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Homogenization of Some Almost Periodic Coercive Functional (**)

Omogeneizzazione di alcuni funzionali coercivi quasi-periodici

Riassunto. — Si dimostra un teorema di omogeneizzazione per integrali $\int_\Omega f(x, Du(x))\,dx$, con Ω aperto limitato di $\mathbb{R}^n$ e $u \in H^{1,p}(\Omega; R^N)$. f è una funzione di Caratheodory che dipende in modo quasi-periodico dalla prima variabile e soddisfa alle condizioni di crescita:

$$|\xi|^p \leq f(x, \xi) \leq C(1 + |\xi|)^p .$$

1. - Introduction

In the last years, much work has been done about Γ-convergence and homogenization of functionals depending on scalar valued functions (see for example [6], [12], [5]), or on vector valued functions ([2], [10], [14]). These papers deal with functionals of the type

$$\int_\Omega f\left(\frac{x}{\varepsilon}, Du(x)\right) dx, \tag{1.1}$$

where Ω is an open bounded subset of R^n, u belongs to $H^{1,p}(\Omega; R^N)$ and f is a Caratheodory function *periodic in the first variable*.

Under suitable growth conditions on f, it is possible to prove the existence of the limit

$$\Gamma(L^p(\Omega; R^N)^-) \lim_{\varepsilon \to 0} \int_\Omega f\left(\frac{x}{\varepsilon}, Du(x)\right) dx = \int_\Omega \varphi(Du(x))\, dx\,, \tag{1.2}$$

for every $u \in H^{1,p}(\Omega; R^N)$, and to give an asymptotic formula for the function φ.

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We shall consider functionals of the form (1.1) with *almost periodic* dependence on the first variable of f. We shall prove an homogenization theorem and give an asymptotic formula for the limit, as has already been done when f is a quadratic form in the second variable (see [11], [16]).

The main result of this paper is the following.

HOMOGENIZATION THEOREM: *Let* $f\colon R^n \times R^{nN} \to [0, +\infty[$ *be a Caratheodory function, quasiconvex in the second variable (see Definition* 4.1), *satisfying the growth condition*

$$|\xi|^p \le f(x, \xi) \le c(1 + |\xi|^p) \tag{1.3}$$

for every $\xi \in R^{nN}$ *and a.a.* $x \in R^n$, *and* p*-almost periodic in the first variable (see Definition* 3.1). *Then for every open bounded subset* Ω *of* R^n, *and every* $u \in H^{1,p}(\Omega; R^N)$ *there exists the limit*

$$\Gamma(L^p(\Omega; R^N)^-) \lim_{\varepsilon \to 0} \int_\Omega f\left(\frac{x}{\varepsilon}, Du(x)\right) dx = \int_\Omega \varphi(Du(x))\, dx\,, \tag{1.4}$$

and φ *is a quasiconvex function, given by the formula*

$$\varphi(\xi) = \lim_{T \to +\infty} \min\left\{\frac{1}{T^n}\int_{[0,T]^n} f(x, Du(x) + \xi)\, dx\,;\ u \in H_0^{1,p}([0, T]^n; R^N)\right\}. \tag{1.5}$$

2. - Γ-CONVERGENCE

Let us recall the definitions of Γ-convergence for functionals defined on a topological space, with values in $\overline{R}$ (as in [7]).

2.1. DEFINITION: Let $I \subseteq \overline{R}$, X a topological space, $E \subseteq X$, $\overline{E}$ the closure of E in X, $(F_i)_{i\in I}$ a family of functionals, each defined on E, with values in $\overline{R}$, and $s \in \overline{I}$. For every $x \in \overline{E}$, we define

$$\Gamma(X^-) \liminf_{i \to s} F_i(x) = \sup_{U \in \mathfrak{I}_X(x)} \liminf_{i \to s} \inf_{y \in U \cap E} F_i(y)\,, \tag{2.1}$$

$$\Gamma(X^-) \limsup_{i \to s} F_i(x) = \sup_{U \in \mathfrak{I}_X(x)} \limsup_{i \to s} \inf_{y \in U \cap E} F_i(y) \tag{2.2}$$

(where $\mathfrak{I}_X(x)$ denotes the family of all neighbourhoods of x in X).

If at a point $x \in \overline{E}$,

$$\Gamma(X^-) \liminf_{i \to s} F_i(x) = \Gamma(X^-) \limsup_{i \to s} F_i(x)\,,$$

the common value will be indicated by

$$\Gamma(X^-) \lim_{i \to s} F_i(X)\,.$$

If the limit $\Gamma(X^-)\lim_{i\to t} F_i(x)$ exists for every $x \in E$, we will say that the functionals F_i $\Gamma(X^-)$-converge as $i \to t$ to the functional $\Gamma(X^-)\lim_{i\to t} F_i$:

2.2. Remark (see Prop. 3.3 of [8]): If X is metric, $I = \mathrm{N}$ and $x \in E$, we have $\lambda = \Gamma(X^-)\lim_{h\to+\infty} F_h(x)$ if and only if:

i) for every sequence $(x_h)_{h\in\mathrm{N}}$ in E converging to x, we have

$$\lambda \leq \liminf_{h\to+\infty} F_h(x_h);$$

ii) there exists a sequence $(x_h)_{h\in\mathrm{N}}$ in E converging to x, such that

$$\lambda \geq \limsup_{h\to+\infty} F_h(x_h).$$

2.3. Remark: If X is metric, $I = R_+$ and $x \in E$, we have

$$\lambda = \Gamma(X^-)\lim_{\varepsilon\to 0} F_\varepsilon(x)$$

if and only if for every sequence $(\varepsilon_h)_{h\in\mathrm{N}}$ of positive real numbers converging to 0, there exists a subsequence $(\varepsilon_{h_k})_{k\in\mathrm{N}}$ such that

$$\lambda = \Gamma(X^-)\lim_{k\to+\infty} F_{\varepsilon_{h_k}}(x).$$

3. - Almost periodicity

We shall consider functions with an almost periodic dependence on one variable. Let us recall the usual definitions of almost periodic functions (see [3], [4]).

3.1. Definition: A function $f\colon R^n \to R$ is *uniformly almost periodic* if it is the uniform limit in R^n of a sequence $(p_h)_{h\in\mathrm{N}}$ of trygonometric polynomials, i.e. functions of the type

$$p_h(x) = \sum_{k=1}^{m_h} a_k \exp[i\lambda_k \cdot x],$$

with $\lambda_1, \ldots, \lambda_{m_h} \in R^n$.

We can give another definition which includes also all periodic functions.

3.2. Definition: A function $f\colon R^n \to R$ is *almost periodic* if to every $\varepsilon > 0$, there corresponds an inclusion length $L_\varepsilon > 0$, such that for every $a \in R^n$, there exists $\tau \in a + [0, L_\varepsilon]^n$ such that for a.a. $x \in R^n$

$$|f(x+\tau) - f(x)| < \varepsilon.$$

3.3. Remark: If f is a continuous function verifying Definition 3.2, then it verifies also Definition 3.1 (see for example [3], pag. 76 and [15]). Conversely, if f is uniformly almost periodic, then it is also almost periodic ([3], pag. 6, and [15]). That is, uniformly almost periodic functions are all the continuous almost periodic functions.

Here we deal with functions $f(x, \xi)$ almost periodic in x, with a kind of uniformity with respect to ξ. Functions of this type, indexed by R^n, have been studied by Fink ([9], Chapter 2, § 7). More precisely, we given the following definition.

3.4. Definition: A function $f\colon R^n \times R^{nN} \to [0, +\infty[$ is called *p-almost periodic* in the first variable, if to every $\varepsilon > 0$ corresponds an inclusion length $L_\varepsilon > 0$ with the following property: for every $a \in R^n$, $\tau \in a + [0, L_\varepsilon]^n$ exists, such that for a.a. $x \in R^n$

$$|f(x, \xi) - f(x + \tau, \xi)| < \varepsilon(1 + |\xi|^p) \tag{3.1}$$

for every $\xi \in R^{nN}$.

The τ's which satisfy (3.1) will be called the ε-quasi periods of f.

A simple example of a p-almost periodic function is

$$f(x, \xi) = (2 + \cos x)|\xi|^p + (1 + \operatorname{sen}\sqrt{2}x)|\xi|^2,$$

with $p > 2$ (see [3], Theorem 5, pag. 5).

3.5. Remark: If f_1, f_2 are p-almost periodic functions such that

$$\frac{f_1(x, \xi)}{1 + |\xi|^p}, \frac{f_2(x, \xi)}{1 + |\xi|^p}$$

are continuous in x, uniformly with respect to ξ, then $f_1 + f_2$ is almost periodic (see [9], pag. 17).

4. - Notations and preliminary results

If A is a Lebesgue measurable subset of R^n, meas (A) will be its Lebesgue measure.

If $0 < \text{meas}(A) < +\infty$, and $f\colon A \to [0, +\infty]$, the number

$$\fint_A f(x)\, dx = \frac{1}{\text{meas}(A)} \int_A f(x)\, dx$$

will be the *mean* of f on A.

Ap_n will be the set of all open bounded subsets of R^n.

4.1. Definition (see Morrey [13]): A continuous function $f\colon R^{nN} \to R$ is *quasiconvex* if for every $\xi \in R^{nN}$, for any open subset of R^n, Ω, and any $u \in C_0^1(\Omega; R^N)$

$$f(\xi) \le \int_\Omega f(\xi + Du(x))\, dx . \tag{4.1}$$

4.2. Remark: If f satisfies the growth condition

$$|\xi|^p \le f(\xi) \le c(1 + |\xi|^p) ,$$

and is quasiconvex, then (4.1) holds for every $\xi \in R^{nN}$, any open subset Ω of R^n, and $u \in H_0^{1,p}(\Omega; R^N)$.

4.3. The class $\mathcal{F}$: In all that follows n, N, c, p are fixed; $n, N \in \mathbf{N}$, $c, p \in$ $\in [1, +\infty[$. We will say that a functional F defined on the pairs (u, Ω), where $\Omega \in Ap_n$ and $u \in H^{1,p}(\Omega; R^N)$, belongs to the class $\mathcal{F} = \mathcal{F}(c, p, n, N)$ if there exists a Caratheodory function $f\colon R^n \times R^{nN} \to [0, +\infty[$ such that

$$\xi \mapsto f(x, \xi) \text{ is quasiconvex for every } x \in R^n ; \tag{4.2}$$

$$|\xi|^p \le f(x, \xi) \le c(1 + |\xi|^p) \tag{4.3}$$

for every $\xi \in R^{nN}$ and a.a. $x \in R^n$;

$$F(u, \Omega) = \int_\Omega f(x, Du(x))\, dx \tag{4.4}$$

for every $\Omega \in Ap_n$ and $u \in H^{1,p}(\Omega, R^N)$.

The class $\mathcal{F}$, equipped with the structure of the $\Gamma(L^{p-})$ convergence is a compact space, in the sense explained by the following theorem.

4.4. Theorem ([10] Theorem 2.4): *If $(F_h)_{h\in\mathbf{N}}$ is a sequence of functionals of the class $\mathcal{F}$, then there exists a subsequence $(F_{h_k})_{k\in\mathbf{N}}$ of $(F_h)_{h\in\mathbf{N}}$ and a functional $F_\infty \in \mathcal{F}$ such that*

$$F_\infty(u, \Omega) = \Gamma(L^p(\Omega; R^N)^-) \lim_{k\to+\infty} F_{h_k}(u, \Omega)$$

for every $\Omega \in Ap_n$ and $u \in H^{1,p}(\Omega; R^N)$.

In the next paragraph we shall use the following result (see for example Proposition III.5 in [1]).

4.5. Proposition: *If $F_h \in \mathcal{F}$ for $h \in \mathbf{N}$ and, for some $\Omega \in Ap_n$, $u \in H^{1,p}(\Omega; R^N)$*

$$F_\infty(u, \Omega) = \Gamma(L^p(\Omega; R^N)^-) \lim_{h\to+\infty} F_h(u, \Omega) ,$$

then there exists a sequence $(u_h)_{h\in N}$ *in* $H_0^{1,p}(\Omega, R^N)$ *converging to* 0 *in* $L^p(\Omega; R^N)$ *such that*

$$F_\infty(u, \Omega) = \lim_{h\to+\infty} F_h(u + u_h, \Omega).$$

5. - Homogenization

We can pass now to the proof of the Homogenization Theorem. We consider, for every positive ε, the functional F_ε defined by

$$F_\varepsilon(u, \Omega) = \int_\Omega f\left(\frac{x}{\varepsilon}, Du(x)\right) dx \tag{5.1}$$

for every $\Omega \in Ap_n$, $u \in H^{1,p}(\Omega; R^N)$, where $f(x, \xi)$ is a fixed Caratheodory function satisfying (4.2), (4.3) and p-almost periodic in x.

By Theorem 4.4, for every sequence of positive numbers $(\varepsilon_h)_{h\in N}$ converging to 0, there exists a subsequence $(\varepsilon_{h_i})_{i\in N}$ of $(\varepsilon_h)_{h\in N}$ and a Caratheodory function φ satisfying (4.2), (4.3), such that the limit

$$\Gamma(L^p(\Omega; R^N)^-) \lim_{i\to+\infty} F_{\varepsilon_{h_i}}(u, \Omega) = \int_\Omega \varphi(x, Du(x))\, dx \tag{5.2}$$

exists for every $\Omega \in Ap_n$ and $u \in H^{1,p}(\Omega; R^N)$. In order to prove the existence of the limit $\Gamma(L^p(\Omega; R^N)^-) \lim_{\varepsilon\to 0} F_\varepsilon$, for every $\Omega \in Ap_n$, it is sufficient to show that the function φ in (5.2) does not depend on the particular (sub)sequence, so that each Γ-convergent (sub)sequence has the same limit (Remark 2.3).

We fix, from now on, a sequence $(\varepsilon_{h_i})_{i\in N}$ satisfying (5.2).

5.1. Proposition: *The function* $\varphi(x, \xi)$ *in* (6.2) *can be chosen independent of* x.

Proof: Let us fix $x_0, y_0 \in R^n$, $r > 0$, $K \in N$, $\xi \in R^{nN}$. Let B be the open ball of center x_0 and radius r, B_K the open ball of center x_0 and radius $r(1 - 1/K)$. Given $\varepsilon > 0$, it is possible to find a sequence $(\tau_i)_{i\in N}$ of ε-quasi periods such that, set $y_i = x_0 + \varepsilon_{h_i}\tau_i$, we have $\lim_{i\to+\infty} y_i = y_0$.

Let $(u_i)_{i\in N}$ be a sequence in $H_0^{1,p}(B, R^N)$ converging to 0 in $L^p(B, R^N)$, such that

$$\int_B \varphi(x, \xi)\, dx = \lim_{i\to+\infty} \int_B f\left(\frac{x}{\varepsilon_{h_i}}, Du_i(x) + \xi\right) dx.$$

If i is large enough, $y_0 + B_K \subset y_i + B$, and then we obtain

$$\int_B \varphi(x, \xi)\, dx \geq \liminf_{i\to+\infty} \int_B f\left(\frac{x}{\varepsilon_{h_i}} + \tau_i, Du_i(x) + \xi\right) dx -$$

$$- \varepsilon \limsup_{i\to+\infty} \int_B (1 + |Du_i(x) + \xi|^p)\, dx = \liminf_{i\to+\infty} \int_{y_i+B} f\left(\frac{x}{\varepsilon_{h_i}}, Du_i(x + x_0 - y_i) + \xi\right) dx -$$

$$-\varepsilon \limsup_{i\to+\infty} \int_B (1+|Du_i+\xi|^p)\,dx \geq \liminf_{i\to+\infty} \int_{x_0+B_0} f\left(\frac{x}{\varepsilon_{h_i}}, Du_i(x+x_0-y_i)+\xi\right)dx -$$

$$-\varepsilon \limsup \int_B (1+|Du_i+\xi|^p)\,dx \geq \int_{x_0+B_0} \varphi(x,\xi)\,dx - \varepsilon \limsup_{i\to+\infty} \int_B (1+|Du_i+\xi|^p)\,dx .$$

As $(u_i)_{i\in N}$ is bounded in $H^{1,p}(B; R^N)$ and ε is arbitrary, we have

$$\int_B \varphi(x,\xi)\,dx \geq \int_{x_0+B_0} \varphi(x,\xi)\,dx .$$

If we let K go to $+\infty$ and we use Beppo Levi's theorem, we obtain

$$\int_B \varphi(x,\xi)\,dx \geq \int_{x_0+B} \varphi(x,\xi)\,dx$$

and then, by symmetry, the equality; so

$$\int_B \varphi(x,\xi)\,dx = \int_B \varphi(x+y_0,\xi)\,dx .$$

By the arbitrarity of r, x_0, y_0 and ξ, we have proven the proposition.

5.2. Definition: For every $t>0$, $\xi \in R^{nN}$, we define (see also [12], [5])

$$(5.3) \qquad g_t(\xi) = \min\Big\{ \int_{[0,t]^n} f(x, Du(x)+\xi)\,dx : u \in H_0^{1,p}([0,t]^n; R^N) \Big\}.$$

We give now the fundamental lemma for the proof of the Homogenization Theorem.

5.3. Lemma: *The limit* $\lim_{t\to+\infty} g_t(\xi)$ *exists for every* $\xi \in R^{nN}$.

Proof: Let $t>0$ and $u_t \in H_0^{1,p}([0,t]^n, R^N)$ such that

$$\int_{[0,t]^n} f(x, Du_t(x)+\xi)\,dx = g_t(\xi) .$$

Let $\varepsilon>0$ and $L_\varepsilon>0$ the inclusion length of f related to ε. If $s>t+L_\varepsilon$, we can construct $u \in H_0^{1,p}([0,s]^n; R^N)$ in the following way: for every n-tuple of integers $(i) \in \{1, ..., [s/(t+L)]\}^n$ let $\tau_{(i)}$ be an ε-quasi period of f, with $\tau_{(i)} \in$ $\in (t+L_\varepsilon)(i) + [0, L_\varepsilon]^n$; set

$$u_s(x) = \begin{cases} u_t(x-\tau_{(i)}) & \text{if } x \in \tau_{(i)} + [0,t]^n , \\ 0 & \text{otherwise} . \end{cases}$$

Using u_s, we can give an estimate of $g_s(\xi)$.

$$g_s(\xi) < \fint_{[0,s[^n} f(x, Du_s(x)+\xi)\,dx = \frac{1}{s^n}\Big(\sum_{(i)} \int_{\tau_{(i)}+[0,t[^n} f(x, Du_t(x-\tau_{(i)})+\xi)\,dx +$$

$$+ \int_{[0,s[^n \setminus \bigcup_{(i)} (\tau_{(i)}+[0,t[^n)} f(x,\xi)\,dx\Big) < \frac{1}{s^n}\Big(\sum_{(i)} \int_{[0,t[^n} \big(f(x+\tau_i, Du_t(x)+\xi) - f(x, Du_t(x)+\xi)\big)\,dx +$$

$$+ \sum_{(i)} \int_{[0,t[^n} f(x, Du_t(x)+\xi)\,dx + c(1+|\xi|^p)\Big(s^n - t^n\Big(\frac{s}{t+L_\varepsilon}-1\Big)^n\Big)\Big) <$$

$$< \frac{t^n}{s^n}\Big[\frac{s}{t+L_\varepsilon}\Big]^n\Big(\varepsilon\frac{1}{t^n}\int_{[0,t[^n}(1+|Du_t(x)+\xi|^p)\,dx + g_t(\xi)\Big) +$$

$$+ c(1+|\xi|^p)\Big(1-\Big(\frac{t}{t+L_\varepsilon}-\frac{t}{s}\Big)^n\Big) < \varepsilon\frac{1}{t^n}\int_{[0,t[^n}(1+f(x, Du_t(x)+\xi))\,dx +$$

$$+ g_t(\xi) + c(1+|\xi|^p)\Big(1-\Big(\frac{t}{t+L_\varepsilon}-\frac{t}{s}\Big)^n\Big) =$$

$$= g_t(\xi)(1+\varepsilon) + \varepsilon + c(1+|\xi|^p)\Big(1-\Big(\frac{t}{t+L_\varepsilon}-\frac{t}{s}\Big)^n\Big).$$

Now, taking the limit first in s and then in t, we get

$$\limsup_{s\to+\infty} g_s(\xi) < (1+\varepsilon)\Big(\liminf_{t\to+\infty} g_t(\xi)+\varepsilon\Big) + 2\varepsilon.$$

As ε can be chosen arbitrary, we have proven the lemma.

5.4. Proposition: *For every $\xi \in R^{nN}$, we have $\lim_{t\to+\infty} g_t(\xi) = \varphi(\xi)$.*

Proof: By Proposition 4.5, there exists a sequence $(u_k)_{k\in N}$ in $H_0^{1,p}((0,1)^n; R^N)$ converging to 0 in $L^p((0,1)^n; R^N)$, such that

$$\varphi(\xi) = \lim_{i\to+\infty} \int_{[0,1]^n} f\Big(\frac{x}{\varepsilon_{h_i}}, Du_k(x)+\xi\Big)\,dx =$$

$$= \lim_{i\to+\infty} (\varepsilon_{h_i})^n \int_{[-1/\varepsilon_{h_i}, 1/\varepsilon_{h_i}]^n} \big(f(x, Du_i(\varepsilon_{h_i}x)+\xi)\big)\,dx > \lim_{i\to+\infty} g_{1/\varepsilon_{h_i}}(\xi);$$

so $\lim_{t\to+\infty} g_t(\xi) < \varphi(\xi)$.

Let, for every $i \in N$, $u_i \in H_0^{1,p}((0,1)^n; R^n)$ such that

$$g_{1/\varepsilon_{h_i}}(\xi) = \int_{[0,1]^n} f\Big(\frac{x}{\varepsilon_{h_i}}, Du_i(x)+\xi\Big)\,dx.$$

The sequence $(u_i)_{i\in N}$ is weakly relatively compact in $H_0^{1,p}((0,1)^n; R^n)$; so we can suppose that it converges weakly to u_∞ in $H_0^{1,p}((0,1)^n; R^n)$.

The quasiconvexity of φ assures that

$$\varphi(\xi) = \int_{[0,1]^n} \varphi(\xi)\, dx = \min\left\{ \int_{[0,1]^n} \varphi(\xi + Du(x))\, dx : u \in H_0^{1,p}((0,1)^n; R^m)\right\};$$

so that

$$\varphi(\xi) < \int_{[0,1]^n} \varphi(Du_\infty(x) + \xi)\, dx < \liminf_{i\to+\infty} \int_{[0,1]^n} f\left(\frac{x}{\varepsilon_{h_i}}, Du_i(x) + \xi\right) dx = \lim_{i\to+\infty} g_i(\xi).$$

Proposition 5.4 shows that φ depends only on f, and not on the particular sequence $(\varepsilon_{h_i})_{i\in N}$. Therefore the Homogenization Theorem is proven.

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BIBLIOGRAPHY

[1] E. ACERBI - G. BUTTAZZO, *On the limit of periodic Riemannian metrics*, to appear on J. Analyse Math.

[2] E. ACERBI - N. FUSCO, *Semicontinuity problems in the calculus of variations*, Arch. Rational Mech. Anal., 86 (1984), 125-145.

[3] A. BESICOVITCH, *Almost Periodic Functions*, Cambridge, 1932.

[4] H. BOHR, *Collected Mathematical Works*, Vol. II and III, Copenhagen, 1952.

[5] A. BRAIDES, *Omogeneizzazione di integrali non coercivi*, Ricerche di Matematica, 32 (1983).

[6] G. BUTTAZZO - G. DAL MASO, *Γ-limits of integral functionals*, J. Analyse Math., 37 (1980), 145-185.

[7] E. DE GIORGI, *G-operators and Γ-convergence*, to appear on the proceedings of the Internati nal Meeting of Math., Warsaw, 1983.

[8] E. DE GIORGI - T. FRANZONI, *Su un tipo di convergenza variazionale*, Rend. Sem. Mat. Brescia, 3 (1979), 63-101.

[9] A. M. FINK, *Almost periodic differential equations*, Lecture Note in Mathematics n. 377, Springer-Verlag, 1974.

[10] N. FUSCO, *On the convergence of integral functionals depending on vector-valued functions*, preprint n. 102, Univ. Napoli, 1983.

[11] S. KOZLOV, *Averaging differential operators with almost periodic rapidly oscillating coefficients*, Math. USSR Sbornik, 35 (1979), 481-498.

[12] P. MARCELLINI, *Periodic solutions and homogenization of nonlinear variational problems*, Ann. Mat. Pura Appl., (4) 117 (1978), 139-152.

[13] C. MORREY, *Quasi-convexity and the semicontinuity of multiple integrals*, Pacific J. Math., 2 (1952), 23-53.

[14] G. MOSCARIELLO - L. NANIA, *On the homogenization of multiple integrals*, to appear on J. Nonlinear, Anal.

[15] M. A. SUBIN: *Differential and pseudodifferential operators in spaces of almost periodic functions*, Mat. Sb. 95 (1974), 560-587.

[16] V. ZHIKOV - S. KOZLOV - O. OLEINIK - KHA T'EN NGOAN, *Averaging and G-convergence of differential operators*, Russian Math. Surveys, 34 (1979), 69-147.