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On the Weyl and Bochner curvature tensor (**)

SUMMARY. — We prove two main theorems, one on the Weyl conformal curvature tensor in a Riemannian manifold and the other on the Bochner curvature tensor in a Kaehlerian manifold. We then give some applications of these theorems. We especially prove a theorem which is a generalization of a theorem of Blair and which shows a close relation between the Weyl curvature tensor and the Bochner curvature tensor.

§ 1. INTRODUCTION AND PRELIMINARIES

Let M^n be an n-dimensional Riemannian manifold ($n \geq 3$) covered by a system of coordinate neighborhoods $\{U ; x^h\}$ and denote by g_{ji}, ∇_j, $K_{kji}{}^h$, K_{ji} and K the metric tensor, the operator of covariant differentiation with respect to g_{ji}, the curvature tensor, the Ricci tensor and the scalar curvature of M^n respectively, where here and in the sequel the indices $h, i, j, k, \cdots$ run over the range $\{1, 2, \cdots, n\}$.

It is well known (Weyl [16]) that the Weyl conformal curvature tensor defined by

$$C_{kji}{}^h = K_{kji}{}^h + \delta_k^h L_{ji} - \delta_j^h L_{ki} + L_k{}^h g_{ji} - L_j{}^h g_{ki}, \tag{1.1}$$

where

$$L_{ji} = -\frac{1}{n-2} K_{ji} + \frac{1}{2(n-1)(n-2)} K g_{ji}, \quad L_k{}^h = L_{kt} g^{th}, \tag{1.2}$$

g^{th} being contravariant components of the metric tensor, is invariant under a conformal change $g_{ji} \to \rho^2 g_{ji}$ of Riemannian metric, ρ being a positive scalar function, and that, for $n = 3$, $C_{kji}{}^h$ vanishes identically and

$$C_{kji} = \nabla_k L_{ji} - \nabla_j L_{ki} \tag{1.3}$$

is invariant under a conformal change of Riemannian metric.

It is also well known (Weyl [16]) that a necessary and sufficient condition for M_n to be conformally flat, that is, to be conformal to a Euclidean space, is that

$$C_{kji}{}^h = 0 \quad \text{for} \quad n > 3$$

and

$$C_{kji} = 0 \quad \text{for} \quad n = 3.$$

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Let M^p be a p-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{V ; y^a\}$ and immersed isometrically into M^n and denote by g_{cb}, ∇_c, $K_{dcb}{}^a$, K_{cb} and K' the metric tensor, the operator of covariant differentiation with respect to g_{cb}, the curvature tensor, the Ricci tensor and the scalar curvature of M^p respectively, where here and in the sequel the indices $a, b, c, \cdots$ run over the range $\{1', 2', \cdots, p'\}$.

We denote the immersion $M^p \to M^n$ by $x^h = x^h(y^a)$, put $B_b{}^h = \partial_b x^h$, where $\partial_b = \partial/\partial y^b$ and denote by $Cn_y{}^h$ $n-p$ mutually orthogonal unit vectors normal to M^p where here and in the sequel the indices x, y, z run over the range $\{(p+1)', \cdots, n'\}$.

If we define the so-called van der Waerden-Bortolotti covariant derivative of $B_b{}^h$ by

$$\text{(1.4)} \qquad \nabla_c B_b{}^h = \partial_c B_b{}^h + \left\{ {h \atop j\ i} \right\} B_c{}^j B_b{}^i - \left\{ {a \atop c\ b} \right\} B_a{}^h ,$$

$\left\{ {h \atop j\ i} \right\}$ and $\left\{ {a \atop c\ b} \right\}$ being Christoffel's symbols of M^n and M^p respectively, we can write equations of Gauss as

$$\text{(1.5)} \qquad \nabla_c B_b{}^h = h_{cb}{}^x C_x{}^h ,$$

where $h_{cb}{}^x$ are the second fundamental tensors of M^p with respect to normals $C_x{}^h$.

It is known (Yano [18]) that the tensor given by

$$\text{(1.6)} \qquad M_{cb}{}^h = h_{cb}{}^h - \frac{1}{p} g^{ed} h_{ed}{}^h g_{cb}$$

is invariant under a conformal change of Riemannian metric of the ambient manifold M^n.

If $h_{cb}{}^x$ vanishes identically the submanifold is said to be totally geodesic and if $M_{cb}{}^h$ vanishes identically the submanifold is said to be totally umbilical.

The relation between the covariant curvature tensors K_{dcba} of M^p and K_{kjih} of M^n is given by

$$\text{(1.7)} \qquad K_{dcba} = K_{kjih} B_d{}^k B_c{}^j B_b{}^i B_a{}^h + h_{dax} h_{cb}{}^x - h_{cax} h_{db}{}^x ,$$

where $h_{cbx} = h_{cb}{}^x$.

In 1948, Bochner [2] (see also Yano and Bochner [23]) proved

THEOREM A. *In a compact orientable conformally flat Riemannian manifold* M^n, $n \geq 4$, *if the Ricci form is positive definite, then we have* $b_t = 0$ $(t = 1, 2, \cdots, n-1)$, b_t *denoting the t-th Betti number of* M^n.

Now let M^n be a real n-dimensional Kaehlerian manifold $(n = 2m \geq 4)$ covered by a system of complex coordinate neighborhoods $\{U ; z^\alpha\}$, where here and in the sequel the indices $\alpha, \beta, \gamma, \cdots$ run over the range $\{1, 2, \cdots, m\}$

and $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \cdots$ the range $(\bar{1} = m+1, \bar{2} = m+2, \cdots, \bar{n} = 2m)$. Then the metric tensor g_{ji} has the components

$$[g_{ji}] = \begin{bmatrix} 0 & g_{\gamma\bar{\delta}} \\ g_{\bar{\gamma}\delta} & 0 \end{bmatrix} \tag{1.8}$$

and g^{ih} the components

$$[g^{ih}] = \begin{bmatrix} 0 & g^{\beta\bar{\alpha}} \\ g^{\bar{\beta}\alpha} & 0 \end{bmatrix}. \tag{1.9}$$

It is well known that there exists a tensor $F_i^{\ h}$ of type (1.1) having the numerical components

$$[F_i^{\ h}] = \begin{bmatrix} \sqrt{-1}\,\delta_\beta^\alpha & 0 \\ 0 & -\sqrt{-1}\,\delta_{\bar{\beta}}^{\bar{\alpha}} \end{bmatrix} \tag{1.10}$$

in any complex coordinate system and satisfying

$$F_j^{\ i} F_i^{\ h} = -\delta_j^h \quad , \quad F_j^{\ t} F_i^{\ s} g_{ts} = g_{ji} \tag{1.11}$$

and

$$\nabla_j F_i^{\ h} = 0 . \tag{1.12}$$

The tensors $F_{ji} = F_j^{\ t} g_{ti}$ and $F^{ih} = g^{it} F_t^{\ h}$ are both skew-symmetric and have the components

$$[F_{ji}] = \begin{bmatrix} 0 & \sqrt{-1}\,g_{\gamma\bar{\delta}} \\ -\sqrt{-1}\,g_{\bar{\gamma}\delta} & 0 \end{bmatrix} \quad \text{and} \quad [F^{ih}] = \begin{bmatrix} 0 & -\sqrt{-1}\,g^{\beta\bar{\alpha}} \\ \sqrt{-1}\,g^{\bar{\beta}\alpha} & 0 \end{bmatrix} \tag{1.13}$$

respectively.

As a formal analogue to the Weyl conformal curvature tensor, Bochner [3] (see also Yano and Bochner [23]) introduced the curvature tensor

$$\begin{aligned} B_{\delta\bar{\gamma}\beta\bar{\alpha}} = {} & K_{\delta\bar{\gamma}\beta\bar{\alpha}} \\ & - \frac{1}{m+2}\,(g_{\delta\bar{\alpha}} K_{\bar{\gamma}\beta} + g_{\delta\bar{\gamma}} K_{\bar{\alpha}\beta} + K_{\delta\bar{\alpha}} g_{\bar{\gamma}\beta} + K_{\delta\bar{\gamma}} g_{\bar{\alpha}\beta}) \\ & - \frac{1}{2(m+1)(m+2)}\,K\,(g_{\delta\bar{\alpha}} g_{\bar{\gamma}\beta} + g_{\delta\bar{\gamma}} g_{\bar{\alpha}\beta}) , \end{aligned} \tag{1.14}$$

which we now call the Bochner curvature tensor.

As a theorem which corresponds to Theorem A, Bochner [3] (see also Yano and Bochner [23]) proved

THEOREM B. *In a compact Kaehlerian manifold* M^n, $n = 2m \geq 4$, *in which the Bochner curvature tensor vanishes and the Ricci form is positive definite, we have* $b_{2l} = 1$, $b_{2l+1} = 0$ $(0 \leq 2l, 2l+1 \leq 2m)$.

Tachibana [13] gave the following tensor expression of the Bochner curvature tensor in a real coordinate system:

$$\begin{aligned} B_{kjih} = {} & K_{kjih} + g_{kh} L_{ji} - g_{jh} L_{ki} + L_{kh} g_{ji} - L_{jh} g_{ki} \\ & + F_{kh} M_{ji} - F_{jh} M_{ki} + M_{kh} F_{ji} - M_{jh} F_{ki} - 2 (F_{kj} M_{ih} + M_{kj} F_{ih}) , \end{aligned} \tag{1.15}$$

where

$$L_{ji} = -\frac{1}{n+4} K_{ji} + \frac{1}{2(n+2)(n+4)} K g_{ji} \quad , \quad L_i{}^h = L_{it} g^{th} ,$$

(1.16)

$$M_{ji} = -L_{jt} F_i{}^t \quad , \quad M_i{}^h = M_{it} g^{th} .$$

Since then the analogy between the Weyl and Bochner curvature tensors has been studied by Chen [5], Ishihara [24], Liu [14], Matsumoto [10], Tachibana [14], Takagi [15], Watanabe [15] and one of the present authors [5], [19], [20], [24].

A submanifold M^p isometrically immersed in a Kaehlerian manifold M^n is said to be invariant or analytic if the transform of the tangent space $T_y(M^p)$ at any point y of M^p by the complex structure tensor $F_i{}^h$ is still tangent to M^p, that is, if we have equations of the form

$$F_i{}^h B_b{}^i = F_b{}^a B_a{}^h . \tag{1.17}$$

In this case M^p is again a Kaehlerian manifold and is minimal in M^n.

A submanifold M^p isometrically immersed in a Kaehlerian manifold M^n is said to be anti-invariant or totally real if the transform of the tangent space $T_y(M^p)$ at any point y of M^p by the complex structure tensor $F_i{}^h$ is normal to M^p, that is, if we have equations of the form

$$F_i{}^h B_b{}^i = -f_b{}^x C_x{}^h . \tag{1.18}$$

Totally real submanifolds of Kaehlerian manifolds have been studied by Blair [1], Chen [4], Houh [6], Kon [7], [25], [26], Ludden [8], [9], Ogiue [4], Okumura [8], [9] and one of the present authors [8], [9], [21], [22], [25], [26].

As a theorem which shows a close relation between the Weyl and Bochner curvature tensors, Blair [1] proved the following

THEOREM C. *Let M^{2n}, $n \geq 4$, be a real 2 n-dimentional Kaehlerian manifold with vanishing Bochner curvature tensor and let M^n be an n-dimensional totally geodesic, totally real submanifold of M^{2n}. Then M^n is conformally flat.*

One of the present authors [21] showed that this theorem of Blair is valid also for a totally umbilical, totally real submanifold M^n of a Kaehlerian manifold M^{2n} with vanishing Bochner curvature tensor.

The purpose of the present paper is to prove the following theorems which show close relations and analogies between the Weyl and the Bochner curvature tensors. To state the theorems we denote the curvature tensors $K_{kji}{}^h$ by $K(X, Y)Z$ and K_{kjih} by $K(X, Y, Z, W) = g(K(X, Y)Z, W)$, X, Y, Z, W being vector fields of the manifold.

THEOREM 1. *Let M^n be an n-dimensional Riemannian manifold $n > 3$. A necessary and sufficient condition for M^n to be conformally flat is that $K(X, Y, Z, W) = 0$ for any mutually orthogonal vectors X, Y, Z and W.* (Cf. Schouten [11] or [12], p. 307).

As a corollary to Theorem 1, we prove

THEOREM 2. *Let M^p be a p-dimensional totally umbilical submanifold of an n-dimensional conformally flat Riemannian manifold M^n $(3 < p < n)$. Then M^p is conformally flat.* (Cf. Yano [18]).

As a theorem corresponding to Theorem 1 we prove

THEOREM 3. *Let M^n be a real n-dimensional Kaehlerian manifold with the complex structure tensor F, $(n \geq 8)$. A necessary and sufficient condition for M^n to be with vanishing Bochner curvature tensor is that $K(X, Y, Z, W) = 0$ for any mutually orthogonal vectors X, Y, Z, W such that FX, FY, FZ, FW are orthogonal to the linear space spanned by X, Y, Z and W.*

Applying Theorem 3, we prove the following two thorems:

THEOREM 4. *Let M^p be a p-dimensional invariant totally geodesic submanifold of an n-dimensional Kaehlerian manifold M^n with vanishing Bochner curvature tensor $(4 \leq p < n, 8 \leq n)$. Then M^p is with vanishing Bochner curvature tensor.* (Cf. Yamaguchi and Sato [17]).

THEOREM 5. *A p-dimensional totally umbilical, totally real submanifold M^p of an n-dimensional Kaehlerian manifold M^n $(4 \leq p < n, 8 \leq n)$ with vanishing Bochner curvature tensor is conformally flat.* (Cf. Blair [1] and Yano [21]).

§ 2. PROOFS OF THEOREMS 1 AND 2

To prove Theorem 1, we need the following

LEMMA 1. *If a Riemannian manifold M^n of dimension $n > 3$ has the curvature tensor of the form*

$$K_{kjih} = g_{kj} A_{ih} + g_{ki} B_{jh} + g_{kh} C_{ji} + g_{ji} D_{kh} + g_{jh} E_{ki} + g_{ih} H_{kj}, \tag{2.1}$$

where A_{ih}, B_{jh}, C_{ji}, D_{kh}, E_{ki} and H_{kj} are local components of certain tensors of (0, 2)-type, then M^n is conformally flat.

Proof of Theorem 1. Let X^k, Y^j, Z^i and W^h be local components of any mutually orthogonal vectors X, Y, Z and W respectively.

If M^n is conformally flat, then the curvature tensor of M^n has the form

$$K_{kjih} = -g_{kh} L_{ji} + g_{jh} L_{ki} - L_{kh} g_{ji} + L_{jh} g_{ki}. \tag{2.2}$$

Hence, it is easy to see that $K_{kjih} X^k Y^j Z^i W^h = 0$, that is, $K(X, Y, Z, W) = 0$.

Conversely, if $K_{kjih} X^k Y^j Z^i W^h = 0$ for any mutually orthogonal vectors X^k, Y^j, Z^i and W^h, then the curvature tensor of M^n has the form (2.1). Consequently, by Lemma 1, M^n is conformally flat.

Proof of Theorem 2. We first of all remember the equations of Gauss (1.6). Since M^p is totally umbilical, we can put $h_{cb}{}^x = g_{cb} h^x$ and then equations of Gauss become

$$\text{(2.3)} \qquad K_{dcba} = K_{kjih} B_d{}^k B_c{}^j B_b{}^i B_a{}^h + (g_{da} g_{cb} - g_{ca} g_{db}) h_x h^x .$$

Transvecting the both sides of (2.3) with any mutually orthogonal vectors X^d, Y^c, Z^b and W^a of M^p, we have

$$\text{(2.4)} \qquad K_{dcba} X^d Y^c Z^b W^a = K_{kjih} (B_d{}^k X^d)(B_c{}^j Y^c)(B_b{}^i Z^b)(B_a{}^h W^a) .$$

By virtue of the assumption that M^n is conformally flat, the right hand side of (2.4) vanishes. Thus, from (2.4), we have

$$K_{dcba} X^d Y^c Z^b W^a = 0$$

for any mutually orthogonal vectors X^d, Y^c, Z^b and W^a. Consequently by Theorem 1, M^p is conformally flat.

§ 3. Proof of Lemma 1

To prove Lemma 1, we need the following

Proposition 1. *If a Riemannian manifold* M^n *has the curvature tensor of the form* (2.1), *then the following relations hold.*

$$\text{(3.1)} \quad \begin{aligned} &\text{(i)} \quad A + H = 0 \quad , \quad B + C + D + E = 0 , \\ &\text{(ii)} \quad A_{ji} = \frac{1}{n} A g_{ji} \quad , \quad H_{ji} = \frac{1}{n} H g_{ji} , \\ &\text{(iii)} \quad B_{ji} = E_{ji} + \frac{1}{n}(B - E) g_{ji} \quad , \quad C_{ji} = - E_{ji} + \frac{1}{n}(C + E) g_{ji} , \\ &\qquad\qquad D_{ji} = - E_{ji} + \frac{1}{n}(D + E) g_{ji} \end{aligned}$$

where $A = g^{ji} A_{ji}$, $B = g^{ji} B_{ji}$, etc.

Proof of Lemma 1. Substituting (3.1) (ii) and (iii) into (2.1) and making use of (3.1) (i), we have

$$\text{(3.2)} \quad \begin{aligned} K_{kjih} &= \frac{1}{n} A g_{kj} g_{ih} + g_{ki}\left(E_{jh} + \frac{1}{n}(B - E) g_{jh}\right) + g_{kh}\left(- E_{ji} + \frac{1}{n}(C + E) g_{ji}\right) \\ &\quad + g_{ji}\left(- E_{kh} + \frac{1}{n}(D + E) g_{kh}\right) + g_{jh} E_{ki} + \frac{1}{n} H g_{ih} g_{kj} \\ &= - g_{kh} E_{ji} + g_{jh} E_{ki} - E_{kh} g_{ji} + E_{jh} g_{ki} + \frac{1}{n}(A + H) g_{kj} g_{ih} \\ &\quad + \frac{1}{n}(C + D + 2E) g_{kh} g_{ji} + \frac{1}{n}(B - E) g_{ki} g_{jh} \\ &= - g_{kh} E_{ji} + g_{jh} E_{ki} - E_{kh} g_{ji} + E_{jh} g_{ki} - \frac{1}{n}(B - E)(g_{kh} g_{ji} - g_{ki} g_{jh}) . \end{aligned}$$

Putting

$$L_{ji} = E_{ji} + \frac{1}{2n}(B - E) g_{ji},$$

we have from (3.2)

$$K_{kjih} = -g_{kh} L_{ji} + g_{jh} L_{ki} - L_{kh} g_{ji} + L_{jh} g_{ki}. \tag{3.3}$$

Transvecting (3.3) with g^{kh}, we have

$$\begin{aligned} K_{ji} &= -nL_{ji} + L_{ji} - Lg_{ji} + L_{ji} \\ &= -(n-2) L_{ji} - Lg_{ji}, \end{aligned} \tag{3.4}$$

where $L = g^{ji} L_{ji}$. Moreover transvecting (3.4) with g^{ji}, whe have

$$K = -2(n-1) L$$

and therefore (3.4) becomes

$$K_{ji} = -(n-2) L_{ji} + \frac{1}{2(n-1)} Kg_{ji}$$

or

$$L_{ji} = -\frac{1}{n-2} K_{ji} + \frac{1}{2(n-1)(n-2)} Kg_{ji}.$$

Consequently, (3.3) shows that M^n is conformally flat.

Proof of Proposition 1. From (2.1), we easily have the following equalities:

(a.1) $g^{kh} K_{khji} = 0 = nA_{ji} + B_{ji} + C_{ij} + D_{ji} + E_{ij} + Hg_{ji}$,

(a.2) $g^{kh} K_{jikh} = 0 = Ag_{ji} + B_{ij} + C_{ij} + D_{ji} + E_{ji} + nH_{ji}$,

(a.3) $g^{kh} K_{kjih} = K_{ji} = A_{ij} + B_{ji} + nC_{ji} + Dg_{ji} + E_{ji} + H_{ij}$,

(a.4) $g^{kh} K_{jkih} = -K_{ji} = A_{ij} + Bg_{ji} + C_{ji} + D_{ji} + nE_{ji} + H_{ji}$,

(a.5) $g^{kh} K_{jhki} = K_{ji} = A_{ji} + B_{ji} + Cg_{ji} + nD_{ji} + E_{ji} + H_{ji}$,

(a.6) $g^{kh} K_{kjhi} = -K_{ji} = A_{ji} + nB_{ji} + C_{ji} + D_{ji} + Eg_{ji} + H_{ij}$.

From (a.1), (a.2), (a.3), (a.4), (a.5) and (a.6), we have

$$n[A] + [B] - [C] + [D] - [E] = 0,$$
$$-[B] - [C] + [D] + [E] + n[H] = 0,$$
$$-[A] + [B] + n[C] + [E] - [H] = 0,$$
$$-[A] + [C] + [D] + n[E] + [H] = 0,$$
$$[A] + [B] + n[D] + [E] + [H] = 0,$$
$$[A] + n[B] + [C] + [D] - [H] = 0$$

respectively, where $[A]=\frac{1}{2}(A_{ji}-A_{ij})$, $[B]=\frac{1}{2}(B_{ji}-B_{ij})$ etc. Regarding the set of these 6 equations as a system of simultaneous equations with respect to $[A]$, $[B]$, $[C]$, $[D]$, $[E]$, $[H]$, we have

$$[A]=[B]=[C]=[D]=[E]=[H]=0\,, \tag{3.5}$$

that is, A_{ji}, B_{ji}, C_{ji}, D_{ji}, E_{ji} and H_{ji} are symmetric tensors, because of

$$\begin{vmatrix} n & 1 & -1 & 1 & -1 & 0 \\ 0 & -1 & -1 & 1 & 1 & n \\ -1 & 1 & n & 0 & 1 & -1 \\ -1 & 0 & 1 & 1 & n & 1 \\ 1 & 1 & 0 & n & 1 & 1 \\ 1 & n & 1 & 1 & 0 & -1 \end{vmatrix} = (n-3)^2(n+2)^2 \neq 0\,.$$

Taking account of (3.5), we have, forming (a.3) + (a.4),

$$2A_{ji}+B_{ji}+Bg_{ji}+(n+1)\,C_{ji}+Dg_{ji}+D_{ji}+(n+1)\,E_{ji}+2H_{ji}=0\,. \tag{3.6}$$

Transvecting (3.6) and (a.1) with g^{ji}, we have

$$2(A+H)+(n+1)(B+C+D+E)=0\,, \tag{3.7}$$

$$n(A+H)+B+C+D+E=0 \tag{3.8}$$

respectively.

From (3.7) and (3.8), (3.1) (i) follows, that is,

$$A+H=0\quad,\quad B+C+D+E=0\,. \tag{3.9}$$

Forming (a.1) — (a.2), we have

$$nA_{ji}-Ag_{ji}+Hg_{ji}-nH_{ji}=0$$

or

$$A_{ji}=H_{ji}+\frac{1}{n}(A-H)\,g_{ji}\,. \tag{3.10}$$

Similarly, forming (a.6) — (a.4) and (a.5) — (a.4), we have

$$B_{ji}=E_{ji}+\frac{1}{n}(B-E)\,g_{ji}\,, \tag{3.11}$$

$$C_{ji}=D_{ji}+\frac{1}{n}(C-D)\,g_{ji} \tag{3.12}$$

respectively. (3.11) is the first equation of (3.1) (iii).

Substituting (3.10), (3.11) and (3.12) into (a.1), we have

$$2\,D_{ji} = -2\,E_{ji} - nH_{ji} + \frac{1}{n}(D + E - B - C - nA)\,g_{ji}$$

or making use of (3.9)

$$D_{ji} = -E_{ji} - \frac{1}{2}\,nH_{ji} + \frac{1}{n}\left(D + E + \frac{1}{2}\,nH\right)g_{ji} \tag{3.13}$$

and therefore (3.12) becomes

$$C_{ji} = -E_{ji} - \frac{1}{2}\,nH_{ji} + \frac{1}{n}\left(C + E + \frac{1}{2}\,nH\right)g_{ji}\,. \tag{3.14}$$

Substituting (3.10), (3.11), (3.13) and (3.14) into (3.6) and using (3.9), we easily have (3.1) (ii), that is,

$$H_{ji} = \frac{1}{n}\,Hg_{ji}$$

and therefore from (3.10), we have

$$A_{ji} = \frac{1}{n}\,Ag_{ji}\,.$$

Making use of $H_{ji} = \frac{1}{n}\,Hg_{ji}$, from (3.13) and (3.14), we obtain the last two equations of (3.1) (iii). Q.D.E.

§ 4. Proofs of Theorems 3, 4 and 5

To prove these theorems, we need the following

LEMMA 2. *If a Kaehlerian manifold* M^n *of real dimension* $n > 4$ *has the curvature tensor of the form*

$$\begin{aligned} K_{kjih} = {} & g_{kj}A_{ih} + g_{ki}B_{jh} + g_{kh}C_{ji} + g_{ji}D_{kh} + g_{jh}E_{ki} + g_{ih}H_{kj} \\ & + F_{kj}P_{ih} + F_{ki}Q_{jh} + F_{kh}R_{ji} + F_{ji}S_{kh} + F_{jh}T_{ki} + F_{ih}V_{kj}\,, \end{aligned} \tag{4.1}$$

where $A_{ih}, B_{jh}, C_{ji}, D_{kh}, E_{ki}, H_{kj}, P_{ih}, Q_{jh}, R_{ji}, S_{kh}, T_{ki}$ *and* V_{kj} *are local components of certain tensors of* (0, 2)*-type, then the Bochner curvature tensor of* M^n *vanishes.*

Proof of Theorem 3. Let X^k, Y^j, Z^i and W^h be respectively local components of any mutually orthogonal vectors X, Y, Z and W such that FX, FY, FZ and FW are orthogonal to the linear space spanned by X, Y, Z and W.

If the Bochner curvature tensor vanishes, then we have

$$\begin{aligned} K_{kjih} = {} & -g_{kh}L_{ji} + g_{jh}L_{ki} - L_{kh}g_{ji} + L_{jh}g_{ki} \\ & - F_{kh}M_{ji} + F_{jh}M_{ki} - M_{kh}F_{ji} + M_{jh}F_{ki} + 2\,(M_{kj}F_{ih} + F_{kj}M_{ih})\,. \end{aligned} \tag{4.2}$$

Hence, it is easy to see that $K_{kjih}X^kY^jZ^iW^h = 0$, that is, $K(X, Y, Z, W) = 0$.

Conversely, if $K_{kjih} X^k Y^j Z^i W^h = 0$ for any mutually orthogonal vectors X^k, Y^j, Z^i and W^h such that $F_t^{\,k} X^t$, $F_t^{\,j} Y^t$, $F_t^{\,i} Z^t$ and $F_t^{\,h} W^t$ are orthogonal to the linear space spanned by X^k, Y^j, Z^i and W^h, then the curvature tensor has the form (4.1).

Consequently, by Lemma 2, the Bochner curvature tensor vanishes.

Proof of Theorem 4. Since M^p is an analytic submanifold, we have (1.17) and therefore $F_{ji} B_c^{\,j} B_b^{\,i} = F_{cb}$. Applying ∇_d to the both sides of this equality and making use of the assumption that M^p is totally geodesic, we have

$$0 = \nabla_d F_{cb},$$

which shows that M^p is a Kaehlerian manifold.

On the other hand, from equations of Gauss, we have

$$K_{dcba} = K_{kjih} B_d^k B_c^j B_b^i B_a^h,$$

because M^p is totally geodesic.

Transvecting the both sides of the last equation with any mutually orthogonalvectors X^d, Y^c, Z^b and W^a such that $F_e^{\,d} X^e$, $F_e^{\,c} Y^e$, $F_e^{\,b} Z^e$ and $F_e^{\,a} W^e$ are orthogonal to the linear space spanned by X^a, Y^b, Z^c and W^d, we have

$$K_{dcba} X^d Y^c Z^b W^a = K_{kjih} (B_d^{\,k} X^d)(B_c^{\,j} Y^c)(B_b^{\,i} Z^b)(B_a^{\,h} W^a).$$

But, since FBX, FBY, FBZ and FBW are also orthogonal to the linear space spanned by BX, BY, BZ and BW and M^n has the vanishing Bochner curvature tensor, the right hand side vanishes. Thus we have

$$K_{dcba} X^d Y^c Z^b W^a = 0$$

for any mutually orthogonal vectors X^d, Y^c, Z^b and W^a such that $F_e^{\,d} X^e$, $F_e^{\,c} Y^e$, $F_e^{\,b} Z^e$ and $F_e^{\,a} W^e$ are orthogonal to the linear space spanned by X^d, Y^c, Z^b and W^a.

Consequently, by Theorem 3, M^p has the vanishing Bochner curvature tensor.

Proof of Theorem 5. By equations of Gauss, we have (2.3). Transvecting the both sides of (2.3) with any mutually orthogonal vectors X^d, Y^c, Z^b and W^a, we have

$$K_{dcba} X^d Y^c Z^b W^a = K_{kjih} (B_d^{\,k} X^d)(B_c^{\,j} Y^c)(B_b^{\,i} Z^b)(B_a^{\,h} W^a). \tag{4.3}$$

But, since submanifold M^p is totally real, FBX, FBY, FBZ and FBW are orthogonal to the linear space spanned by BX, BY, BZ and BW and the Bochner curvature tensor of M^n vanishes, the right hand side of (4.3) vanishes. Hence, we have

$$K_{dcba} X^d Y^c Z^b W^a = 0$$

for any mutually orthogonal vectors X^d, Y^c, Z^b and W^a. Consequently, by Theorem 1, M^p is conformally flat.

§ 5. Proof of Lemma 2

To prove Lemma 2, we need the following

Proposition 2. *If a Kaehlerian manifold M^n of complex dimension $m \geq 3$ has the curvature tensor of the form* (4.1), *then the following relations hold in a complex coordinate system.*

$$\text{(i)}\quad A_{\beta\bar\alpha} = \frac{1}{m} A_1 g_{\beta\bar\alpha}, \qquad H_{\beta\bar\alpha} = \frac{1}{m} H_1 g_{\beta\bar\alpha},$$

$$\text{(5.1)}\quad \text{(ii)}\quad C_{\bar\alpha\beta} + E_{\beta\bar\alpha} = \frac{1}{m}(C_2 + E_1) g_{\beta\bar\alpha}, \qquad D_{\beta\bar\alpha} + E_{\beta\bar\alpha} = \frac{1}{m}(D_1 + E_1) g_{\beta\bar\alpha},$$

$$\text{(iii)}\quad P_{\beta\bar\alpha} - V_{\beta\bar\alpha} = \frac{1}{m}(P_1 - V_1) g_{\beta\bar\alpha}, \qquad R_{\bar\alpha\beta} + S_{\beta\bar\alpha} = \frac{1}{m}(R_2 + S_1) g_{\beta\bar\alpha},$$

where $A_1 = g^{\beta\bar\alpha} A_{\beta\bar\alpha}$, $H_1 = g^{\beta\bar\alpha} H_{\beta\bar\alpha}$ *etc.*, $C_2 = g^{\bar\beta\alpha} C_{\bar\beta\alpha}$, $R_2 = g^{\bar\beta\alpha} R_{\bar\beta\alpha}$ *and the indices* $\alpha, \beta, \cdots$ *and* $\bar\alpha, \bar\beta, \cdots$ *run over the ranges* $\{1, 2, \cdots, m\}$ *and* $\{\bar 1, \bar 2, \cdots, \bar m\}$ *respectively.*

Proof of Lemma 2. In our complex coordinate system, (4.1) can be written as

$$\text{(5.2)}\quad K_{\delta\bar\gamma\beta\bar\alpha} = g_{\delta\bar\gamma} A_{\beta\bar\alpha} + g_{\delta\bar\alpha} C_{\bar\gamma\beta} + g_{\bar\gamma\beta} D_{\delta\bar\alpha} + g_{\beta\bar\alpha} H_{\delta\bar\gamma}$$
$$+ i g_{\delta\bar\gamma} P_{\beta\bar\alpha} + i g_{\delta\bar\alpha} R_{\bar\gamma\beta} - i g_{\bar\gamma\beta} S_{\delta\bar\alpha} + i g_{\beta\bar\alpha} V_{\delta\bar\gamma},$$

where $i = \sqrt{-1}$.

From Proposition 2, we have

$$A_{\beta\bar\alpha} = \frac{1}{m} A_1 g_{\beta\bar\alpha}, \qquad C_{\bar\gamma\beta} = -E_{\beta\bar\gamma} + \frac{1}{m}(C_2 + E_1) g_{\beta\bar\gamma},$$

$$D_{\delta\bar\alpha} = -E_{\delta\bar\alpha} + \frac{1}{m}(D_1 + E_1) g_{\delta\bar\alpha}, \qquad H_{\delta\bar\gamma} = \frac{1}{m} H_1 g_{\delta\bar\gamma},$$

$$P_{\beta\bar\alpha} = V_{\beta\bar\alpha} + \frac{1}{m}(P_1 - V_1) g_{\beta\bar\alpha}, \qquad R_{\bar\gamma\beta} = -S_{\beta\bar\gamma} + \frac{1}{m}(R_2 + S_1) g_{\beta\bar\gamma}$$

and thus substituting these equations into (5.2), we have

$$\text{(5.3)}\quad K_{\delta\bar\gamma\beta\bar\alpha} = \frac{1}{m}(A_1 + H_1 + i(P_1 - V_1)) g_{\delta\bar\gamma} g_{\beta\bar\alpha}$$
$$+ \frac{1}{m}(C_2 + 2E_1 + D_1 + i(R_2 + S_1)) g_{\delta\bar\alpha} g_{\bar\gamma\beta} - (g_{\delta\bar\alpha} E_{\beta\bar\gamma} + g_{\bar\gamma\beta} E_{\delta\bar\alpha})$$
$$- i(g_{\delta\bar\alpha} S_{\beta\bar\gamma} + g_{\bar\gamma\beta} S_{\delta\bar\alpha}) + i(g_{\beta\bar\alpha} V_{\delta\bar\gamma} + g_{\delta\bar\gamma} V_{\beta\bar\alpha}).$$

Since $K_{\delta\bar{\gamma}\beta\bar{\alpha}} = K_{\beta\bar{\gamma}\delta\bar{\alpha}}$, from

$$2\,K_{\delta\bar{\gamma}\beta\bar{\alpha}} = K_{\delta\bar{\gamma}\beta\bar{\alpha}} + K_{\beta\bar{\gamma}\delta\bar{\alpha}},$$

we have

$$\begin{aligned}(5.4)\quad 2\,K_{\delta\bar{\gamma}\beta\bar{\alpha}} = {} & \frac{1}{m}(A_1 + C_2 + 2\,E_1 + D_1 + H_1 + i(P_1 + R_2 + S_1 - V_1))\,g_{\delta\bar{\gamma}}\,g_{\beta\bar{\alpha}} \\ & + \frac{1}{m}(A_1 + C_2 + 2\,E_1 + D_1 + H_1 + i(P_1 + R_2 + S_1 - V_1))\,g_{\delta\bar{\alpha}}\,g_{\bar{\gamma}\beta} \\ & - g_{\delta\bar{\alpha}}(E_{\beta\bar{\gamma}} + i\,S_{\beta\bar{\gamma}} - i\,V_{\beta\bar{\gamma}}) - g_{\beta\bar{\gamma}}(E_{\delta\bar{\alpha}} + i\,S_{\delta\bar{\alpha}} - i\,V_{\delta\bar{\alpha}}) \\ & - g_{\beta\bar{\alpha}}(E_{\delta\bar{\gamma}} + i\,S_{\delta\bar{\gamma}} - i\,V_{\delta\bar{\gamma}}) - g_{\delta\bar{\gamma}}(E_{\beta\bar{\alpha}} + i\,S_{\beta\bar{\alpha}} - i\,V_{\beta\bar{\alpha}}).\end{aligned}$$

If we put

$$\lambda = \frac{1}{m}(A_1 + C_2 + 2\,E_1 + D_1 + H_1 + i(P_1 + R_2 + S_1 - V_1)),$$

$$L_{\beta\bar{\alpha}} = -\frac{1}{8}\lambda\,g_{\beta\bar{\alpha}} + \frac{1}{4}(E_{\beta\bar{\alpha}} + i\,S_{\beta\bar{\alpha}} - i\,V_{\beta\bar{\alpha}}),$$

then (5.4) becomes

$$(5.5)\qquad K_{\delta\bar{\gamma}\beta\bar{\alpha}} + 2\,g_{\delta\bar{\alpha}}\,L_{\beta\bar{\gamma}} + 2\,g_{\bar{\gamma}\beta}\,L_{\delta\bar{\alpha}} + 2\,g_{\beta\bar{\alpha}}\,L_{\delta\bar{\gamma}} + 2\,g_{\delta\bar{\gamma}}\,L_{\beta\bar{\alpha}} = 0.$$

Transvecting (5.5) with $g^{\delta\bar{\alpha}}$, we have

$$K_{\bar{\gamma}\beta} + 2\,m\,L_{\beta\bar{\gamma}} + 2\,g_{\bar{\gamma}\beta}\,L_1 + 2\,L_{\beta\bar{\gamma}} + 2\,L_{\beta\bar{\gamma}} = 0$$

or

$$(5.6)\qquad K_{\bar{\gamma}\beta} + 2\,L_1\,g_{\bar{\gamma}\beta} + 2\,(m+2)\,L_{\beta\bar{\gamma}} = 0,$$

where $L_1 = g^{\beta\bar{\gamma}}\,L_{\beta\bar{\gamma}}$.

Moreover, transvecting (5.6) with $g^{\bar{\gamma}\beta}$, we have

$$\frac{1}{2}\,K + 2\,mL_1 + 2\,(m+2)\,L_1 = 0$$

or

$$L_1 = -\frac{1}{8\,(m+1)}\,K$$

and therefore from (5.6) we have

$$(5.7)\qquad L_{\beta\bar{\gamma}} = -\frac{1}{2\,(m+2)}\,K_{\beta\bar{\gamma}} + \frac{1}{8\,(m+1)\,(m+2)}\,K\,g_{\bar{\gamma}\beta}.$$

Thus, substituting (5.7) into (5.5), we have

$$K_{\delta\bar{\gamma}\beta\bar{\alpha}} - \frac{1}{m+2}\left(g_{\delta\bar{\gamma}} K_{\beta\bar{\alpha}} + g_{\delta\bar{\alpha}} K_{\beta\bar{\gamma}} + g_{\beta\bar{\alpha}} K_{\delta\bar{\gamma}} + g_{\beta\bar{\gamma}} K_{\delta\bar{\alpha}}\right)$$
$$- \frac{1}{2(m+1)(m+2)} K \left(g_{\delta\bar{\gamma}} g_{\beta\bar{\alpha}} + g_{\delta\bar{\alpha}} g_{\beta\bar{\gamma}}\right) = 0 ,$$

which shows that the Bochner curvature tensor vanishes.

Proof of Proposition 2. From (4.1), we have the following equalities:

(b.1) $$g^{kh} K_{kjih} = K_{ji} = A_{ij} + B_{ji} + 2m C_{ji} + D g_{ji} + E_{ji} + H_{ij}$$
$$- F_j^{\ t} P_{it} - F_i^{\ t} Q_{jt} + 0 + S F_{ji} + F_j^{\ t} T_{ti} + F_i^{\ t} V_{tj} ,$$

(b.2) $$g^{kh} K_{jkhi} = K_{ji} = A_{ji} + B_{ji} + C g_{ji} + 2n D_{ji} + E_{ji} + H_{ji}$$
$$+ F_j^{\ t} P_{ti} + F_j^{\ t} Q_{ti} + R F_{ji} + 0 - F_i^{\ t} T_{jt} - F_i^{\ t} V_{jt} ,$$

(b.3) $$g^{kh} K_{kjhi} = -K_{ji} = A_{ji} + 2m B_{ji} + C_{ji} + D_{ji} + E g_{ji} + H_{ij}$$
$$- F_j^{\ t} P_{ti} + 0 - F_i^{\ t} R_{jt} + F_j^{\ t} S_{ti} + T F_{ji} - F_i^{\ t} V_{tj} ,$$

(b.4) $$g^{kh} K_{jkih} = -K_{ji} = A_{ij} + B g_{ji} + C_{ji} + D_{ji} + 2m E_{ji} + H_{ji}$$
$$+ F_j^{\ t} P_{it} + Q F_{ji} + F_j^{\ t} R_{ti} - F_i^{\ t} S_{jt} + 0 + F_i^{\ t} V_{jt} ,$$

(b.5) $$g^{kh} K_{khji} = 0 = 2m A_{ji} + B_{ji} + C_{ij} + D_{ji} + E_{ij} + H g_{ji}$$
$$+ 0 - F_j^{\ t} Q_{ti} - F_i^{\ t} R_{tj} - F_j^{\ t} S_{ti} - F_i^{\ t} T_{tj} + V F_{ji} ,$$

(b.6) $$g^{kh} K_{jikh} = 0 = A g_{ji} + B_{ji} + C_{ij} + D_{ji} + E_{ji} + 2m H_{ji}$$
$$+ P F_{ji} + F_j^{\ t} Q_{it} + F_j^{\ t} R_{it} + F_i^{\ t} S_{jt} + F_i^{\ t} T_{jt} + 0 ,$$

where $A = g^{ji} A_{ji}$, $B = g^{ji} B_{ji}$, etc.

In a Kaehlerian manifold, the following relations are valid:

$$F^{kh} K_{kjih} = F_{jt} K_i^{\ t} \quad , \quad F^{kh} K_{khji} = -2 F_{jt} K_i^{\ t} .$$

Hence, again from (4.1) we have the following equalities:

(b.7) $$F^{kh} K_{kjih} = F_{jt} K_i^{\ t} = F_j^{\ t} A_{it} + F_i^{\ t} B_{jt} + 0 + \hat{D} g_{ji} - F_j^{\ t} E_{ti} - F_i^{\ t} H_{tj}$$
$$+ P_{ij} + Q_{ji} + 2m R_{ji} + \hat{S} F_{ji} + T_{ji} + V_{ij} ,$$

(b.8) $$F^{kh} K_{jkhi} = F_{jt} K_i^{\ t} = F_j^{\ t} A_{ij} - F_j^{\ t} B_{ti} + \hat{C} g_{ji} + 0 + F_i^{\ t} E_{jt} - F_i^{\ t} H_{jt}$$
$$- P_{ji} + Q_{ji} + \hat{R} F_{ji} + 2m S_{ji} + T_{ji} - V_{ji} ,$$

(b.9) $F^{kh}K_{khji} = -2F_{ji}K_t^t = 0 + F_j^tB_{ti} + F_i^tC_{tj} - F_j^tD_{ti} - F_i^tE_{tj} + \bar{H}g_{ji}$
$+ 2mP_{ji} + Q_{ji} + R_{ij} - S_{ji} - T_{ij} + \bar{V}F_{ji}$,

(b.10) $F^{kh}K_{jikh} = -2F_{ji}K_t^t = \bar{A}g_{ji} + F_j^tB_{it} - F_j^tC_{it} + F_i^tD_{jt} - F_i^tE_{jt} + 0$
$+ \bar{P}F_{ji} - Q_{ij} + R_{ij} - S_{ji} + T_{ji} + 2mV_{ji}$.

(b.11) $F^{kh}K_{kjhi} = -F_{ji}K_t^t = F_j^tA_{ti} + 0 + F_i^tC_{jt} - F_j^tD_{ti} + \bar{E}g_{ji} - F_i^tH_{tj}$
$+ P_{ji} + 2mQ_{ji} + R_{ji} + S_{ji} + \bar{T}F_{ji} - V_{ij}$.

(b.12) $F^{kh}K_{jkih} = -F_{ji}K_t^t = F_j^tA_{it} + \bar{B}g_{ji} - F_j^tC_{ti} + F_i^tD_{jt} + 0 - F_i^tH_{jt}$
$- P_{ij} + \bar{Q}F_{ji} + R_{ji} + S_{ji} + 2mT_{ji} + V_{ji}$,

where $\bar{A} = F^{ji}A_{ji}$, $\bar{B} = F^{ji}B_{ji}$, etc.

Forming (b.1) + (b.2) + (b.3) + (b.4) + (b.5) + (b.6), we have

(5.8) $((2m+2)A_{ji} + 2A_{ij}) + ((2m+3)B_{ji} + B_{ij}) + ((2m+2)C_{ji} + 2C_{ij})$
$+ (2m+4)D_{ji} + ((2m+3)E_{ji} + E_{ij}) + ((2m+2)H_{ji} + 2H_{ij})$
$+ F_j^t(T_{ti} + Q_{it} + R_{it} + R_{ti}) - F_i^t(T_{tj} + Q_{jt} + R_{jt} + R_{tj})$
$+ (A + B + C + D + E + H)g_{ji} + (P + Q + R + S + T + V)F_{ji} = 0.$

Transvecting (5.8) with g^{ji}, we have

(5.9) $$A + B + C + D + E + H = 0$$

and therefore taking account of (5.9), we have, from (5.8),

(5.10) $$A_{(ji)} + B_{(ji)} + C_{(ji)} + D_{(ji)} + E_{(ji)} + H_{(ji)} = 0,$$

where $A_{(ji)} = \frac{1}{2}(A_{ji} + A_{ij})$, etc.

Transvecting (b.1), (b.3) with g^{ji} and (b.7), (b.11) with F^{ji}, we have

(5.11) $K = (A + H) + (B + E) + 2m(C + D) - (\bar{P} + \bar{V}) - (\bar{Q} + \bar{T})$,

(5.12) $-K = (A + H) + 2m(B + E) + (C + D) + (\bar{P} + \bar{V}) - (\bar{R} + \bar{S})$,

(5.13) $K = (A + H) - (B + E) - (\bar{P} + \bar{V}) + 2m(\bar{R} + \bar{S}) + (\bar{Q} + \bar{T})$,

(5.14) $K = -(A + H) + (C + D) - (\bar{P} + \bar{V}) - (\bar{R} + \bar{S}) - 2m(\bar{Q} + \bar{T})$

respectively. Forming (5.11) − (5.13) and (5.12) + (5.13), we have

(5.15) $0 = 2(B + E) + 2m(C + D) - 2m(\bar{R} + \bar{S}) - 2(\bar{Q} + \bar{T})$,

(5.16) $0 = 2m(B + E) + 2(C + D) - 2(\bar{R} + \bar{S}) - 2m(\bar{Q} + \bar{T})$

respectively and consequently forming (5.15)$\times m$ — (5.16), we see that

$$C + D = \acute{R} + \bar{S} \quad , \quad B + E = \check{Q} + \hat{T} . \tag{5.17}$$

Forming (5.11) + (5.12), we have

$$0 = 2(A + H) + (2m + 1)(B + E + C + D) - (\check{Q} + \acute{R} + \bar{S} + \hat{T})$$

or making use of (5.9)

$$0 = (1 - 2m)(A + H) - (\check{Q} + \acute{R} + \bar{S} + \hat{T}) \tag{5.18}$$

and from (5.9) and (5.17), we have

$$-(A + H) = \check{Q} + \acute{R} + \bar{S} + \hat{T} . \tag{5.19}$$

Thus, from (5.18) and (5.19), we have

$$(2 - 2m)(A + H) = 0 ,$$

from which

$$A + H = 0 . \tag{5.20}$$

Consequently, from (5.9) and (5.18), we have

$$B + E + C + D = 0 \quad , \quad \check{Q} + \acute{R} + \bar{S} + \hat{T} = 0 . \tag{5.21}$$

Next, transvecting (b.1), (b.2), (b.3), (b.4), (b.5) and (b.6) with F^{ji} we have

$$-(\bar{A} + V) + (\breve{B} + T) + 2m(\check{C} + S) + (\dot{E} + Q) - (\hat{H} + P) = 0 ,$$

$$(\bar{A} + V) + (\breve{B} + T) + 2m(\bar{D} + R) + (\dot{E} + Q) + (\hat{H} + P) = 0 ,$$

$$(\bar{A} + V) + 2m(\breve{B} + T) + (\check{C} + S) + (\bar{D} + R) - (\hat{H} + P) = 0 ,$$

$$-(\bar{A} + V) + (\check{C} + S) + (\bar{D} + R) + 2m(\dot{E} + Q) + (\hat{H} + P) = 0 ,$$

$$2m(\bar{A} + V) + (\breve{B} + T) - (\check{C} + S) + (\bar{D} + R) - (\dot{E} + Q) = 0 ,$$

$$-(\breve{B} + T) - (\check{C} + S) + (\bar{D} + R) + (\dot{E} + Q) + 2m(\hat{H} + P) = 0$$

respectively.

Regarding the set of these 6 equations as a system of simultaneous equations with respect to $\bar{A} + V , \breve{B} + T , \check{C} + S , \bar{D} + R , \dot{E} + Q , \hat{H} + P$, we have

$$\bar{A} + V = \breve{B} + T = \check{C} + S = \bar{D} + R = \dot{E} + Q = \hat{H} + P = 0 , \tag{5.22}$$

because of

$$\begin{vmatrix} -1 & 1 & 2m & 0 & 1 & -1 \\ 1 & 1 & 0 & 2m & 1 & 1 \\ 1 & 2m & 1 & 1 & 0 & -1 \\ -1 & 0 & 1 & 1 & 2m & 1 \\ 2m & 1 & -1 & 1 & -1 & 0 \\ 0 & -1 & -1 & 1 & 1 & 2m \end{vmatrix} = 64\,(m-1)^3\,(m+1)^3 = 0\,.$$

Now, in our complex coordinate system, (b.1) can be written as

$$(5.23)\quad K_{\alpha\beta} = A_{\beta\bar\alpha} + B_{\alpha\beta} + 2\,m\,C_{\alpha\beta} + Dg_{\alpha\beta} + E_{\alpha\beta} + H_{\beta\bar\alpha} + i\,P_{\beta\bar\alpha} - i\,Q_{\alpha\beta}$$
$$- i\,Sg_{\beta\bar\alpha} - i\,T_{\alpha\beta} + i\,V_{\beta\bar\alpha}\,,$$

$$(5.24)\quad K_{\beta\bar\alpha} = A_{\alpha\beta} + B_{\beta\bar\alpha} + 2\,m\,C_{\beta\bar\alpha} + Dg_{\beta\bar\alpha} + E_{\beta\bar\alpha} + H_{\alpha\beta} - i\,P_{\alpha\beta} + i\,Q_{\beta\bar\alpha}$$
$$+ i\,Sg_{\alpha\beta} + i\,T_{\beta\bar\alpha} - i\,V_{\alpha\beta}\,.$$

Forming (5.23) + (5.24) and (5.23) − (5.24), we have

$$(c.1)_1\quad K_{\beta\bar\alpha} = (A) + (B) + 2\,m\,(C) + Dg_{\beta\bar\alpha} + (E) + (H) + i\,[P] + i\,[Q]$$
$$+ i\,[T] + i\,[V]\,,$$

$$(c.1)_2\quad 0 = [A] - [B] - 2\,m\,[C] - [E] + [H] + i\,(P) - i\,(Q) - i\,Sg_{\beta\bar\alpha}$$
$$- i\,(T) + i\,(V)$$

respectively, where

$$(A) = \frac{1}{2}\,(A_{\beta\bar\alpha} + A_{\alpha\beta})\quad,\quad [A] = \frac{1}{2}\,(A_{\beta\bar\alpha} - A_{\alpha\beta}),\quad \text{etc.}$$

Similarly, from (b.2), (b.3), ⋯, (b.12), we have the following relations.

$$(c.2)_1\quad K_{\beta\bar\alpha} = (A) + (B) + Cg_{\beta\bar\alpha} + 2\,m\,(D) + (E) + (H) + i\,[P]$$
$$+ i\,[Q] + i\,[T] + i\,[V]\,,$$

$$(c.2)_2\quad 0 = -[A] - [B] - 2\,m\,[D] - [E] - [H] - i\,(P) - i\,(Q)$$
$$- i\,Rg_{\beta\bar\alpha} - i\,(T) - i\,(V)\,,$$

$$(c.3)_1\quad -K_{\beta\bar\alpha} = (A) + 2\,m\,(B) + (C) + (D) + Eg_{\beta\bar\alpha} + (H) - i\,[P] + i\,[R]$$
$$+ i\,[S] - i\,[V]\,,$$

$$(c.3)_2\quad 0 = -[A] - 2\,m\,[B] - [C] - [D] + [H] + i\,(P) - i\,(R) - i\,(S)$$
$$- i\,T_{\beta\bar\alpha} - i\,(V)\,,$$

$$(c.4)_1 \quad -K_{\beta\lambda} = (A) + Bg_{\beta\lambda} + (C) + (D) + 2m(E) + (H) - i[P] + i[R] + i[S] - i[V],$$

$$(c.4)_2 \quad 0 = [A] - [C] - [D] - 2m[E] - [H] - i(P) - iQg_{\beta\lambda} - i(R) - i(S) + i(V),$$

$$(c.5)_1 \quad 0 = 2m(A) + (B) + (C) + (D) + (E) + Hg_{\beta\lambda} - i[Q] - i[R] - i[S] - i[T],$$

$$(c.5)_2 \quad 0 = -2m[A] - [B] + [C] - [D] + [E] + i(Q) - i(R) + i(S) - i(T) - iVg_{\beta\lambda},$$

$$(c.6)_1 \quad 0 = Ag_{\beta\lambda} + (B) + (C) + (D) + (E) + 2m(H) - i[Q] - i[R] - i[S] - i[T],$$

$$(c.6)_2 \quad 0 = [B] + [C] - [D] - [E] - 2m[H] - iPg_{\beta\lambda} - i(Q) - i(R) + i(S) + i(T),$$

$$(c.7)_1 \quad K_{\beta\lambda} = (A) - (B) - (E) + (H) + i[P] - i[Q] - 2mi[R] + \bar{S}g_{\beta\lambda} - i[T] + i[V],$$

$$(c.7)_2 \quad 0 = [A] + [B] + i\bar{D}g_{\beta\lambda} + [E] + [H] + i(P) + i(Q) + 2mi(R) + i(T) + i(V),$$

$$(c.8)_1 \quad K_{\beta\lambda} = (A) - (B) - (E) + (H) + i[P] - i[Q] + \bar{R}g_{\beta\lambda} - 2mi[S] - i[T] + i[V],$$

$$(c.8)_2 \quad 0 = -[A] + [B] + i\bar{C}g_{\beta\lambda} + [E] - [H] - i(P) + i(Q) + 2mi(S) + i(T) - i(V),$$

$$(c.9)_1 \quad -2K_{\beta\lambda} = (B) - (C) - (D) + (E) - 2mi[P] - i[Q] + i[R] + i[S] - i[T] + \bar{V}g_{\beta\lambda},$$

$$(c.9)_2 \quad 0 = -[B] - [C] + [D] + [E] + i\bar{H}g_{\beta\lambda} + 2mi(P) + i(Q) + i(R) - i(S) - i(T),$$

$$(c.10)_1 \quad -2K_{\beta\lambda} = (B) - (C) - (D) + (E) + \bar{P}g_{\beta\lambda} - i[Q] + i[R] + i[S] - i[T] - 2mi[V],$$

$$(c.10)_2 \quad 0 = i\bar{A}g_{\beta\lambda} + [B] - [C] + [D] - [E] - i(Q) + i(R) - i(S) + i(T) + 2mi(V),$$

$$(c.11)_1 \quad -K_{\beta\bar{\alpha}} = (A) - (C) - (D) + (H) - i\,[P] - 2\,mi\,[Q] - i\,[R] - i\,[S] + \tilde{T}g_{\beta\bar{\alpha}} - i\,[V]\,,$$

$$(c.11)_2 \quad 0 = -[A] + [C] + [D] + i\,\tilde{E}g_{\beta\bar{\alpha}} + [H] + i\,(P) + 2\,mi\,(Q) + i\,(R) + i\,(S) - i\,(V)\,,$$

$$(c.12)_1 \quad -K_{\beta\bar{\alpha}} = (A) - (C) - (D) + (H) - i\,[P] + \tilde{Q}g_{\beta\bar{\alpha}} - i\,[R] - i\,[S] - 2\,mi\,[T] - i\,[V]\,,$$

$$(c.12)_2 \quad 0 = [A] + i\,\tilde{B}g_{\beta\bar{\alpha}} + [C] + [D] - [H] - i\,(P) + i\,(R) + i\,(S) + 2\,mi\,(T) + i\,(V)$$

respectively.

Next, forming $(c.5)_1 - (c.6)_1$, we have

$$0 = 2\,m\,(A) - Ag_{\beta\bar{\alpha}} - 2\,m\,(H) + Hg_{\beta\bar{\alpha}}$$

or

$$(A) - \frac{1}{2m}\,Ag_{\beta\bar{\alpha}} - (H) + \frac{1}{2m}\,Hg_{\beta\bar{\alpha}} = 0\,,$$

that is,

$$(d.1) \quad (c.5)_1 - (c.6)_1 \rightarrow (A) - \frac{1}{2m}\,Ag_{\beta\bar{\alpha}} - \left((H) - \frac{1}{2m}\,Hg_{\beta\bar{\alpha}}\right) = 0\,.$$

Similarly, we have

$$(d.2) \quad (c.3)_1 - (c.4)_1 \rightarrow (B) - \frac{1}{2m}\,Bg_{\beta\bar{\alpha}} - \left((E) - \frac{1}{2m}\,Eg_{\beta\bar{\alpha}}\right) = 0\,,$$

$$(d.3) \quad (c.1)_1 - (c.2)_1 \rightarrow (C) - \frac{1}{2m}\,Cg_{\beta\bar{\alpha}} - \left((D) - \frac{1}{2m}\,Dg_{\beta\bar{\alpha}}\right) = 0\,,$$

$$(d.4) \quad (c.9)_1 - (c.10)_1 \rightarrow [P] - \frac{i}{2m}\,\tilde{P}g_{\beta\bar{\alpha}} - \left([V] - \frac{i}{2m}\,\tilde{V}g_{\beta\bar{\alpha}}\right) = 0\,,$$

$$(d.5) \quad (c.11)_1 - (c.12)_1 \rightarrow [Q] - \frac{1}{2m}\,\tilde{Q}g_{\beta\bar{\alpha}} - \left([T] - \frac{i}{2m}\,\tilde{T}g_{\beta\bar{\alpha}}\right) = 0\,,$$

$$(d.6) \quad (c.7)_1 - (c.8)_1 \rightarrow [R] - \frac{i}{2m}\,\tilde{R}g_{\beta\bar{\alpha}} - \left([S] - \frac{i}{2m}\,\tilde{S}g_{\beta\bar{\alpha}}\right) = 0\,,$$

$$(d.7) \quad (c.5)_2 + (c.10)_2 \rightarrow [A] - \frac{i}{2m}\,\tilde{A}g_{\beta\bar{\alpha}} - i\left((V) - \frac{1}{2m}\,Vg_{\beta\bar{\alpha}}\right) = 0\,,$$

$$(d.8) \quad (c.3)_2 + (c.12)_2 \rightarrow [B] - \frac{i}{2m}\,\tilde{B}g_{\beta\bar{\alpha}} - i\left((T) - \frac{1}{2m}\,Tg_{\beta\bar{\alpha}}\right) = 0\,,$$

$$(d.9) \quad (c.1)_2 + (c.8)_2 \rightarrow [C] - \frac{i}{2m}\,\tilde{C}g_{\beta\bar{\alpha}} - i\left((S) - \frac{1}{2m}\,Sg_{\beta\bar{\alpha}}\right) = 0\,,$$

(d.10) $(c.2)_2 + (c.7)_2 \rightarrow [D] - \frac{i}{2m}\bar{D}g_{\beta\alpha} - i\left((R) - \frac{1}{2m}Rg_{\beta\alpha}\right) = 0\,,$

(d.11) $(c.4)_2 + (c.11)_2 \rightarrow [E] - \frac{i}{2m}\bar{E}g_{\beta\alpha} - i\left((Q) - \frac{1}{2m}Qg_{\beta\alpha}\right) = 0\,,$

(d.12) $(c.6)_2 + (c.9)_2 \rightarrow [H] - \frac{i}{2m}\bar{H}g_{\beta\alpha} - i\left((P) - \frac{1}{2m}Pg_{\beta\alpha}\right) = 0\,.$

Thus, making use of (5.22), from (d.7), (d.8), (d.11) and (d.12) we have,

$$i(V) = [A] - \frac{i}{m}\bar{A}g_{\beta\alpha} \quad , \quad i(T) = [B] - \frac{i}{m}\bar{B}g_{\beta\alpha}\,,$$

$$i(Q) = [E] - \frac{i}{m}\bar{E}g_{\beta\alpha} \quad , \quad i(P) = [H] - \frac{i}{m}\bar{H}g_{\beta}$$

respectively.

Substituting these equations and $S = -\bar{C}$ into $(c.1)_2$, we have

(5.25) $$[A] - \frac{i}{2m}\bar{A}g_{\beta\alpha} - \left([B] - \frac{i}{2m}\bar{B}g_{\beta\alpha}\right) - m\left([C] - \frac{i}{2m}\bar{C}g_{\beta\alpha}\right)$$
$$- \left([E] - \frac{i}{2m}\bar{E}g_{\beta\alpha}\right) + \left([H] - \frac{i}{2m}\bar{H}g_{\beta\alpha}\right) = 0\,.$$

Putting

$$\{A\} = [A] - \frac{i}{2m}\bar{A}g_{\beta\alpha} \quad , \quad \{B\} = [B] - \frac{i}{2m}\bar{B}g_{\beta\alpha}, \cdots, \{H\} = [H] - \frac{i}{2m}\bar{H}g_{\beta\alpha}\,,$$

we can write (5.25) as

$$\{A\} - \{B\} - m\{C\} - \{E\} + \{H\} = 0\,.$$

Similarly from $(c.2)_2, (c.3)_2, (c.4)_2, (c.6)_2$ and $(c.8)_2$, we have

$$\{A\} + \{B\} + m\{D\} + \{E\} + \{H\} = 0\,,$$

$$\{A\} + m\{B\} + \{C\} + \{D\} - \{H\} = 0\,,$$

$$\{A\} - \{C\} - \{D\} - m\{E\} - \{H\} = 0\,,$$

$$m\{A\} + \{B\} - \{C\} + \{D\} - \{E\} = 0\,,$$

$$\{B\} + \{C\} - \{D\} - \{E\} - m\{H\} = 0$$

respectively.

Regarding the set of these 6 equations as a system of simultaneous equations with respect to $\{A\}, \{B\}, \{C\}, \{D\}, \{E\}$ and $\{H\}$, we have

$$\{A\} = \{B\} = \{C\} = \{D\} = \{E\} = \{H\} = 0$$

because of

$$\begin{vmatrix} 1 & -1 & -m & 0 & -1 & 1 \\ 1 & 1 & 0 & m & 1 & 1 \\ 1 & m & 1 & 1 & 0 & -1 \\ 1 & 0 & -1 & -1 & -m & -1 \\ m & 1 & -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 & -1 & -m \end{vmatrix} = -(m-2)^3 (m+2)^3 \neq 0 ,$$

or

$$(5.26) \quad [A] - \frac{i}{2m} \bar{A} g_{\beta\bar{\alpha}} = 0 \ , \ [B] - \frac{i}{2m} \bar{B} g_{\beta\bar{\alpha}} = 0 \ , \ [C] - \frac{i}{2m} \bar{C} g_{\beta\bar{\alpha}} = 0 ,$$

$$[D] - \frac{i}{2m} \bar{D} g_{\beta\bar{\alpha}} = 0 \ , \ [E] - \frac{i}{2m} \bar{E} g_{\beta\bar{\alpha}} = 0 \ , \ [H] - \frac{i}{2m} \bar{H} g_{\beta\bar{\alpha}} = 0 .$$

Thus, from (d.7), (d.8), (d.9), (d.10), (d.11) and (d.12), we have respectively

$$(5.27) \quad (V) - \frac{1}{2m} V g_{\beta\bar{\alpha}} = 0 \ , \ (T) - \frac{1}{2m} T g_{\beta\bar{\alpha}} = 0 \ , \ (S) - \frac{1}{2m} S g_{\beta\bar{\alpha}} = 0 ,$$

$$(R) - \frac{1}{2m} R g_{\beta\bar{\alpha}} = 0 \ , \ (Q) - \frac{1}{2m} Q g_{\beta\bar{\alpha}} = 0 \ , \ (P) - \frac{1}{2m} P g_{\beta\bar{\alpha}} = 0 .$$

On the other hand, forming $(c.5)_1 + (c.6)_1$, we have

$$(5.28) \quad 0 = m((A) + (H)) + (B) + (C) + (D) + (E) + \frac{1}{2}(A + H) g_{\beta\bar{\alpha}} - i([Q] + [R] + [S] + [T])$$

and making use of (5.10) and (5.20),

$$(5.29) \quad (A) + (B) + (C) + (D) + (E) + (H) = 0 , \quad A + H = 0 .$$

We can reduce (5.28) to

$$(5.30) \quad 0 = (m-1)((A) + (H)) - i([Q] + [R] + [S] + [T]) .$$

Similarly forming $(c.7)_1 + (c.8)_1 + (c.11)_1 + (c.12)_1$, we have

$$0 = 2((A) + (H)) - ((B) + (C) + (D) + (E)) + \frac{1}{2}(\bar{Q} + \bar{R} + \bar{S} + \bar{T}) g_{\beta\bar{\alpha}} - i(m+1)([Q] + [R] + [S] + [T])$$

and making use of (5.10) and (5.21), we can reduce the last equation to

(5.31) $$0 = 3\,((A) + (H)) - i\,(m+1)\,([Q] + [R] + [S] + [T]) \,.$$

Forming $(m+1) \times (5.30) - (5.31)$, we have

$$0 = (m^2 - 4)\,((A) + (H))$$

or

(5.32) $$(A) + (H) = 0 \,.$$

Hence, from (5.29) and (5.30), we have respectively

(5.33) $$(B) + (C) + (D) + (E) = 0 \,, \quad [Q] + [R] + [S] + [T] = 0 \,.$$

Consequently, making use of $(A) + (H) = 0$ and $A + H = 0$, we have from (d.1)

(5.34) $$(A) - \frac{1}{2m} A g_{\beta\bar{\alpha}} = 0 \quad , \quad (H) - \frac{1}{2m} H g_{\beta\bar{\alpha}} = 0 \,.$$

Making use of $(B) + (C) = -(D) - (E)$ and $B + C = -D - E$, we have from (d.2) and (d.3)

(5.35) $$(B) + (C) = \frac{1}{2m} (B + C)\, g_{\beta\bar{\alpha}} \quad , \quad (D) + (E) = \frac{1}{2m} (D + E)\, g_{\beta\bar{\alpha}} \,.$$

From $(c.1)_1 + (c.4)_1$, we have

$$0 = 2\,((A) + (H)) + (B) + (C) + (D) + (E) + 2m\,((C) + (E)) + (B + D)\, g_{\beta\bar{\alpha}} + i\,([Q] + [R] + [S] + [T])$$

or making use of (5.32) and (5.33),

$$2m\,((C) + (E)) + (B + D)\, g_{\beta\bar{\alpha}} = 0 \,,$$

that is,

$$(C) + (E) = -\frac{1}{2m} (B + D)\, g_{\beta\bar{\alpha}}$$

or since $B + D = -C - E$,

(5.36) $$(C) + (E) = \frac{1}{2m} (C + E)\, g_{\beta\bar{\alpha}} \,.$$

Now, from (5.26) we have

$$A_{\beta\bar{\alpha}} - A_{\bar{\alpha}\beta} - \frac{i}{m} \bar{A} g_{\beta\bar{\alpha}} = 0 \,.$$

But since

$$\bar{A} = F^{ji} A_{ji} = -ig^{\beta\bar{\alpha}} A_{\beta\bar{\alpha}} + ig^{\alpha\bar{\beta}} A_{\alpha\bar{\beta}} = -i(A_1 - A_2),$$

the last equation becomes

$$A_{\beta\bar{\alpha}} - A_{\alpha\bar{\beta}} - \frac{1}{m}(A_1 - A_2)\, g_{\beta\bar{\alpha}} = 0 . \tag{5.37}$$

On the other hand, from (5.34), we have

$$A_{\beta\bar{\alpha}} + A_{\alpha\bar{\beta}} - \frac{1}{m}(A_1 + A_2)\, g_{\beta\bar{\alpha}} = 0 . \tag{5.38}$$

Hence, forming (5.37) + (5.38), we have (5.1) (i), that is,

$$A_{\beta\bar{\alpha}} = \frac{1}{m} A_1 g_{\beta\bar{\alpha}}$$

and similarly

$$H_{\beta\bar{\alpha}} = \frac{1}{m} H_1 g_{\beta\bar{\alpha}} .$$

Next from (5.26), taking account of $i\bar{C} = C_1 - C_2$ and $i\bar{E} = E_1 - E_2$, we have

$$C_{\beta\bar{\alpha}} - C_{\alpha\bar{\beta}} = \frac{1}{m}(C_1 - C_2)\, g_{\beta\bar{\alpha}} \quad , \quad E_{\beta\bar{\alpha}} - E_{\alpha\bar{\beta}} = \frac{1}{m}(E_1 - E_2)\, g_{\beta\bar{\alpha}}$$

and therefore

$$C_{\beta\bar{\alpha}} - C_{\alpha\bar{\beta}} - (E_{\beta\bar{\alpha}} - E_{\alpha\bar{\beta}}) = \frac{1}{m}(C_1 - C_2 - E_1 + E_2)\, g_{\beta\bar{\alpha}} \tag{5.39}$$

and by (5.36)

$$C_{\beta\bar{\alpha}} + C_{\alpha\bar{\beta}} + E_{\beta\bar{\alpha}} + E_{\alpha\bar{\beta}} = \frac{1}{m}(C_1 + C_2 + E_1 + E_2)\, g_{\beta\bar{\alpha}} . \tag{5.40}$$

Hence forming (5.40) — (5.39), we have (5.1) (ii), that is,

$$C_{\alpha\bar{\beta}} + E_{\beta\bar{\alpha}} = \frac{1}{m}(C_2 + E_1)\, g_{\beta\bar{\alpha}} . \tag{5.41}$$

Similarly from (5.26) and (5.33), we have

$$D_{\beta\bar{\alpha}} - D_{\alpha\bar{\beta}} + E_{\beta\bar{\alpha}} - E_{\beta\bar{\alpha}} = \frac{1}{m}(D_1 - D_2 + E_1 - E_2)\, g_{\beta\bar{\alpha}} , \tag{5.42}$$

$$D_{\beta\bar{\alpha}} + D_{\alpha\bar{\beta}} + E_{\beta\bar{\alpha}} + E_{\alpha\bar{\beta}} = \frac{1}{m}(D_1 + D_2 + E_1 + E_2)\, g_{\beta\bar{\alpha}} . \tag{5.43}$$

respectively and consequently forming (5.42) + (5.43), we have

$$D_{\beta\bar{\alpha}} + E_{\beta\bar{\alpha}} = \frac{1}{m}(D_1 + E_1)\, g_{\beta\bar{\alpha}}.$$

Finally, for (5.1) (iii), from (d.4) and (5.27), we have

$$(5.44) \qquad P_{\beta\bar{\alpha}} - P_{\alpha\bar{\beta}} - (V_{\beta\bar{\alpha}} - V_{\alpha\bar{\beta}}) = \frac{1}{m}(P_1 - P_2 - V_1 + V_2)\, g_{\beta\bar{\alpha}},$$

$$(5.45) \qquad P_{\beta\bar{\alpha}} + P_{\alpha\bar{\beta}} - (V_{\beta\bar{\alpha}} + V_{\alpha\bar{\beta}}) = \frac{1}{m}(P_1 + P_2 - V_1 - V_2)\, g_{\beta\bar{\alpha}}$$

respectively and consequently forming (5.44) + (5.45), we have

$$P_{\beta\bar{\alpha}} - V_{\beta\bar{\alpha}} = \frac{1}{m}(P_1 - V_1)\, g_{\beta\bar{\alpha}}.$$

Similarly, from (d.6) and (5.27), we have

$$(5.46) \qquad R_{\beta\bar{\alpha}} - R_{\bar{\alpha}\beta} - (S_{\beta\bar{\alpha}} - S_{\alpha\bar{\beta}}) = \frac{1}{m}(R_1 - R_2 - S_1 + S_2)\, g_{\beta\bar{\alpha}},$$

$$(5.47) \qquad R_{\beta\bar{\alpha}} + R_{\bar{\alpha}\beta} + (S_{\beta\bar{\alpha}} + S_{\alpha\bar{\beta}}) = \frac{1}{m}(R_1 + R_2 + S_1 + S_2)\, g_{\beta\bar{\alpha}}$$

respectively and consequently forming (5.47) — (5.46), we have

$$R_{\alpha\bar{\beta}} + S_{\beta\bar{\alpha}} = \frac{1}{m}(R_2 + S_1)\, g_{\beta\bar{\alpha}}.$$

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