\xi, \eta, \xi, \rho, g)$ with respect to $\xi, \eta, \rho$ and $g$ satisfy a Lipschitz condition in $E = R \times S$ with respect to the variables $\xi, \rho$ and $g$. Let $H$ be the greatest of the 12 Lipschitz constants in (2.5). We obtain then the following estimates, $\forall \tau, \epsilon \\ A$:  

$$\begin{align*}
|f_1(\xi, \eta, \tau_k(\xi, \eta)) - f_1(\xi, \eta, \tau_k(\xi, \eta))| &< H|\Delta \xi(\xi, \eta)|; \\
|f_1(\xi, \eta, \tau_k(\xi, \eta)) \tau_{11}(\xi, \eta) - f_1(\xi, \eta, \tau_k(\xi, \eta)) \tau_{11}(\xi, \eta)| &< \\
&< \left| \left| f_1(\xi, \eta, \tau_k(\xi, \eta)) - f_1(\xi, \eta, \tau_k(\xi, \eta)) \right| \tau_{11}(\xi, \eta) \right| + \\
&+ \left| f_1(\xi, \eta, \tau_k(\xi, \eta)) \tau_{11}(\xi, \eta) - \tau_{11}(\xi, \eta) \right| < H|\Delta \xi(\xi, \eta)| \cdot A_1 + L|\Delta \xi(\xi, \eta)|.
\end{align*}$$

In an analogous way we obtain:

$$\begin{align*}
|f_1(\xi, \eta, \tau_k(\xi, \eta)) \tau_{11}(\xi, \eta) - f_1(\xi, \eta, \tau_k(\xi, \eta)) \tau_{11}(\xi, \eta)| < \\
< H|\Delta \xi(\xi, \eta)| \cdot A_1 + L|\Delta \xi(\xi, \eta)|.
\end{align*}$$

$$\begin{align*}
|f_1(\xi, \eta, \tau_k(\xi, \eta)) \tau_{11}(\xi, \eta) - f_1(\xi, \eta, \tau_k(\xi, \eta)) \tau_{11}(\xi, \eta)| < \\
< H|\Delta \xi(\xi, \eta)| \cdot A_1 + L|\Delta \xi(\xi, \eta)|.
\end{align*}$$

Observe that, by (2.27),

$$\begin{align*}
\Delta w_\epsilon = \int_{\epsilon}^{1} \left[ f_1(\xi, \beta, \tau_k(\xi, \beta)) - f_1(\xi, \beta, \tau_k(\xi, \beta)) + \\
+ f_1(\xi, \beta, \tau_k(\xi, \beta)) \tau_{11}(\xi, \beta) - f_1(\xi, \beta, \tau_k(\xi, \beta)) \tau_{11}(\xi, \beta) + \\
+ f_1(\xi, \beta, \tau_k(\xi, \beta)) \tau_{11}(\xi, \beta) - f_1(\xi, \beta, \tau_k(\xi, \beta)) \tau_{11}(\xi, \beta) \right] d\beta.
\end{align*}$$
Adding (3.19)-(3.22) and integrating from 0 to \(\eta\):

\[
(3.23) \quad |\Delta w_{4s}(\xi, \eta)| = \left| \int_0^\eta \{ F_{1s}(\xi, \beta) - F_{2s}(\xi, \beta) \} \, d\beta \right| <
\]

\[
< H(1 + A_1 + M_2 + M) \int_0^\eta |\Delta z(\xi, \beta)| \, d\beta +
\]

\[
+ L \int_0^\eta (|\Delta z_{2s}(\xi, \beta)| + |\Delta z_{3s}(\xi, \beta)| + |\Delta z_{4s}(\xi, \beta)|) \, d\beta .
\]

We can now multiply by \( \exp \left[ -\varrho \eta \right] \) and evaluate the norm:

\[
(3.24) \quad |\Delta w_{4s}| < (H(1 + A_1 + M_2 + M) |\Delta z_{1s}| +
\]

\[
+ \left( \frac{L}{\varrho(\varrho)} \right) (|\Delta z_{2s}| + |\Delta z_{3s}| + |\Delta z_{4s}|) .
\]

\( e) \) The last term to take into account is

\[
(3.25) \quad |\Delta w_{3s}(\xi, \eta)| = |F_3(\xi, \eta) - F_2(\xi, \eta)| < L |\Delta z(\xi, \eta)| .
\]

Multiplying by \( \varrho \exp \left[ -\varrho \eta \right] \) and calculating the norm we have

\[
(3.26) \quad \varrho |\Delta w_{3s}| < \varrho L |\Delta z_{1s}| .
\]

\( f) \) Let us finally add (3.14), (3.16), (3.18), (3.24) and (3.26). We have by (2.8):

\[
(3.27) \quad \| \Delta w \| <
\]

\[
< (1/\varrho(\varrho)) \left\{ \frac{L(1 + A_1 + M_2 + M_2 + M)}{\varrho} + (M(\varrho) + 1) + H(1 + A_1 + M_2 + M) \right\} |\Delta z_{1s}| +
\]

\[
+ L(\varrho(\varrho)) \left( |\Delta z_{2s}| + |\Delta z_{3s}| + |\Delta z_{4s}| \right) =
\]

\[
= \left[ (C(\varrho) + L(\varrho(\varrho)) \right) |\Delta z_{1s}| + (L(\varrho) \left( |\Delta z_{2s}| + |\Delta z_{3s}| + |\Delta z_{4s}| \right) \right] < KL |\Delta z| ,
\]

where \( C \) denotes the constant in \( \{ \} \) and

\[
(3.28) \quad K = \max \left\{ \frac{(C(\varrho) + L(\varrho(\varrho))}{}, L(\varrho) \right\} .
\]

By (3.12) it is possible to choose \( \varrho \) so that \( L(\varrho(\varrho)) < 1 \); for \( \varrho \) large enough we shall finally obtain \( K < 1 \). Hence the contraction mapping theorem applies, and the existence and uniqueness of a fixed point \( z_1 \equiv A \) is proved. Let \( \Gamma_1 : \xi = \gamma_1(\eta) \) be the line which is implicitly defined by

\[
(3.29) \quad G_1(\xi, \eta) = A'(\xi) + \int_0^\xi f(\xi, \beta, y_1(\xi, \beta)) \, d\beta = 0 ,
\]
and let $Z_1$ be the domain given by setting $f_1$ in (2.15). It is immediate to verify that the pair $(\gamma_1, \gamma_1)$ gives a solution of the $II^*$ problem in $R$.

The uniqueness of the solution of the $II^*$ problem does not follow directly from the previous theorem. This solution, in fact, as defined in §1, is $C^1(Z)$, whereas $u(\xi, \eta)$ belongs to a smaller class. Thus we have to prove a regularity theorem, from which the uniqueness will follow at once, at least in a suitable rectangle $R$.

**LEMMA 2:** Let $y(\xi, \eta)$ and $\Gamma$ give a solution of the $II^*$ problem in a rectangle $R$, where (2.1), (2.4) and (2.5) hold (see fig. 2). There exists then a rectangle $R \subset R$ such that, letting $Z = R \cap \overline{Z}$, $y(\xi, \eta)$ and $\Gamma$ are of class $C^1$ in $Z$.

![Fig. 2.](image)

**PROOF:** After restricting, if needed, $R$ so that $y(\xi, \eta) \in S$, $\forall (\xi, \eta) \in Z$, what is possible because $y(0, 0) = 0$, we have:

\[(3.30) \quad y_2(\xi, \eta) = G(\xi, \eta) = A(\xi) + \int \frac{y}{\beta} f(\xi, \beta, y(\xi, \beta)) \, d\beta.\]

Thus we observe that, beyond $y_1, y_2$ and $y_{x\beta}$, also $y_{\beta\beta} = G_{\eta}(\xi, \eta) = f(\xi, \eta, y(\xi, \eta))$ is continuous in $Z$. Let us consider the application $\delta: C^0(Z) \to C^0(Z)$ given by setting $\nu = \delta \rho$ where

\[(3.31) \quad \nu(\xi, \eta) = A(\xi) + \int \frac{y}{\beta} f(\xi, \beta, y(\xi, \beta), \rho(\xi, \beta), y_{x}(\xi, \beta)) \, d\beta.\]

A classical argument shows that $\delta$ has a unique fixed point in $C^0(Z)$, which
agrees, by (3.30), with \( y_0(\xi, \eta) \). Let now construct the set \( A \) as in (2.6), observing that the inequalities

\[
\begin{align*}
\max_{\xi} |y(\xi, \eta)| &< A_0, \\
\max_{\xi} |y_0(\xi, \eta)| &< A_1, \\
\max_{\xi} |y_0(\xi, \eta)| &< M_1,
\end{align*}
\]

(3.32)

hold; the remaining constants are chosen as in Lemma 1. We have, in particular:

\[ -M < y_0(\xi, \eta) < 0. \]

(3.33)

Lemma 1 allows us to choose \( R \) so that the application \( \tau \) defined in (2.17) is \( A \rightarrow A \). Let \( Z = \mathcal{Z} \cap R \), and \( X_1 = C^4(Z) \), with the norm

\[ \|p\| = \|p\|_0 + |p_t|_0 + \sigma |p_{tt}|_0 \]

(3.34)

similar to (2.8), where

\[ [u]_0 = \max_{\xi} \{\exp [-q(\eta)] u(\xi, \eta)\} \]

(3.35)

A new admissible function set \( A' \) is now introduced:

\[ A' = \{p \in X_1 | p(\xi, 0) = \mathcal{A}'(\xi), \forall \xi \in [0, a]; |p(\xi, \eta)| < A_1; 0 < p_t(\xi, \eta) < M_2; -M < p_{tt}(\xi, \eta) < 0\} . \]

(3.36)

The same argument of steps \( a), b), c) \) of Lemma 1 (with \( p \) and \( \nu \) instead of \( \xi_t \) and \( u_t \) respectively) shows that \( \delta : A' \rightarrow A' \), and

\[ p_t(\xi, \eta) > \epsilon > 0. \]

(3.37)

We can now proceed as in steps \( b), d), e) \) of Theorem 1, taking into account that in (3.31) \( p \) appears only in the fourth argument of \( f \), while the third and the fifth one are fixed. We obtain thus the following simpler estimates:

\[ |A_{\xi}|_0 < (L/\eta)|A p|_0, \]

(3.38)

\[ |A_{tt}|_0 < (M/\eta)(1 + A_1 + M_2 + M)|A p|_0 + (L/\eta)|A p|_0, \]

(3.39)

\[ \sigma |A_{tt}|_0 < L \sigma |A p|_0. \]

(3.40)

Adding (3.38)-(3.40) we deduce that, for suitable \( \varrho \) and \( \sigma \), \( \delta \) is a contraction in the norm (3.34) too. Note that (3.12) is no longer required. Hence the
fixed point \( y \in A \subset C^1(Z) \), and we have, by (3.37):

\[
(3.41) \quad y_d(\xi, \eta) = G_d(\xi, \eta) > \epsilon > 0 \quad \text{in } Z.
\]

It is immediate now to verify that the line \( L: \xi = \gamma(\eta) \) (that is \( G(\xi, \eta) = 0 \)) is of class \( C^1 \), and that

\[
(3.42) \quad \gamma'(\eta) = -\frac{G_x(\gamma(\eta), \eta)}{G_{\xi}(\gamma(\eta), \eta)} > 0 \quad \text{in } [0, b'].
\]

Let us introduce the rectangle \( R' = [0, a] \times [0, b'] \), and divide \( Z \) into \( Z' \cup Z'' \) as in fig. 3. Note that the solution \( y \) is given, in \( Z' \), also by the formula

\[
(3.43) \quad y(\xi, \eta) = -\int_{\gamma} f(x, \beta, y(x, \beta)) \, d\Gamma,
\]

because it is a solution of a Cauchy problem too, with data vanishing on \( \Gamma \). In \( Z'' \) a simple Darboux problem must be solved. If \( (\xi, \eta) \in R' \setminus Z' \), the same formula (3.43) gives an integral equation which allows us to extend the function \( y(\xi, \eta) \) to the whole of \( R' \). By known results (see Cinquina Cibrario, Cinquina [4], pag. 161) we deduce the

**Theorem 3 (Regularity):** Every solution \( y \) of a \( \Pi^* \) problem, under the hypotheses (2.1), (2.4) and (2.5), can be extended to a function \( \tilde{y} \in C^1(R') \), satisfying in \( R' \) a Cauchy problem on \( \Gamma' \), with vanishing data.
The function
\begin{equation}
G(\xi, \eta) = A'(\xi) + \int_{\xi}^{\eta} f(\xi, \beta, \tilde{y}(\xi, \beta)) \, d\beta
\end{equation}
is now defined on the whole of $R'$, where it is of class $C^1$, and (3.43) may be rewritten as
\begin{equation}
\tilde{y}(\xi, \eta) = -\int_{\mu(\xi)}^{\gamma(\xi)} d\xi \left[ f(x, \beta, \tilde{y}(x, \beta)) \right] \, d\beta = \\
= \int_{\gamma(\mu)}^{\xi} \left[ A'(x) + \int_{\mu(\xi)}^{\gamma(\xi)} f(x, \beta, \tilde{y}(x, \beta)) \, d\beta \right] \, dx = \int_{\gamma(\mu)}^{\xi} G(x, \eta) \, dx,
\end{equation}
where $\eta = \varphi(\xi)$ is the inverse function of $\xi = \gamma(\eta)$.

Remark: If $b' < b$ the fixed point of $\tau$, given in Theorem 2, does not belong to $C^2(R)$. Indeed $y_{\bar{y}y}$ is not continuous at the point $(a, b')$, and consequently on the segment $\eta = b'$. This depends on the fact that the function $g(\eta)$ given by (2.20) is not of class $C^1$ in $[0, b']$; indeed $g$ represents $I'$ only in $[0, b']$. A further restriction on $b$ (as in the next section) would obviously eliminate such an unnatural feature.

Let us now finally prove the

Theorem 4 (Uniqueness): Given two solutions $y_1$ and $y_2$ of the II* problem, corresponding to the same datum $A(\xi)$, let $R'_1$ and $R'_2$ be the rectangles obtained in Theorem 3. Then $y_1 = y_2$ in $R' = R'_1 \cap R'_2$.

Proof: Suppose $\tilde{y}_1 \neq \tilde{y}_2$. This implies $\gamma_1(\eta) = \gamma_2(\eta)$, by the uniqueness of the solution of the Cauchy problem (3.43). Let
\begin{equation}
\begin{cases}
\bar{\eta} = \max \{ \eta | y_1(\beta) = y_2(\beta) \text{ in } [0, \eta] \} > 0, \\
\bar{\xi} = \gamma_1(\bar{\eta}) = \gamma_2(\bar{\eta}) .
\end{cases}
\end{equation}

In the rectangle $R_0 = \{(\xi, \eta) \in R^2 | \xi > \bar{\xi}, \eta < \bar{\eta}\}$ we have again $\tilde{y}_1 = \tilde{y}_2$, because they are both solutions of the same Darboux problem. In particular:
\begin{equation}
\tilde{y}_1(\xi, \eta) = \tilde{y}_2(\xi, \eta) = \overline{A}(\xi) \quad \text{for } \bar{\xi} < \xi < a',
\end{equation}
and
\begin{equation}
\overline{A}'(\xi) > 0 \quad \text{for } \bar{\xi} < \xi
\end{equation}
by (3.41). Setting
\begin{equation}
R = \{(\xi, \eta) \in R^2 | \xi > \bar{\xi}, \eta > \bar{\eta}\},
\end{equation}
let
\[ \overline{A}_0 = \max_{\tilde{\eta}} \max \{ \tilde{f}_{1}(\tilde{\xi}, \eta), \tilde{f}_{2}(\tilde{\xi}, \eta) \}, \]
\[ \overline{A}_1 = \max_{\tilde{\eta}} \max \{ \tilde{f}_{11}(\tilde{\xi}, \eta), \tilde{f}_{12}(\tilde{\xi}, \eta) \}, \]
\[ M_1 = \max_{\tilde{\eta}} \max \{ \tilde{f}_{11}(\tilde{\xi}, \eta), \tilde{f}_{12}(\tilde{\xi}, \eta) \}, \]
\[ \overline{M}_2 > \max_{\tilde{\eta}} \max \{ \tilde{f}_{111}(\tilde{\xi}, \eta), \tilde{f}_{112}(\tilde{\xi}, \eta) \}. \]

From these constants we can construct, as in Lemma 1, a set \( \tilde{\mathcal{A}} \) of admissible functions, and a corresponding rectangle \( R \). But \( \tilde{y}_1 \) and \( \tilde{y}_1 \) (restricted to \( R \)) are both fixed points of the transformation \( \tau \), so that they coincide on \( R \), against the hypothesis (3.46).

4. - The case \( f(0,0,0,0,0) = 0 \)

Let \( A(\tilde{\xi}) \) satisfy (2.1) \((\Rightarrow \text{(2.2)})\). Set
\[ (4.1) \quad B_0 = 2A_0, \quad B_1 = 2A_1, \quad M_1 > 0; \]
\[ (4.2) \quad S' = [0, B_0] \times [-B_1, B_1] \times [-M_1, M_1], \quad \tilde{E}' = \tilde{R} \times S'; \]
and assume:
\[ (4.3) \quad f \in C^1(\tilde{E}), \quad f(0,0,0,0,0) = 0, \]
\[ (4.4) \quad f_\varepsilon, f_\delta, f_* \text{ Lipschitz continuous in } \tilde{E} \text{ with respect to } \varepsilon. \]

We shall define a new set of admissible functions
\[ (4.5) \quad \mathcal{B} = \{ \xi = \xi(\tilde{\xi}, \eta) | \xi \in C^2(\tilde{R}), \xi(\tilde{\xi}, 0) = A(\tilde{\xi}), \forall \tilde{\xi} \in [0, a], \]
\[ \xi_0(0,0) = \xi_0(0,0) = \xi_0(0,0) = 0, \quad 0 < \xi(\tilde{\xi}, \eta) < B_\varepsilon, \]
\[ |\xi(\tilde{\xi}, \eta)| < B_1, \quad |\xi_\varepsilon(\xi, \eta)| < M_1, \quad 0 < |\xi(\tilde{\xi}, \eta)| < M_2, \]
\[ |\xi_\varepsilon(\xi, \eta)| < M, \quad |\xi_\varepsilon(\xi, \eta)| < M_3. \]

\( \mathcal{B} \) is a closed subset of \( \mathcal{X} \) and \( C^2 \). We give the norm in \( C^2 \) (which is consistent with the norm in \( \mathcal{X} \))
\[ (4.6) \quad \|u\|_2 = \|u\|_1 + \|u_\varepsilon\|_0 + \sigma \|u_\varepsilon\|_1 + \sigma \|u_\varepsilon\|_0 = \|u\| + \sigma \|u_\varepsilon\|_0, \]
where the norms in the right hand side are given in (2.8), (2.9), (2.10). We
suppose also that

\begin{equation}
F(\xi, \eta) = f(\xi, \eta, z(\xi, \eta))
\end{equation}

is strictly decreasing in \( \eta \) \( \forall \xi \in [0, a] \) and \( \forall \eta \in \mathcal{B} \).

It is obvious that this hypothesis is satisfied, at least for a rectangle \( R \) small enough, if

\begin{equation}
f_a(0, 0, 0, 0, 0) < 0
\end{equation}

and \( f_a \) satisfies a Lipschitz condition with respect to all its variables.

We have in fact, \( \forall \eta \in \mathcal{B} \)

\[
\frac{\partial}{\partial \eta} f(\xi, \eta, z(\xi, \eta), z_\eta(\xi, \eta), z_\xi(\xi, \eta)) |_{(\xi, \eta, 0, 0, 0)} = f_a(0, 0, 0, 0, 0) = d < 0.
\]

Let \( K \) the greatest of the Lipschitz constants of \( f_a \); by a classical argument we obtain

\[
\left| \frac{\partial}{\partial \eta} f(\xi, \eta, z(\xi, \eta), z_\eta(\xi, \eta), z_\xi(\xi, \eta)) - f_a(0, 0, 0, 0, 0) \right| < \frac{\partial}{\partial \eta} f(\xi, \eta, z(\xi, \eta), z_\eta(\xi, \eta), z_\xi(\xi, \eta)) < K(a + b + B_1 + B_4 + M_4) + L(M_1 + M + M_3)
\]

and therefore

\begin{equation}
\frac{\partial}{\partial \eta} f(\xi, \eta, z(\xi, \eta), z_\eta(\xi, \eta), z_\xi(\xi, \eta)) < d + K(a + b + B_1 + B_4 + M_4) + L(M_1 + M + M_3).
\end{equation}

But \( z(0, 0) = z_\xi(0, 0) = z_\eta(0, 0) = z_\xi(0, 0) = z_\eta(0, 0) = 0 \) and we can choose \( R \) as small as we want; so it is always possible to reduce \( B_4, B_1, M_4, M, M_3 \) so that the right hand side of (4.9) becomes negative.

Hypothesis (4.7) allows us to define the application \( \tau \) given in (2.17). Now it is possible to prove the following Lemma (similar to Lemma 1):

**Lemma 3:** Under the hypotheses (2.1), (4.3), (4.4), (4.7) it is possible to choose \( a, b, M_2, M, M_3 \) such that

\[ \tau: \mathcal{B} \rightarrow \mathcal{B}. \]

**Proof:** Let \( M = \text{Max} |f(\xi, \eta, z, p, q)|, \) and \( L_1, L_2, L_3, L_4, L_5 \) be the Lipschitz constants of \( f \) with respect to the five variables. We choose \( M_2 > A_2, 0 < c < a_2, M_3 > M_2 \varepsilon \) and we set

\begin{align}
K_1 &= L_1 + L_3 B_1 + L_4 M_2 + L_4 M, \\
K_2 &= L_2 + L_3 M_1 + L_4 M + L_4 M_2.
\end{align}
(4.12) \[ a = \min \{ \varepsilon, M/4, (M_2 - M^2)/K_2 \} \]
(4.13) \[ b = \min \{ \varepsilon, (a_2 - \varepsilon)/K_1, (M_2 - a_2)/K_1, A_0/\alpha M, A_1/M, \alpha M \} \]

We obtain in \( R \) (as in Lemma 1) \( \varepsilon < \psi(\xi, \eta) < M_2 \). In fact it suffices to put \( K_1 \) instead of \( K \) in (2.28). Let us now consider \( G(\xi, \eta) \). We have, by (4.7):

\[
G(0, \eta) = \int_{\xi}^{\eta} f(0, \beta, z(0, \beta)) \, d\beta < 0, \quad 0 < \eta < b,
\]
and

\[
G(\xi, 0) = A'(\xi) > 0, \quad 0 < \xi < a,
\]

\[
G(\xi, 0) > \varepsilon > 0 \quad \text{as in \S 2}. 
\]

The equation \( G(\xi, \eta) = 0 \) defines implicitly a line \( \Gamma: \xi = \gamma(\eta) \), with

\[
\gamma \in C^1([0, b']), \quad \gamma'(\eta) = -f(\gamma(\eta), \eta, z(\gamma(\eta), \eta))/G_1(\gamma(\eta), \eta).
\]

We note that \( \gamma'(\eta) > 0, \forall \eta > 0 \). We want to show that \( \gamma'(\eta) < 0 \) is inconsistent with the definition of \( \Gamma \); in fact under this hypothesis we have

\[
f(\gamma(\tilde{\eta}), \tilde{\eta}, z(\gamma(\tilde{\eta}), \tilde{\eta})) > 0
\]

and therefore, by (4.7),

\[
f(\gamma(\tilde{\eta}), \eta, z(\gamma(\tilde{\eta}), \eta)) > 0 \quad \forall \eta < \tilde{\eta}
\]

and

\[
G(\gamma(\tilde{\eta}), \tilde{\eta}) = A'(\gamma(\tilde{\eta})) + \int_{\xi}^{\tilde{\eta}} f(\gamma(\tilde{\eta}), \beta, z(\gamma(\tilde{\eta}), \beta)) \, d\beta > 0,
\]

which is absurd. We have moreover

(4.14) \[ 0 < \gamma'(\eta) < M/\varepsilon \]

from which it follows, by the last of (4.13),

(4.15) \[ Mb/\varepsilon < a. \]

We obtain so \( b = b' \), \( \gamma \in C^1([0, b']) \) and (see (2.19)) \( v_n \in C^1(R) \). We can verify, as in Lemma 1, that

(4.16) \[ |v_1(\xi, \eta)| < A'(\xi) + \int_{\xi}^{\eta} f(\xi, \beta, z(\xi, \beta)) \, d\beta < A_1 + bM < B_1 \]
and, with a straightforward computation,

\[ |w_0(\xi, \eta)| < M_1, \]
\[ |w_{2i}(\xi, \eta)| < M, \]
\[ 0 < w(\xi, \eta) < B_0. \]

Lastly we prove the estimate of \( w_{0i} \), which is given by the formula (see (2.19))

\[
(4.17) \quad w_{0i}(\xi, \eta) = \frac{\partial}{\partial \eta} \int_{x(\xi)}^{y(\xi)} f(x, \eta, z(x, \eta)) \, dx =
\]
\[
= \int_{x(\xi)}^{y(\xi)} (f_1 + f_2 \xi + f_3 \eta \xi + f_4 \xi \eta \xi) \, dx - f(y(\eta), \eta, z(y(\eta), \eta)) \eta'(\eta).
\]

We have thus

\[
(4.18) \quad |w_{0i}(\xi, \eta)| < aK_2 + M^2/c < M_3.
\]

We can state now the following

**Corollary 2:** Let \( \zeta_1, \zeta_2 \in \mathcal{B} \) and \( g_1(\eta), g_2(\eta) \) the corresponding lines defined in (2.20). Under the hypotheses (2.1), (4.3), (4.4), (4.7) we have \( \| \Delta g \|_0 < L \| \eta \| \| \Delta \zeta_1 \|_0 \).

It corresponds closely to Corollary 1, and can be proved identically.

We shall now repeat, with few modifications, the argument of § 3. Let us now prove

**Theorem 5 (Existence):** Let \( R = [0, a] \times [0, b] \) and assume that \( a, b \) are given by (4.12), (4.13); let moreover \( a \) satisfy (3.12), where \( L = \max \{ L_2, L_4, L_6 \} \).

Under the hypotheses (2.1), (4.3), (4.4), (4.7), the II* problem has at least one solution in \( R \).

**Proof:** \( \tau \) maps \( \mathcal{B} \) into itself as a contraction in the norm (4.6), for \( \eta \) and \( \sigma \) suitably chosen. We observe that (3.14), (3.16), (3.18) and (3.26) are true in \( \mathcal{B} \). Let \( E' = R \times S' \), with \( S' \) defined in (4.2), and let \( H \) be the greatest of the Lipschitz constants of all the first derivatives of \( f(\xi, \eta, \xi, \rho, \eta) \) with respect to \( (\xi, \rho, \eta) \) in \( E' \). We have, as in (3.24),

\[
(4.19) \quad \| \Delta w | \|_0 < H(1 + B_1 + M_2 + M) \| \eta \| \| \Delta \zeta | \|_0 + L \| \eta \| \| \Delta \zeta | \|_1 + \\
+ L \| \eta \| \| \Delta \zeta | \|_0 + L \| \eta \| \| \Delta \zeta | \|_n.
\]
We have to prove now the estimate of $|\Delta w_{xx}|_\delta$; $\Delta w_{xx}$ is given by

$$
(4.20) \quad \Delta w_{xx}(x, \eta) = \frac{\partial}{\partial x} \int_{\gamma_{\delta}(x)}^{\gamma_{\delta}(x)} \Delta F(x, \eta) \, dx + \frac{\partial}{\partial y} \int_{\gamma_{\delta}(y)}^{\gamma_{\delta}(y)} \frac{\partial F}{\partial x}(x, \eta) \, dx = 
$$

$$
= \int_{\gamma_{\delta}(x)}^{\gamma_{\delta}(x)} \Delta F(x, \eta) \, dx - \Delta F(\gamma_{\delta}(x), \eta) \gamma_{\delta}'(x) + \int_{\gamma_{\delta}(y)}^{\gamma_{\delta}(y)} \frac{\partial F}{\partial x}(x, \eta) \, dx + 
$$

$$
+ F(\gamma_{\delta}(x), \eta) \gamma_{\delta}'(x) - F(\gamma_{\delta}(y), \eta) \gamma_{\delta}'(y).
$$

The estimate of $\Delta F_{x}$ is analogous to the previous one for $\Delta F_{y}$ in (3.19)-(3.22), bearing in mind that in $R$ it is:

$$
|F_{14}(x, \eta)| < L_1 + L(A_1 + M_2 + M),
$$

$$
|F_{15}(x, \eta)| < L_2 + L(M_1 + M_2 + M),
$$

and that $|\gamma_{\delta}'(\eta)| < M/\delta$ by (4.14). It follows then that

$$
(4.21) \quad |\Delta w_{xx}(x, \eta)| \leq \int_{0}^{\delta} \left\{ H(1 + M_1 + M + M_2) |\Delta x(x, \eta)| + 
$$

$$
+ L[|\Delta z_{x}(x, \eta)| + |\Delta z_{y}(x, \eta)| + |\Delta z_{yx}(x, \eta)|] \, dx + L(M/c) |\Delta y(\eta, \eta, \eta)| + 
$$

$$
+ (L_2 + L(M_1 + M + M_2) + M(L_1 + L(A_1 + M_2 + M)) |\gamma_{\delta}'(\eta)|) \right\} \, dx.
$$

Denoting by $D$ the last constant in $\{}$, and keeping the norms, we obtain thus:

$$
(4.22) \quad \sigma |\Delta w_{xx}| \leq \sigma \{ H(1 + M_1 + M + M_2) + L \} |\Delta x| + 
$$

$$
+ L \sigma |\Delta z_{x}| + L \sigma |\Delta z_{y}| + L \sigma |\Delta z_{yx}| + D M/c |\Delta y| \leq C |\Delta x|,
$$

We add (3.14), (3.16), (3.18), (3.26), (4.19) and (4.22) and we obtain, for $\sigma$ small and $\delta$ large enough:

$$
(4.23) \quad \|\Delta w\|_{2} < C \|\Delta x\|_{2}, \quad \text{with} \ C < 1,
$$

and the thesis follows. In order to prove uniqueness and regularity of the solution of the $II^*$ problem (at least in a suitable rectangle $R$) we must prove some theorems analogous of Lemma 2, also under the different hypotheses made on $f$. We obtain, in fact:

**Lemma 4:** Let $y$, $\Gamma$ be a solution of the $II^*$ problem in $\overline{R}$, where (2.1), (4.3), (4.4), (4.7) hold; there exists then a rectangle $R \subset \overline{R}$ such that, letting $Z = \overline{Z} \cap R$, $y_{\delta}(x, \eta)$ and $\Gamma$ are of class $C^1$ in $Z$ and $y'(\eta) > 0$, for $\eta > 0$. 

Then we can extend uniquely \( y \) in \( R' \) using (3.45), that is:

\[
\tilde{y}(\xi, \eta) = \int_{\gamma(\eta)}^\xi \tilde{G}(x, \eta) \, dx.
\]

\( I' \) is however tangent to the characteristic \( \xi = 0 \) in \((0, 0)\), so the classical results on the solution of a Cauchy problem are not sufficient to guarantee that \( \tilde{y} \in C^2(R') \). We must prove therefore the property directly, by means of the techniques used in Lemma 2 for the continuity of \( y_{12} \).

**Theorem 6:** Every solution \( y \) of \( II^* \), under the hypotheses (2.1), (4.3), (4.4), (4.7), can be extended to a function \( \tilde{y} \in C^2(R') \), defined in (4.24), satisfying in \( R' \) a Cauchy problem with vanishing data.

**Proof:** We already verified that \( y_{12} \in C^0(Z) \); the same proof works in order to get that \( y_{12} \in C^0(R') \). We show also that \( y_{40} \in C^0(R') \).

Let

\[
B_0 = \max_{R} |\tilde{y}(\xi, \eta)|, \quad B_1 = \max_{R} |\tilde{y}_1(\xi, \eta)|, \quad M_1 = \max_{R} |\tilde{y}_2(\xi, \eta)|
\]

and \( s' \) given by (4.2), \( M = \max_{R \times s'} |f(\xi, \eta, \xi, \eta, \xi, \eta)| \),

\[
B = \max_{R' \times s'} \{ |g(\xi, \eta)| < M_1, \ C |\tilde{y}_1(\xi, \eta) < M, \ C |\tilde{y}_2(\xi, \eta) < M_3 \}.
\]

We introduce in \( C^1(R') \) the norm

\[
\|g\| = |\xi| + |\eta| + \sigma |q_1| + \sigma |q_2|
\]

with

\[
|\mu| = \max_{R} \{ \exp[-q|\xi - \gamma(\eta)|] \} |\tilde{y}(\xi, \eta)|
\]

Let \( \omega : \mathcal{B} \to C^1(R') \) be the map defined by \( \omega \tilde{y} = \tilde{v} \), with

\[
\tilde{v}(\xi, \eta) = \int_{\gamma(\eta)}^\xi \tilde{f}(x, \eta, \tilde{y}(x, \eta), \tilde{y}_1(x, \eta), \mu(x, \eta)) \, dx.
\]

Since \( R, R' \) are constructed by Lemma 3, we have immediately

\[
\omega : \mathcal{B} \to \mathcal{B}'.
\]

We obtain also, using (3.18), (4.19), (4.20), and noting that only the last
argumen is varying,

\begin{align}
(4.30) \quad |\Delta v(\xi, \eta)| &< L_2 \left| \int_{\gamma(\phi)} |\Delta u(x, \eta)| \, dx \right|, \\
(4.31) \quad |\Delta v_1(\xi, \eta)| &< L_2 |\Delta u(\xi, \eta)|, \\
(4.32) \quad |\Delta v_2(\xi, \eta)| &< H(1 + M_1 + M + M_3) \left| \int_{\gamma(\phi)} |\Delta u(x, \eta)| \, dx \right| + \\
&\quad + L_1 \left| \int_{\gamma(\phi)} |\Delta u_1(x, \eta)| \, dx \right| + L_2 |\Delta u(\gamma, \eta)| |\gamma'(\eta)|,
\end{align}

then, multiplying by the exponential weight all the last three formulas and multiplying also (4.31), (4.32) by \( \sigma \), we have, by addition:

\begin{align}
(4.33) \quad \| \Delta v \|_{\gamma} &< (L_2 + M H(1 + M_1 + M + M_3)) |\gamma| + L_1 |\gamma| + L_2 M |\gamma| \sigma |\Delta u|_{\gamma} + \\
&\quad + L_2 M \sigma |\gamma| |\Delta u|_{\gamma} + L_2 \sigma |\gamma| |\Delta u|_{\gamma}.
\end{align}

Therefore \( \omega \) is a contraction too, for a suitable choice of \( \rho, \sigma \), and we prove that \( \gamma_1 \), unique fixed point of \( \omega \), belongs to \( C^1 \). We have lastly the uniqueness theorem

**Theorem 7**: Let \( \gamma_1 \) and \( \gamma_2 \) be solutions of the \( \Pi \) problem. There exists then a rectangle \( R' \) such that \( \gamma_1(\xi, \eta) = \gamma_2(\xi, \eta) \) in \( R' \).

The proof is the same of Theorem 4.

5. - Support Domains

\textbf{a)} Let us now consider the problem (1.2)-(1.5), with

\begin{align}
(5.1) \quad A(\xi) &\in C^2([0, \overline{a}]), \quad B(\eta) \in C^2([0, \overline{b}]), \quad A(0) = B(0), \\
(5.2) \quad f &\in C^{1,1}(\overline{R} \times S),
\end{align}

where \( S \) is a parallelepiped in \( \mathbb{R}_+ \times \mathbb{R}^2 \) as in (2.3). A necessary condition for the existence of a support domain is

\begin{align}
(5.3) \quad A(0) = A'(0) = B(0) = B'(0) = 0.
\end{align}

While it is easy to verify (see [3]) that, if (5.3) does not hold, the solution of Darboux problem with data \( A(\xi) \) and \( B(\eta) \) satisfies, at least locally, the unilateral condition (1.3), and is strictly positive in an open rectangle \( (0, \overline{a}) \times \)}
\( \times (0, b') \). Suppose therefore that (5.3), (2.4), (2.5) hold, and moreover that

\[
A'(0) > 0, \quad B'(0) > 0.
\]

It is then possible to solve the \( \Pi^*_1 \) problem with respect to the datum \( A(\xi) \), and the analogous \( \Pi^*_2 \) one relative to \( B(\eta) \), and obtain two lines

\[
\begin{align*}
I_1: & \quad \xi = \gamma_1(\eta) \iff \eta = \varphi_1(\xi), \\
I_2: & \quad \eta = \gamma_2(\xi).
\end{align*}
\]

If there exists an interval \([0, \bar{\xi}]\) such that

\[
\varphi_1(\xi) < \gamma_2(\xi) \quad \text{for} \quad 0 < \xi < \bar{\xi},
\]

then the function

\[
y(\xi, \eta) = \begin{cases} 
\gamma_1(\xi, \eta) & \text{in} \quad Z_1 = \{0 < \xi < \bar{\xi}, 0 < \eta < \varphi_1(\xi)\}, \\
0 & \text{in} \quad \Delta = \{0 < \xi < \bar{\xi}, \varphi_1(\xi) < \eta < \gamma_2(\xi)\}, \\
\gamma_2(\xi, \eta) & \text{in} \quad Z_2 = \{0 < \xi < \bar{\xi}, \gamma_2(\xi) < \eta < b_2\},
\end{cases}
\]

satisfies the unilateral problem, and admits \( \Delta \) as support domain. (5.7) gives obviously an extension law for the unilateral problem. Since

\[
\gamma'_1(0) = -f(0, 0, 0, 0, 0)/A'(0), \quad \gamma'_2(0) = -f(0, 0, 0, 0, 0)/B'(0),
\]

the condition

\[
A'(0)B'(0) < f^2(0, 0, 0, 0, 0)
\]

implies (5.6). If, on the other hand,

\[
A'(0)B'(0) > f^2(0, 0, 0, 0, 0)
\]

the solution of the free Darboux problem is again strictly positive in \((0, a') \times \times (0, b')\), and no support domain exists. Moreover, if, instead of (5.4),

\[
A'(0) > 0, \quad B(\eta) = 0
\]

holds, we get the simpler solution

\[
y(\xi, \eta) = \begin{cases} 
\gamma_1(\xi, \eta) & \text{in} \quad Z_1, \\
0 & \text{in} \quad \Delta = \mathcal{R} \setminus Z_1,
\end{cases}
\]

(an analogous case holds if \( A(\xi) = 0 \)).
b) Let us now turn to the case considered in § 4. It is \( f = 0 \) at the origin. The hypothesis (5.4) again implies existence and uniqueness of the solution of the \( H^1 \) and \( H^2 \) problems; (5.6), however, can no longer hold. In fact, \( I' \), is necessarily tangent, at the origin, to the \( \eta \)-axis, and \( I'_\xi \) to the \( \xi \)-axis, so that no support domain exists. Suppose, on the other hand, that (5.11) holds. Then (5.12) gives a solution of the support problem. The case \( B'(0) = 0, B'(\eta) \neq 0 \) for the quasi-linear equation is still open; for the linear one see Amerio [3].

REFERENCES