SEppo Heikkilä(*)

On mild and strict solutions
of nonlinear functional differential equations (**) 

Summary. — This paper deals with mild and strict solutions to a nonlinear perturbed functional differential equation, of parabolic or of hyperbolic type, in a Banach space. Some results concerning existence and uniqueness of these solutions are derived. Their bounds and dependence on initial mapping and parameter are investigated by a comparison method with minimal solutions of comparison equations as estimating functions.

1. Introduction

This paper investigates the nonlinear functional differential equation

\[ x'(t) + A(t)x(t) = f(t, x_t) \]

in a Banach space. In case (1) is of certain parabolic or of hyperbolic type a Bompiani-Walter type existence and uniqueness theorem is verified for mild solutions of (1). The method of successive approximations is used, as in [2] for the equation

\[ x'(t) = f(t, x_t) \]

Furthermore, some estimates are derived for solutions of (1), with minimal solutions of scalar integral equations as estimators. Also strict solutions of (1) in the parabolic case and mild solutions of

\[ x'(t) + Ax(t) = f(t, x_t) \]

where \( A \) is the infinitesimal generator of a strongly continuous semigroup in a Banach space, are studied.

All the solutions mentioned above satisfy an integral equation of the form

\[ x(t) = y(t) + \int_0^T U(t^*, s)f(s, x_s)\, ds, \quad -\tau \leq t \leq T, \]

where \( t^* = \max \{t, 0\} \), whence most of the results obtained for these solutions will be first derived for solutions of (4).

(*) University of Oulu – Dept. of Applied Mathematics and Statistics – Finland.
2. Notations

Let $X$ be a Banach space with norm $\|\cdot\|$ and $B(X)$ be the Banach space of all bounded linear maps on $X$, with the norm of uniform operator topology. Denote by $C(Z)$ and $C_+(Z)$ the sets of all continuous mappings from a topological space $Z$ into $X$ and into the nonnegative reals $\mathbb{R}_+$, respectively. In case $Z = [a, b], a \leq b$, these sets are endowed with the uniform topology. The uniform norm of $\mathcal{U} = C[-\tau, 0], \tau \geq 0$, is denoted by $\|\cdot\|_{\mathcal{U}}$. Given $J = [0, T], T > 0$, and $u, v \in C_+(J)$, define

$$u \leq v \quad \text{iff} \quad u(t) \leq v(t) \quad \text{for each} \quad t \in J,$$

and for $y \in C[-\tau, T]$ and $t \in J$ define $y_t \in \mathcal{C}$ by

$$y_t(s) = y(t + s), \quad -\tau \leq s \leq 0.$$

Let $U : [0 \leq s \leq t \leq T] \to B(X)$ be given and assume that $U(t, s)$ is strongly continuous, i.e., $(t, s) \mapsto U(t, s)x$ is continuous for each $x \in X$. The Uniform Boundedness Theorem implies that the constant $\alpha$ given by

$$(2.1) \quad \alpha = \sup \{ \|U(t, s)\| \mid 0 \leq s \leq t \leq T \}$$

is finite. Given $f \in C(J \times \mathcal{U})$ we can define a mapping $F$ on $C[-\tau, T]$ by

$$(2.2) \quad Fx(t) = \int_0^{t+} U(t^+, s) f(s, x_s) \, ds, \quad -\tau \leq t \leq T.$$

In view of this definition (4) can be written in the form

$$x(t) = y(t) + Fx(t).$$

In (2.2) $t^+$ denotes $\max \{ t, 0 \}$, and analogous to that we shall denote $t^- = \min \{ t, 0 \}$.

3. An existence and uniqueness theorem

In this section $U$ is assumed to be a mapping from $[0 \leq s \leq t \leq T]$ into $B(X)$ such that $U(t, s)$ is strongly continuous. $\alpha$ denotes the constant defined by (2.1).

Theorem 3.1. Assume that $f \in C(J \times \mathcal{U})$ and for $(t, \varphi), (t, \overline{\varphi}) \in J \times \mathcal{U}$

$$(3.1) \quad \|f(t, \varphi) - f(t, \overline{\varphi})\| \leq g(t, \|\varphi - \overline{\varphi}\|_{\mathcal{U}})$$

where $g \in C_+(J \times \mathbb{R}_+)$ satisfying

(i) $g(t, r)$ is nondecreasing in $r \in \mathbb{R}_+$ for each $t \in J$;
(ii) for each \( v \in C(J) \) the integral equation

\[
 u(t) = v(t) + \int_0^t g(s, u(s)) \, ds
\]

has a solution on \( J \);

(iii) \( u(t) = 0 \) is the only solution of (3.2) with \( v(t) = 0 \).

Then for each \( y \in C[-\tau, T] \) the successive approximations

\[
x^{n+1} = y + Fx^n, \quad n \in \mathbb{N} = \{1, 2, \ldots\},
\]

with \( F \) given by (2.2) and with any \( x^i \in C[-\tau, T] \) as the first approximation, converge on \( [-\tau, T] \) uniformly to a unique solution of (4).

Proof. Let \( x^i \in C[-\tau, T] \) be given. From (3.1) one can easily deduce that

\[
\| Fx_i - F\bar{x}_i \|_0 \leq \alpha \int_0^t g(t, \| \bar{x}_i - \bar{x} \|_0) \, ds, \quad t \in J,
\]

whenever \( x, \bar{x} \in C[-\tau, T] \), whence for each \( n \in \mathbb{N} \) and \( t \in J \)

\[
\| x^{n+1} - x^n \|_0 \leq v(t) + \alpha \int_0^t g(s, \| x^n - x^n \|_0) \, ds
\]

where

\[
v(t) = \| y_i - x^n \|_0 + \alpha \int_0^t \| f(s, x^n) \| \, ds.
\]

Applying (i) one can obtain from (3.5) by induction that

\[
\| x^{n+1} - x^n \|_0 \leq u(t), \quad t \in J,
\]

where \( u \) is any solution of (3.2) with \( v \) given by (3.6).

The uniform convergence of the sequence \( (x^n) \) to a solution of (4) and the uniqueness of this solution can then be proved, by the method of Walter, as in [2] in case of the equation (2).

**Remarks 3.1.** A local existence and uniqueness theorem is obtained without the hypothesis (ii). On the other hand, (ii) and the continuity of \( g \) can be replaced by the Caratheodory conditions (cfr. [1], p. 665). Similarly, instead of \( f \) being continuous it suffices that \( f(t, \varphi) \) is continuous in \( \varphi \in \mathcal{C} \) for almost every \( t \in J \), \( f(t, y) \) is strongly measurable in \( t \in J \) for each \( y \in C[-\tau, T] \), and that for each \( M > 0 \) there is a Lebesgue integrable function \( h: J \to \mathbb{R}^+ \) such that

\[
\| f(t, \varphi) \| \leq h(t).
\]
whenever \((t, \varphi) \in J \times \mathcal{C}, \| \varphi \|_0 \leq M\). These hypotheses and the Dominated Convergence Theorem for Bochner integrals imply that the integral in (2.2) exists in the Bochner sense whenever \(x \in C[\tau, T]\), and that the uniform limit of the successive approximations \(x^n\) is a solution of (4).

If the equation (3.2) has for \(v(t) = 0\) the zero function as the only solution when \(\alpha = 1\), the same holds trivially also when \(0 \leq \alpha < 1\). But this is no longer true when \(\alpha > 1\) as we see from the following example \(^{(1)}\).

Given \(\alpha > 1\), define \(g \in C_+ (J \times \mathbb{R})\) by

\[
g(t, r) = \begin{cases} \frac{2t}{\alpha} & \text{for } r \geq t, t \in J; \\ \frac{2r}{\alpha t} & \text{for } 0 \leq r < t, 0 < t \leq T. \end{cases}
\]

It is easy to show that \(u(t) = 0\) is the only solution of (3.2) when \(\alpha = 1\) and \(v(t) = 0\), whereas

\[
u(t) = Ce^t, \quad t \in J,
\]

is for each \(C \in [0, 1]\) a solution of (3.2) when \(\alpha = \alpha\) and \(v(t) = 0\) (cfr. [1], p. 676).

4. INEQUALITIES

Let the mapping \(U\) be as in the preceding section, and let \(x\) denote the solution of (4), with the given \(y \in C[\tau, T]\).

**Lemma 4.1.** Let \(g \in C_+ (J \times \mathbb{R})\) satisfy the hypotheses (i) and (ii) of Theorem 3.1, and let \(v \in C_+ (J)\) be given. Then the equation (3.2) has the minimal solution \(u\). If \((y^n)\) is a sequence in \(C[\tau, T]\) such that \(y^n \to y\) uniformly on \([\tau, T]\) and that for each \(t \in J\)

\[
\| y^n_t \|_0 \leq v(t) \quad \text{and} \quad \| y^{n+1}_t \|_0 \leq v(t) + \int_{\tau}^{t} g(s, \| y^n_s \|_0) \, ds, \quad n \in \mathbb{N},
\]

then

\[
\| y_t \|_0 \leq u(t), \quad t \in J.
\]

**Proof.** The existence of the minimal solution \(u\) of (3.2) is trivially verified by the method of successive approximations (cfr. [2], Lemma 2.1). By the monotonicity of \(g\), the sequence \((y^n)\), satisfying (4.1), is bounded above by \(u\), as we see by induction. This implies (4.2).

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\(^{(1)}\) This example is suggested by Dr. J. Tienari.
As a consequence of Theorem 3.1 and Lemma 4.1 we have

**Theorem 4.1.** With the hypotheses of Theorem 3.1, and for \( z \in C \left[ -\tau, T \right] \) and \( \tau \in C, (J) \) satisfying

\[
\| x_t - y_t - Fz_t \|_0 \leq \psi(t), \quad t \in J,
\]

we have

\[
\| x_t - x_0 \|_0 \leq u(t), \quad t \in J,
\]

where \( u(t) \) is the minimal solution of (3.2).

**Proof.** Let \((x^*)\) be the sequence of the successive approximations, with \( x^0 = y + Fz \) as the first approximation, converging to \( x \). Then it is easy to verify that the conditions (4.1) are satisfied by \( y^* = x^0 \), whence (4.4) follows from Lemma 4.1.

**Corollary 4.1.** If the hypotheses of Theorem 3.1 hold and if \( \bar{x} \) and \( x \) are the solutions of (4) with given \( y \) and \( \bar{y} \) from \( C \left[ -\tau, T \right] \), respectively, then

\[
\| \bar{x}_t - x_t \|_0 \leq u(t), \quad t \in J,
\]

where \( u(t) \) is the minimal solution of (3.2) with \( \psi(t) = \| x_t - x_0 \|_0 \).

**Proof.** Choose \( z = \bar{x} \) in Theorem 4.1.

**Theorem 4.2.** Assume that the hypotheses of Theorem 3.1 hold, and that for some \( z \in C \left[ -\tau, T \right] \)

\[
\| f(t, \varphi) \| \leq q(t, \| \varphi - z_t \|_0), \quad (t, \varphi) \in J \times \mathbb{R},
\]

where \( q \in C(J \times \mathbb{R}_+) \) satisfying the hypotheses (i) and (ii) given for \( g \) in Theorem 3.1. Then

\[
\| x_t - z_t \|_0 \leq w(t), \quad t \in J,
\]

where \( w(t) \) is the minimal solution of

\[
w(t) = \| y_t - z_t \|_0 + \int_0^t q(s, w(s)) \, ds.
\]

**Proof.** (4.1) holds for \( y^* = x^0 - z, x^1 = y, g = q \) and \( \psi(t) = \| x_t - z_t \|_0, t \in J \).

An example of evaluation functions \( q \) in Theorem 4.2 is

\[
q(t, s) = \| f(t, z_t) \| + zg(t, s), \quad (t, s) \in J \times \mathbb{R}_+,
\]

where \( g \) is the evaluation function in Theorem 3.1. With the choices \( z = 0 \) and \( z = y \), we get estimates on the norm of \( x \) and on its distance from \( y \), respectively.
Theorem 4.3. Assume that $f$ depends on a parameter belonging to a non-empty set $Y$, and satisfies the hypotheses of Theorem 3.1 for given $\mu, \bar{\mu} \in Y$. If $x$ and $\bar{x}$ denote the corresponding solutions of (4) with the same $y \in C [-\tau, T]$, then

$$
\|x_t - \bar{x}_t\| \leq u(t), \quad t \in J,
$$

where $u$ is the minimal solution of (3.2) with:

$$
\nu(t) = a \int_0^t \|f(s, \bar{x}_s, \mu) - f(s, x_s, \mu)\| \, ds.
$$

Proof. Let $(x^n)$ denote the sequence of the successive approximations, with $x^0 = y + F\bar{x}$ as the first approximation, converging to $x$. Then (4.1) holds for $y^n = x^n - \bar{x}$.

Remarks 4.1. The results derived in this section hold also under the weaker hypotheses given in Remarks 3.1 for $f$ and $g$. For $q$ the hypotheses of $g$, without (iii), are sufficient.

The closed interval $J = [0, T]$ can be replaced in this and in preceding section by $[0, T], 0 < T \leq \infty$, provided that $U$ is assumed to be bounded in $\{0 \leq s \leq t < T\}$. The convergence of the successive approximations $x^n$ is then uniform on each compact subset of $[-\tau, T]$. In particular, if the hypotheses of Theorem 4.2 hold with $J = [0, T]$, and if the minimal solution of (4.8) is bounded for bounded $y \in C [-\tau, T]$, then also the solution $x$ of (4) is bounded. If further $y$ and $x$ have limits from the left as $t$ tends to $T$, the same holds also for $x$.

To see that the boundedness of the minimal solution of (4.8) is a weaker condition than the boundedness of the corresponding maximal solution, define

$$
q(t, r) = \begin{cases} 
\frac{1}{a} e^{-t} & \text{for } 0 \leq r < 2 (1 - e^{-t}), \\
\frac{1}{a} (e^{-t} + |r - 2 (1 - e^{-t})|) & \text{for } r \geq 2 (1 - e^{-t}), \quad t \geq 0.
\end{cases}
$$

This $q$ has the properties required in Theorem 4.2, and the integral equation

$$
\nu(t) = 1 - e^{-t} + a \int_0^t q(s, \nu(s)) \, ds
$$

has a bounded minimal solution

$$\nu_+ (t) = 2 (1 - e^{-t}),$$

but an unbounded maximal solution

$$\nu^* (t) = 2 (1 - e^{-t}) + \frac{t^2}{4}.$$
5. ON MILD SOLUTIONS

Let \{A(t) \mid t \in J\} be a family of closed linear operators from a dense subset of X into X. Assume that

1) there exist \(0 < \alpha \in \left(\frac{\pi}{2}, \pi\right)\) and \(C > 0\) such that \([\lambda I + A(t)]^{-1}\) exists, belongs to \(B(X)\), and \(\| [\lambda I + A(t)]^{-1} \| \leq \frac{C}{1 + |\lambda|} \) whenever \(t \in J\) and \(\lambda\) is a complex number for which \(|\arg \lambda| \leq \alpha\) (I denotes the identity operator of X).

The operators \(-A(t)\) form then a parabolic family in the sense that \(-A(t)\) is for each \(t \in J\) the infinitesimal generator of an analytic semigroup (cfr. [3], Remark 3.1.1).

Assume further that

2) \(t \mapsto A(t) (A(x))^{-1}\) is for each \(x \in J\) a uniformly Hölder continuous mapping from \(J\) into \(B(X)\); the Hölder constant and exponent being independent of \(x \in J\).

The hypotheses above ensure (see [3], Theorem 3.2.1) that the evolution equation

\[
y'(t) + A(t)y(t) = 0
\]

has a unique fundamental solution \(U : [0 \leq t \leq T] \rightarrow B(X)\), and that for each \(x_0 \in X\) the mapping \(t \mapsto U(t, 0)x_0\) is the unique solution of \((1')\) with the value \(x_0\) at \(t = 0\). This motivates us to call the mapping \(y \in C[-\tau, T]\), defined by

\[
y(t) = \sum_{\sigma = 0}^{\infty} \frac{\tau^\sigma}{\sigma!} y_n(t - \tau), \quad \text{for} \quad -\tau \leq t \leq T,
\]

the solution of \((1')\) with the given \(\varphi_0 \in X\) as the initial mapping.

By a **mild solution** of the functional differential equation

\[
x'(t) + A(t)x(t) = f(t, x(t))
\]

with the given \(\varphi_0 \in X\) as the initial mapping, we mean a solution of the integral equation

\[
x(t) = U(t, 0) \varphi_0(t^*) + \int_0^T U(t, s)f(x, s) \, ds, \quad \text{for} \quad -\tau \leq t \leq T.
\]

From \((5.1)\) and \((5.2)\) it follows that the study of mild solutions of \((1)\) is reduced to the study of \((4)\). Furthermore, the fundamental solution \(U(t, x)\) of \((1')\) is strongly continuous (cfr. [3], Lemma 3.4.3), whence

all the results derived in sections 3 and 4 for solutions of \((4)\) hold also for mild solutions of \((1)\), when \(y\) in \((4)\) is considered as the solution of \((1')\) with a given initial mapping, and \(U\) as the fundamental solution of \((1')\).
Particularly we have

**Theorem 5.1.** Assume \( f \in C(J \times \mathbb{R}) \) and for \((t, \varphi), (t, \overline{\varphi}) \in J \times \mathbb{R}\)

\[
\| f(t, \varphi) - f(t, \overline{\varphi}) \| \leq g(t, \| \varphi - \overline{\varphi} \|)
\]

where \( g \in C(J \times \mathbb{R}) \), \( g(t, r) \) is nondecreasing in \( r \in \mathbb{R} \), for each \( t \in J \), the initial value problem

\[
u'(t) = a(g(t, u(t))) \quad , \quad u(0) = r_0, \quad \text{with } a \text{ given by (2.1)},
\]

has for each \( r_0 \geq 0 \) a solution on \( J \), and \( u(t) = 0 \) is the only solution of (5.3) with \( r_0 = 0 \). Then for each \( \varphi_0 \in \mathbb{K} \) the functional differential equation (1) has a unique mild solution \( x \) on \([-\tau, T]\) with \( \varphi_0 \) as the initial mapping. Moreover, \( x \) depends continuously on \( \varphi_0 \).

**Proof.** Let \( \varphi_0 \in \mathbb{K} \) be given. It is easy to see that the hypotheses of Theorem 3.1 hold for \( f \) (cfr. [2], Lemma 2.1). This proves the existence and uniqueness of the mild solution \( x \) of (1) with \( \varphi_0 \) as the initial mapping.

Let \( \overline{\varphi}_0 \) be another initial mapping from \( \mathbb{K} \) and \( x \) be the corresponding mild solution of (1). If \( y \) and \( \overline{x} \) denote the solutions of (1) with \( \varphi_0 \) and \( \overline{\varphi}_0 \) as the initial mappings, respectively, then it follows from (5.1) that for each \( t \in J \)

\[
\| y_t - \overline{x}_t \| \leq \| \varphi_0 - \overline{\varphi}_0 \| + x \| \varphi_0(0) - \overline{\varphi}_0(0) \| .
\]

This, together with Corollary 4.1, implies that

\[
\| x_t - \overline{x}_t \| \leq u(t) \leq u(T), \quad t \in J,
\]

where \( u \) is the minimal solution of (5.3) with

\[
r_0 = \| \varphi_0 - \overline{\varphi}_0 \| + x \| \varphi_0(0) - \overline{\varphi}_0(0) \| .
\]

If now \( \overline{\varphi}_0 \to \varphi_0 \) in \( \mathbb{K} \), then \( r_0 \to 0 \) and, since the zero function is the only solution of (5.3) with \( r_0 = 0 \), then

\[
u(T) \to 0 \quad \text{as } \overline{\varphi}_0 \to \varphi_0
\]

(cfr. [7], Theorem II.8.VIII). Hence, by (5.4), \( \overline{x}(t) \to x(t) \) uniformly on \([-\tau, T]\) as \( \overline{\varphi}_0 \to \varphi_0 \) in \( \mathbb{K} \).

From Theorem 4.3 we obtain

**Theorem 5.2.** Let \( Y \) be a metric space and assume that \( f \in C(J \times \mathbb{R} \times Y) \) and \( f \) satisfies for each fixed \( \mu \in Y \) the hypotheses of Theorem 5.1. Then the mild solution of (1), with the given initial mapping, depends continuously on the parameter \( \mu \).

**Proof.** Let \( x \) and \( \overline{x} \) denote the mild solutions of (1) corresponding to given \( \varphi_0 \in \mathbb{K} \) and \( \mu, \overline{\mu} \in Y \). Since the set \( \{(t, \overline{x}, \mu) | 0 \leq t \leq T\} \) is compact in the product space \( J \times \mathbb{K} \times Y \), where \( f \) is continuous, one can show by elementary analysis (cfr. [5], p. 34) that \( \overline{v} \), given by (4.10), tends to zero uniformly on \( J \) as \( \overline{\mu} \) tends to \( \mu \) in \( Y \). The conclusion follows from (4.9).
As a consequence of Theorem 5.1 we get

**Corollary 5.1.** The conclusions of Theorem 5.1 hold if \( f \in C (J \times \mathbb{C}) \) and if for all \((t, \varphi), (t, \overline{\varphi}) \in J \times \mathbb{C}\)

\[
\| f(t, \varphi) - f(t, \overline{\varphi}) \| \leq p(t) \varphi (\| \varphi - \overline{\varphi} \|) 
\]

(5.6)

where \( p \in C_+ (J), \varphi \in C_+ (\mathbb{R}_+), \varphi \) is nondecreasing, and the integrals \( \int_1^t \frac{dr}{\varphi (r)} \)

and \( \int_1^\infty \frac{dr}{\varphi (r)} \) diverge.

**Proof.** The hypotheses of Theorem 5.1 hold with

\( g(t, r) = p(t) \varphi (r) \) for \((t, r) \in J \times \mathbb{R}_+ \).

The most important special case of Corollary 5.1 is obtained when \( \varphi \) is the identity mapping on \( \mathbb{R}_+ \), in which case the Osgood condition (5.6) is reduced to the Lipschitz condition.

By Remarks 3.1 it suffices that \( p \) in (5.6) is a Lebesgue integrable function from \( J \) into \( \mathbb{R}_+ \).

6. **ON STRICT SOLUTIONS**

By a **strict solution** of (1) with the given initial mapping \( \varphi_0 \in \mathbb{C} \) we mean a mapping \( x \in C [-\tau, T] \) which equals to \( \varphi_0 \) on \([-\tau, 0]\) and which is strongly continuously differentiable and satisfies (1) on \( (0, T) \).

**Theorem 6.1.** Let \( \{ \Lambda(t) \mid t \in J \} \) be a family of operators in \( B (X) \) satisfying the hypotheses 1) and 2) (p. 7) and being uniformly bounded over \( t \in J \). Assume further that \( f \) satisfies the hypotheses of Theorem 5.1. Then for each \( \varphi_0 \in \mathbb{C} \) the equation (1) has on \([-\tau, T]\) a unique strict solution with \( \varphi_0 \) as the initial mapping.

**Proof.** By Theorem 5.1, (1) has a unique mild solution \( x \) with the given initial mapping \( \varphi_0 \in \mathbb{C} \). In view of Lemmas 3.5.1 and 3.5.2 of [3], with \( u_0 = \varphi_0 (0) \) and \( f(t) = f(t, x_t) \), one can deduce that \( x \) is also a strict solution of (1). The uniqueness follows from Theorem 5.1, because for any strict solution \( x \) of (1) the mapping \( t \mapsto f(t, x_t) \) is continuous, whence \( x \) is also a mild solution of (1) (cfr. [4], Vol. II, p. 250).

If the operators \( \Lambda(t) \) are not assumed to be bounded we have

**Theorem 6.2.** Let \( \Lambda(t), t \in J \) be closed linear operators from a dense subset \( D \) of \( X \) into \( X \) satisfying the hypotheses 1) and 2) and

3) \( \lim \sup \| \Lambda(t) x(t) \| < \infty \) for \( x \in C (J), \) \( \text{Im} \ x \subset D \).
Assume further that \( f \) is a mapping from \( J \times \mathcal{C} \) into \( X \) and for all \( (t, \varphi), (t', \varphi') \in J \times \mathcal{C} \)

\[
\| f(t, \varphi) - f(t', \varphi') \| \leq \check{\psi} (|s-t| + \| \varphi - \varphi' \|_0)
\]

where \( \check{\psi} \in C_+(\mathbb{R}_+) \), \( \check{\psi} \) is nondecreasing, the integrals \( \int_0^t \frac{dr}{\check{\psi}(r)} \) and \( \int_t^a \frac{dr}{\check{\psi}(r)} \) diverge, and the integral \( \int_0^t \frac{\check{\psi}(r)}{r} \, dr \) converges. Then to each Lipschitz continuous initial mapping \( \varphi_0 : [-\tau, 0] \to X \) the equations (1) has on \([-\tau, T]\) a unique and Lipschitz continuous strict solution.

**Proof.** Let \( \varphi_0 : [-\tau, 0] \to X \) be a Lipschitz continuous mapping. From (6.1) it follows that \( f \) is continuous and satisfies the Osgood condition (5.6) with \( \rho(t) = 1 \). Thus by Corollary 5.1 the equation (1) has a unique mild solution \( x \) with \( \varphi_0 \) as the initial mapping.

The additional hypothesis 3), the continuous differentiability of the solution \( y \) of (1') on \((0, T] \), the Lipschitz continuity of \( \varphi_0 \) the boundedness of \( U \) and the continuity of \( t \to f(t, x_t) \) imply that \( x \) is Lipschitz continuous on \([-\tau, T] \). In particular, there is \( k > 0 \) such that

\[
\| x_t - x_s \|_0 \leq k |t - s|, \quad 0 \leq s < t \leq T.
\]

From the proof of Theorem 3.2.2 in [3] one can easily deduce that \( x \) is a strict solution of (1) if the integral \( \int_a^t \| f(s, x_s) - f(t, x_t) \| \, ds \) converges for \( t \in J \). This condition holds, since by (6.1) and (6.2)

\[
\| f(s, x_s) - f(t, x_t) \| \leq \check{\psi} \left( \frac{(k+1)}{|t-s|} |s-t| \right), \quad 0 \leq s < t \leq T,
\]

and since the integral \( \int_0^t \frac{\check{\psi}(r)}{r} \, dr \) converges. Thus the mild solution \( x \) of (1) is also a strict solution. The uniqueness can be verified as in Theorem 6.1.

**Remark 6.1.** The hypotheses of Theorem 6.2 for \( f \) hold particularly if \( f \) is Lipschitz continuous in both its arguments, i.e. if (6.1) holds with

\[
\check{\psi}(r) = Mr \quad (M > 0).
\]

The same hypotheses are valid also when \( \check{\psi} \) in (6.1) is given by

\[
\check{\psi}(0) = 0; \quad \check{\psi}(r) = Mr, \quad r > 1/e;
\]

\[
\check{\psi}(r) = Mr \prod_{n=1}^\infty \log \left( \frac{1}{r} \right), 1/exp_\ast (e) < r \leq 1/exp_\ast (1), \quad n \in \mathbb{N},
\]
where \( \log \) and \( \exp \) denote \( n \)-fold iterated logarithm and exponential functions, respectively. This can be verified by elementary analysis.

A counter-example. The hypotheses given for \( \hat{\psi} \) in Corollary 5.1 do not imply the convergence of the integral \( \int_{0}^{1} \frac{\hat{\psi}(r)}{r} \, dr \). To see this, define the sequences \( (a_n) \) and \( (b_n) \) by

\[
a_1 = b_1 = 1 \quad ; \quad a_{n+1} = a_n \exp (-b_n) \quad \text{and} \quad b_{n+1} = 1/(a_{n+1} b_n), \quad n \in \mathbb{N}.
\]

It is easy to see that \( \lim_{n \to \infty} a_n = 0 \) and that

\[
\int_{b_{n+1}}^{a_n} \frac{dr}{b_n r} = 1 \quad \text{for each} \quad n \in \mathbb{N}.
\]

Thus the equations

\[
\hat{\psi}(0) = 0 \quad ; \quad \hat{\psi}(r) = r, \quad r > 1 ;
\]

\[
\hat{\psi}(r) = b_{2i-1} r, \quad a_{2i} < r \leq a_{2i-1}, \quad i \in \mathbb{N};
\]

\[
\hat{\psi}(r) = 1/b_{2i}, \quad a_{2i} < r \leq a_{2i+1}, \quad i \in \mathbb{N},
\]

define a continuous and nondecreasing (and also subadditive) function \( \hat{\psi} : \mathbb{R}_{+} \to \mathbb{R}_{+} \) for which

\[
\int_{0}^{1} \frac{dr}{\hat{\psi}(r)} = \int_{1}^{\infty} \frac{dr}{\hat{\psi}(r)} = \infty, \quad \text{and also} \quad \int_{0}^{1} \frac{\hat{\psi}(r)}{r} \, dr = \infty.
\]

Thus the proof of Theorem 6.2 fails if \( f \) is only Osgood continuous in \( J \times \mathcal{C} \).

7. Further consequences

Let \( A(t), t \in J \) be closed linear operators from a dense subset of \( X \) into \( X \). Assume that \( \{ -A(t) \mid t \in J \} \) is a hyperbolic family in the sense that each of its members is the infinitesimal generator of a contraction semigroup, and that \( t \mapsto A(t)(A(0))^{-1} \) is continuously differentiable mapping from \( J \) into \( B(X) \). The evolution equation (1') has also in this case a unique fundamental solution \( U : [0 \leq t \leq T] \to B(X) \) (cfr. [4], 12.5). Hence

all the results derived above for the mild solution of (1) in the parabolic case hold in the hyperbolic case in question.

Let now \( \{ E(t) \mid t \in J \} \) be a strongly continuous semigroup in \( X \), and let \( -A \) be its infinitesimal generator. If \( f \in C(J \times \mathcal{C}) \) and if the equation

\[
x'(t) + Ax(t) = f(t, x_t)
\]
has a strict solution $x$ with the given initial mapping $\varphi_0 \in \mathcal{V}$, then $x$ satisfies also the equation

$$
(7.1) \quad x(t) = E(t^+) \varphi_0(t^+) + \int_{t^-}^{t^+} E(t^+ - s) f(s, x_s) \, ds, \quad -\tau \leq t \leq T
$$

(cfr. [6]). A solution of (7.1) is therefore called a mild (or generalized) solution of (3).

Comparing (7.1) with (4) and (5.2) we deduce that

all the results derived above for solutions of (4) and for mild solutions of (1) can be restated for mild solutions of (3) by the choices

$$
(7.2) \quad y(t) = E(t^+) \varphi_0(t^+), \quad -\tau \leq t \leq T,
$$

and

$$
(7.3) \quad U(t, s) = E(t - s), \quad 0 \leq s \leq t \leq T.
$$

From the defining properties of $\{E(t)\}$ it namely follows that $y \in C[-\tau, T]$ and that $U(t, s)$ is strongly continuous. In particular, Theorem 2.2 of [6] follows as a consequence of Corollary 5.1. By Remarks 4.1 the interval $J = [0, T]$ can be replaced by $[0, \infty)$ if $t \mapsto E(t)$ is bounded, for ex. if $\{E(t)\}, \, t \geq 0$, is a contraction semigroup.

In the special case $E(t) = I$ = the identity operator of $X$ the considerations above are reduced to the considerations of solutions

$$
(7.4) \quad x(t) = \varphi_0(t^+) + \int_{t^-}^{t^+} f(s, x_s) \, ds
$$

of the functional differential equation

$$
(2) \quad x'(t) = f(t, x_t).
$$

REFERENCES