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Homogenization of Some Almost Periodic Coercive Functional (**) 

Omgogeneizzazione di alcuni funzionali coercivi quasi-periodici 

RIASSUNTO. — Si dimostra un teorema di omogeneizzazione per integrali \( \int_{\Omega} f(x, Du(x)) \, dx \), con \( \Omega \) aperto limitato di \( \mathbb{R}^n \) e \( u \in H^{1,p}(\Omega; \mathbb{R}^n) \), \( f \) è una funzione di Caratheodory che dipende in modo quasi-periodico dalla prima variabile e soddisfa alle condizioni di crescita:

\[
|\xi|^p \leq f(x, \xi) \leq C(1 + |\xi|)^p.
\]

1. - INTRODUCTION

In the last years, much work has been done about \( I \)-convergence and homogenization of functionals depending on scalar valued functions (see for example [6], [12], [5]), or on vector valued functions ([2], [10], [14]). These papers deal with functionals of the type

\[
\int_{\Omega} f \left( \frac{x}{\epsilon}, Du(x) \right) \, dx,
\]

where \( \Omega \) is an open bounded subset of \( \mathbb{R}^n \), \( u \) belongs to \( H^{1,p}(\Omega; \mathbb{R}^n) \) and \( f \) is a Caratheodory function periodic in the first variable.

Under suitable growth conditions on \( f \), it is possible to prove the existence of the limit

\[
I( [L(\Omega; \mathbb{R}^n)]^\ast ) \lim_{\epsilon \to 0} \int_{\Omega} f \left( \frac{x}{\epsilon}, Du(x) \right) \, dx = \int_{\Omega} \psi(Du(x)) \, dx,
\]

for every \( u \in H^{1,p}(\Omega; \mathbb{R}^n) \), and to give an asymptotic formula for the function \( \psi \).

(*) Indirizzo dell'A.: Scuola Normale Superiore, 56100 Pisa.
(**) Memoria presentata il 14 dicembre 1984 da Il. De Giorgi, uno dei XI.

ISSN-0392-4106
We shall consider functionals of the form (1.1) with almost periodic dependence on the first variable of $f$. We shall prove an homogenization theorem and give an asymptotic formula for the limit, as has already been done when $f$ is a quadratic form in the second variable (see [11], [16]).

The main result of this paper is the following.

**Homogenization Theorem:** Let $f: \mathbb{R}^n \times \mathbb{R}^{n^2} \to [0, +\infty]$ be a Carathéodory function, quasiconvex in the second variable (see Definition 4.1), satisfying the growth condition

$$|\xi|^p < f(x, \xi) < c(1 + |\xi|^p)$$

for every $\xi \in \mathbb{R}^{n^2}$ and a.e. $x \in \mathbb{R}^n$, and $p$-almost periodic in the first variable (see Definition 3.1). Then for every open bounded subset $\Omega$ of $\mathbb{R}^n$, and every $u \in \mathcal{H}^{1,p}(\Omega; \mathbb{R}^{n^2})$ there exists the limit

$$\Gamma(L^p(\Omega; \mathbb{R}^{n^2})) \lim_{T \to \infty} \int_{\Omega} f \left( \frac{x}{T}, D_w(x) \right) dx = \int_{\Omega} \psi(D_w(x)) dx,$$

and $\psi$ is a quasiconvex function, given by the formula

$$\psi(\xi) = \lim_{T \to \infty} \min \left\{ \frac{1}{T^2} \int_{(0, T)} f \left( \frac{x}{T}, D_w(x) + \xi \right) dx; u \in \mathcal{H}^{1,p}(\{0, T\}^n; \mathbb{R}^{n^2}) \right\}.$$

2. **$\Gamma$-Convergence**

Let us recall the definitions of $\Gamma$-convergence for functionals defined on a topological space, with values in $\mathbb{R}$ (as in [7]).

2.1. **Definition:** Let $I \subseteq \mathbb{R}$, $X$ a topological space, $E \subseteq X$, $\bar{E}$ the closure of $E$ in $X$, $(F_i)_{i \in I}$ a family of functionals, each defined on $E$, with values in $\mathbb{R}$, and $i \in I$. For every $x \in E$, we define

$$\Gamma(X^-) \liminf_{i \to \infty} F_i(x) = \sup_{i \in \mathbb{R}} \inf_{y \in \mathcal{J}_x(y)} \liminf_{i \to \infty} F_i(y),$$

$$\Gamma(X^-) \limsup_{i \to \infty} F_i(x) = \sup_{i \in \mathbb{R}} \sup_{y \in \mathcal{J}_x(y)} \limsup_{i \to \infty} F_i(y)$$

(where $\mathcal{J}_x(x)$ denotes the family of all neighbourhoods of $x$ in $X$).

If at a point $x \in \bar{E}$,

$$\Gamma(X^-) \liminf_{i \to \infty} F_i(x) = \Gamma(X^-) \limsup_{i \to \infty} F_i(x),$$

the common value will be indicated by

$$\Gamma(X^-) \lim_{i \to \infty} F_i(x).$$
If the limit \( \Gamma(X^n) \lim_{i \to \infty} F_i(\infty) \) exists for every \( \infty \in E \), we will say that the functionals \( F_i \), \( \Gamma(X^n) \)-converge as \( i \to \infty \) to the functional \( \Gamma(X^n) \lim_{i \to \infty} F_i \):

2.2. REMARK (see Prop. 3.3 of [8]): If \( X \) is metric, \( I = N \) and \( \infty \in E \), we have \( \lambda = \Gamma(X^n) \lim_{h \to \infty} F_h(\infty) \) if and only if:

i) for every sequence \((x_h)_{h \in N}\) in \( E \) converging to \( \infty \), we have
\[
\lambda < \lim \inf_{h \to \infty} F_h(x_h);
\]

ii) there exists a sequence \((x_h)_{h \in N}\) in \( E \) converging to \( \infty \), such that
\[
\lambda > \lim \sup_{h \to \infty} F_h(x_h);
\]

2.3. REMARK: If \( X \) is metric, \( I = R_+ \) and \( \infty \in E \), we have
\[
\lambda = \Gamma(X^n) \lim_{i \to \infty} F_i(\infty)
\]
if and only if for every sequence \((\epsilon_h)_{h \in N}\) of positive real numbers converging to \( 0 \), there exists a subsequence \((\epsilon_{h_i})_{i \in N}\) such that
\[
\lambda = \Gamma(X^n) \lim_{i \to \infty} F_{\epsilon_{h_i}}(\infty).
\]

3. - ASYMPTOTIC PERIODICITY

We shall consider functions with an almost periodic dependence on one variable. Let us recall the usual definitions of almost periodic functions (see [3], [4]).

3.1. DEFINITION: A function \( f: R^n \to R \) is uniformly almost periodic if it is the uniform limit in \( R^n \) of a sequence \((p_h)_{h \in N}\) of trigonometric polynomials, i.e. functions of the type
\[
p_h(x) = \sum_{k=1}^{n} a_k \exp \{ i \lambda_k \cdot x \},
\]
with \( \lambda_1, \ldots, \lambda_n \in R^n \).

We can give another definition which includes also all periodic functions.

3.2. DEFINITION: A function \( f: R^n \to R \) is almost periodic if to every \( \epsilon > 0 \), there corresponds an inclusion length \( L_\epsilon > 0 \), such that for every \( a \in R^n \), there exists \( \tau \in [0, L_\epsilon]^n \) such that for a.a. \( x \in R^n \)
\[
|f(x + \tau) - f(x)| < \epsilon.
\]
3.3. Remark: If \( f \) is a continuous function verifying Definition 3.2, then it verifies also Definition 3.1 (see for example [3], pag. 76 and [15]). Conversely, if \( f \) is uniformly almost periodic, then it is also almost periodic ([3], pag. 6, and [15]). That is, uniformly almost periodic functions are all the continuous almost periodic functions.

Here we deal with functions \( f(x, \xi) \) almost periodic in \( x \), with a kind of uniformity with respect to \( \xi \). Functions of this type, indexed by \( \mathbb{R}^{n} \), have been studied by Fink ([9], Chapter 2, § 7). More precisely, we given the following definition.

3.4. Definition: A function \( f: \mathbb{R}^{n} \times \mathbb{R}^{m} \to [0, + \infty] \) is called \( p \)-almost periodic in the first variable, if for every \( \epsilon > 0 \) corresponds an inclusion length \( L_{\epsilon} > 0 \) with the following property: for every \( \alpha \in \mathbb{R}^{n}, \tau \in \alpha + [0, L_{\epsilon}]^{n} \) exists, such that for a.a. \( x \in \mathbb{R}^{n} \)

\[
(f(x, \xi) - f(x + \tau, \xi)) < \epsilon (1 + |\xi|^p)
\]

for every \( \xi \in \mathbb{R}^{m} \).

The \( \tau \)'s which satisfy (3.1) will be called the \( \epsilon \)-quasi periods of \( f \).

A simple example of a \( p \)-almost periodic function is

\[
f(x, \xi) = (2 + \cos \omega|\xi|^p + (1 + \sqrt{2}\omega|\xi|^p),
\]

with \( p > 2 \) (see [3], Theorem 5, pag. 5).

3.5. Remark: If \( f_1, f_2 \) are \( p \)-almost periodic functions such that

\[
\frac{f_1(x, \xi)}{1 + |\xi|^p}, \frac{f_2(x, \xi)}{1 + |\xi|^p}
\]

are continuous in \( x \), uniformly with respect to \( \xi \), then \( f_1 + f_2 \) is almost periodic (see [9], pag. 17).

4. - Notations and preliminary results

If \( A \) is a Lebesgue measurable subset of \( \mathbb{R}^{n} \), \( \text{meas} (A) \) will be its Lebesgue measure.

If \( 0 < \text{meas} (A) < + \infty \), and \( f: A \to [0, + \infty] \), the number

\[
\frac{1}{\text{meas} (A)} \int_A f(x) \, dx
\]

will be the \textit{mean} of \( f \) on \( A \).

\( \mathcal{A} \) will be the set of all open bounded subsets of \( \mathbb{R}^{n} \).
4.1. Definition (see Morrey [13]): A continuous function $f: \mathbb{R}^{n*} \to \mathbb{R}$ is quasiconvex if for every $\xi \in \mathbb{R}^{n*}$, for any open subset of $\mathbb{R}^n$, $\Omega$, and any $u \in C^2(\Omega; \mathbb{R}^n)$

$$f(\xi) < \sup_{\mathcal{B}} f(\xi + D_u(\xi)) \, dx.$$  \hfill (4.1)

4.2. Remark: If $f$ satisfies the growth condition

$$|\xi|^p < f(\xi) < c(1 + |\xi|^p),$$

and is quasiconvex, then (4.1) holds for every $\xi \in \mathbb{R}^{n*}$, any open subset $\Omega$ of $\mathbb{R}^n$, and $u \in H^{1,p}(\Omega; \mathbb{R}^n)$.

4.3. The class $\mathcal{F}$: In all that follows, $n, N, \epsilon, p$ are fixed; $n, N \in \mathbb{N}, \epsilon, p \in [1, +\infty]$. We will say that a functional $F$ defined on the pairs $(u, \Omega)$, where $\Omega \in Ap_n$ and $u \in H^{1,p}(\Omega ; \mathbb{R}^n)$, belongs to the class $\mathcal{F} = \mathcal{F}(\epsilon, p, n, N)$ if there exists a Caratheodory function $f: \mathbb{R}^n \times \mathbb{R}^{n*} \to [0, +\infty]$ such that

$$\xi \mapsto f(x, \xi) \text{ is quasiconvex for every } x \in \mathbb{R}^n;$$

$$|\xi|^p < f(x, \xi) < c(1 + |\xi|^p)$$

for every $\xi \in \mathbb{R}^{n*}$ and a.a. $x \in \mathbb{R}^n$;

$$F(u, \Omega) = \int_{\Omega} f(x, D_u(x)) \, dx;$$

for every $\Omega \in Ap_n$ and $u \in H^{1,p}(\Omega, \mathbb{R}^n)$.

The class $\mathcal{F}$, equipped with the structure of the $\Gamma(L^{n*})$ convergence is a compact space, in the sense explained by the following theorem.

4.4. Theorem ([10] Theorem 2.4): If $(F_h)_{h \in \mathbb{N}}$ is a sequence of functionals of the class $\mathcal{F}$, then there exists a subsequence $(F_h)_{h \in \mathbb{N}}$ of $(F_h)_{h \in \mathbb{N}}$ and a functional $F_u \in \mathcal{F}$ such that

$$F_h(u, \Omega) = \Gamma(L^{n*}(\Omega ; \mathbb{R}^n)) \lim_{h \to +\infty} F_h(u, \Omega)$$

for every $\Omega \in Ap_n$ and $u \in H^{1,p}(\Omega ; \mathbb{R}^n)$.

In the next paragraph we shall use the following result (see for example Proposition III.5 in [11]).

4.5. Proposition: If $F_h \in \mathcal{F}$ for $h \in \mathbb{N}$ and, for some $\Omega \in Ap_n$, $u \in H^{1,p}(\Omega ; \mathbb{R}^n)$

$$F_h(u, \Omega) = \Gamma(L^{n*}(\Omega ; \mathbb{R}^n)) \lim_{h \to +\infty} F_h(u, \Omega),$$
then there exists a sequence \((u_k)_{k \in \mathbb{N}}\) in \(H_0^{1,p}(\Omega, \mathbb{R}^n)\) converging to 0 in \(L^p(\Omega; \mathbb{R}^n)\) such that
\[
F_\varepsilon(u, \Omega) = \lim_{\varepsilon \to 0^+} F_\varepsilon(u + u_k, \Omega).
\]

5. - Homogenization

We can pass now to the proof of the Homogenization Theorem. We consider, for every positive \(\varepsilon\), the functional \(F_\varepsilon\) defined by
\[
F_\varepsilon(u, \Omega) = \int_\Omega f\left(\frac{x}{\varepsilon}, D u(x)\right) \, dx
\]
for every \(\Omega \in \mathcal{A}_p, u \in H^{1,p}(\Omega; \mathbb{R}^n)\), where \(f(x, \xi)\) is a fixed Caratheodory function satisfying (4.2), (4.3) and \(p\)-almost periodic in \(\xi\).

By Theorem 4.4, for every sequence of positive numbers \((\varepsilon_k)_{k \in \mathbb{N}}\) converging to 0, there exists a subsequence \((\varepsilon_{k_n})_{n \in \mathbb{N}}\) of \((\varepsilon_k)_{k \in \mathbb{N}}\) and a Caratheodory function \(\varphi\) satisfying (4.2), (4.3), such that the limit
\[
\lim_{\varepsilon_0 \to 0^+} \int_\Omega \varphi(\frac{x}{\varepsilon_0}, D u(x)) \, dx
\]
exists for every \(\Omega \in \mathcal{A}_p\) and \(u \in H^{1,p}(\Omega; \mathbb{R}^n)\). In order to prove the existence of the limit \(\Gamma^{L^p(\Omega; \mathbb{R}^n)}\) \lim \(\varepsilon_0 \to 0^+\) \(F_\varepsilon\), for every \(\Omega \in \mathcal{A}_p\), it is sufficient to show that the function \(\varphi\) in (5.2) does not depend on the particular (sub)sequence, so that each \(\Gamma\)-convergent (sub)sequence has the same limit (Remark 2.3).

We fix, from now on, a sequence \((\varepsilon_{k_n})_{n \in \mathbb{N}}\) satisfying (5.2).

5.1. PROPOSITION: The function \(\varphi(\xi, \xi)\) in (6.2) can be chosen independent of \(\xi\).

PROOF: Let us fix \(x_0, y_0 \in \mathbb{R}^n, r > 0, K \in \mathbb{N}, \xi \in \mathbb{R}^m\). Let \(B\) be the open ball of center \(x_0\) and radius \(r, B_k\) the open ball of center \(x_0\) and radius \(r(1 - 1/K)\). Given \(\varepsilon > 0\), it is possible to find a sequence \((\varepsilon_{k_n})_{n \in \mathbb{N}}\) of \(\varepsilon\)-quasi periods such that, set \(y_i = x_0 + \varepsilon_i \tau_i\), we have \(\lim_{i \to +\infty} y_i = y_0\).

Let \((u_k)_{k \in \mathbb{N}}\) be a sequence in \(H_0^{1,p}(B, \mathbb{R}^n)\) converging to 0 in \(L^p(B, \mathbb{R}^n)\), such that
\[
\int_B \varphi(\xi, \xi) \, dx = \lim_{\varepsilon_0 \to 0^+} \int_B f\left(\frac{x}{\varepsilon_0}, D u(x) + \xi\right) \, dx.
\]

If \(i\) is large enough, \(y_i + B_k \subseteq y_i + B\), and then we obtain
\[
\int_B \varphi(\xi, \xi) \, dx \geq \inf_{\varepsilon_0 \to 0^+} \int_B f\left(\frac{x}{\varepsilon_0} + \varepsilon_i, D u(x) + \xi\right) \, dx -
\leq \varepsilon \lim_{\varepsilon_0 \to 0^+} \int_B \left(1 + |D u(x) + \xi|^p\right) \, dx \leq \lim_{\varepsilon_0 \to 0^+} \int_B f\left(\frac{x}{\varepsilon_0}, D u(x + x_0 - y_i) + \xi\right) \, dx -
\]
\[ -e \limsup_{t \to +\infty} \int_B (1 + |D\upsilon_t + \xi|^p) \, dx \geq \liminf_{t \to +\infty} \int_B \left( \frac{N}{2} \int_{B_t} \frac{x_i - x_{i_0}}{r} \right) \, dx - e \limsup_{t \to +\infty} \int_B (1 + |D\upsilon_t + \xi|^p) \, dx. \]

As \((\upsilon_t)_{t \in \mathbb{N}}\) is bounded in \(H^{1,p}(B; \mathcal{R}^p)\) and \(e\) is arbitrary, we have

\[ \int_B \psi(x, \xi) \, dx \geq \int_{B_{t_0} + B_t} \psi(x, \xi) \, dx. \]

If we let \(K\) go to \(+\infty\) and we use Beppo Levi's theorem, we obtain

\[ \int_B \psi(x, \xi) \, dx \geq \int_{B_{t_0} + B} \psi(x, \xi) \, dx \]

and then, by symmetry, the equality; so

\[ \int_B \psi(x, \xi) \, dx = \int_{B_{t_0} + B} \psi(x, y_{t_0}, \xi) \, dx. \]

By the arbitrariness of \(r, x_{t_0}, y_{t_0}\) and \(\xi\), we have proven the proposition.

5.2. Definition: For every \(t > 0, \xi \in \mathcal{R}^n\), we define (see also [12], [5])

\begin{equation}
(5.3) \quad g_t(\xi) = \min \left\{ \int_{[0,t]^n} f(x, D\upsilon_t(x) + \xi) \, dx : \upsilon_t \in H^{1,p}_p([0,t]^n; \mathcal{R}^p) \right\}.
\end{equation}

We give now the fundamental lemma for the proof of the Homogenization Theorem.

5.3. Lemma: The limit \(\lim_{t \to +\infty} g_t(\xi)\) exists for every \(\xi \in \mathcal{R}^n\).

Proof: Let \(t > 0\) and \(\upsilon_t \in H^{1,p}_p([0,t]^n; \mathcal{R}^p)\) such that

\[ \int_{[0,t]^n} f(x, D\upsilon_t(x) + \xi) \, dx = g_t(\xi). \]

Let \(e > 0\) and \(L_0 > 0\) the inclusion length of \(f\) related to \(e\). If \(s > t + L_0\), we can construct \(\upsilon \in H^{1,p}_p([0,s]^n; \mathcal{R}^p)\) in the following way: for every \(n\)-tuple of integers \((i) \in \{1, \ldots, [s(t + L_0)]\}^n\) let \(\tau_{i} \in \mathcal{R}^n\) be an \(e\)-quasi period of \(f_i\) with \(\tau_{i} \in (t + L_0)(i) + [0, L_0]^n\); set

\[ u_i(x) = \begin{cases} 
\upsilon_i(x - \tau_i) & \text{if } x \in \tau_i + [0, t]^n, \\
0 & \text{otherwise}.
\end{cases} \]
Using \( u_n \), we can give an estimate of \( g_\varepsilon(\xi) \).

\[
g_\varepsilon(\xi) \leq \int_{\mathbb{R}^n} f(x, Du_n(x) + \varepsilon) \, dx + \frac{1}{r^n} \left( \sum_{i=0}^{n} \int_{10r_i^n} f(x, Du_n(x) + \varepsilon) \, dx \right) + \frac{1}{r^n} \left( \sum_{i=0}^{n} \int_{0,10r_i^n} f(x, Du_n(x) + \varepsilon) \, dx \right) \]

\[+ \sum_{i=0}^{n} \int_{10r_i^n} f(x, Du_n(x) + \varepsilon) \, dx + \varepsilon(1 + |\xi|^p) \left( 1 - \frac{r}{|\xi + L_n - i|} \right)^p \leq \frac{1}{r^n} \left( \int_{10r_i^n} f(x, Du_n(x) + \varepsilon) \, dx + g_\varepsilon(\xi) \right) \]

\[+ \varepsilon(1 + |\xi|^p) \left( 1 - \frac{r}{|\xi + L_n - i|} \right)^p \leq \frac{1}{r^n} \left( \int_{10r_i^n} f(x, Du_n(x) + \varepsilon) \, dx + g_\varepsilon(\xi) + \varepsilon(1 + |\xi|^p) \left( 1 - \frac{r}{|\xi + L_n - i|} \right)^p \right) \]

\[= g_\varepsilon(\xi)(1 + \varepsilon) + \varepsilon + \varepsilon(1 + |\xi|^p) \left( 1 - \frac{r}{|\xi + L_n - i|} \right)^p \]

Now, taking the limit first in \( \varepsilon \) and then in \( i \), we get

\[
\limsup_{\varepsilon \to 0} g_\varepsilon(\xi) \leq (1 + \varepsilon) \liminf_{i \to +\infty} g_\varepsilon(\xi) + \varepsilon + \varepsilon(1 + |\xi|^p) \left( 1 - \frac{r}{|\xi + L_n - i|} \right)^p.
\]

As \( \varepsilon \) can be chosen arbitrary, we have proven the lemma.

5.4. \textbf{Proposition:} \ For every \( \xi \in \mathbb{R}^n \), we have \( \lim g_\varepsilon(\xi) = q(\xi) \).

\textbf{Proof:} \ By Proposition 4.5, there exists a sequence \( (u_i)_i \) in \( H^{1,p}_\alpha((0,1)^n; \mathbb{R}^n) \) converging to 0 in \( L^p((0,1)^n; \mathbb{R}^n) \), such that

\[
q(\xi) = \lim_{i \to +\infty} \int_{(0,1)^n} f(\xi, Du_i(x) + \xi) \, dx = \lim_{i \to +\infty} \left( \varepsilon_i \right)^{1/p} \int_{\beta_i} (f(\xi, Du_i(x) + \xi) - f(\xi, Du_i(x)) \, dx \right) \]

\[\leq \lim_{i \to +\infty} \left( \varepsilon_i \right)^{1/p} \int_{\beta_i} f(\xi, Du_i(x) + \xi) \, dx = \lim_{i \to +\infty} g_{\beta_i}(\xi) \]

so \( \lim g_\varepsilon(\xi) \leq q(\xi) \).

Let, for every \( i \in \mathbb{N} \), \( u_i \in H^{1,p}_\alpha((0,1)^n; \mathbb{R}^n) \) such that

\[
\beta_i u_n(\xi) = \int_{(0,1)^n} f(\xi, Du_i(x) + \xi) \, dx.
\]

The sequence \( (u_i)_i \) is weakly relatively compact in \( H^{1,p}_\alpha((0,1)^n; \mathbb{R}^n) \); so we can suppose that it converges weakly to \( u_n \) in \( H^{1,p}_\alpha((0,1)^n; \mathbb{R}^n) \).
The quasiconvexity of $\varphi$ assures that
\[
\varphi(\varepsilon) = \int_{[0,1]^n} \varphi(\varepsilon) \, dx = \min \left\{ \int_{[0,1]^n} \varphi(\varepsilon + Du(x)) \, dx : u \in H_0^{1,n}((0, 1)^n ; R^n) \right\};
\]
so that
\[
\varphi(\varepsilon) \leq \int_{[0,1]^n} \varphi(Du(x) + \varepsilon) \, dx < \liminf_{k \to +\infty} \int_{[0,1]^n} f \left( \frac{x}{e_{nk}}, Du(x) + \varepsilon \right) \, dx = \lim_{k \to +\infty} g_k(\varepsilon),
\]
Proposition 5.4 shows that $\varphi$ depends only on $f$, and not on the particular sequence $(e_{nk})_{k \in \mathbb{N}}$. Therefore the Homogenization Theorem is proven.

**ACKNOWLEDGEMENTS:** I would like to thank Prof. Ennio de Giorgi for having suggested me this object of research and Giuseppe Buttazzo and Gianni Dal Maso for having helped me in writing this paper.

**BIBLIOGRAPHY**