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On mild and strict solutions of nonlinear functional differential equations (**)

SUMMARY. — This paper deals with mild and strict solutions to a nonlinear perturbed functional differential equation, of parabolic or of hyperbolic type, in a Banach space. Some results concerning existence and uniqueness of these solutions are derived. Their bounds and dependence on initial mapping and parameter are investigated by a comparison method with minimal solutions of comparison equations as estimating functions.

1. INTRODUCTION

This paper investigates the nonlinear functional differential equation

$$x'(t) + A(t)\,x(t) = f(t, x_t) \tag{1}$$

in a Banach space. In case (1) is of certain parabolic or of hyperbolic type a Bompiani-Walter type existence and uniqueness theorem is verified for mild solutions of (1). The method of successive approximations is used, as in [2] for the equation

$$x'(t) = f(t, x_t). \tag{2}$$

Furthermore, some estimates are derived for solutions of (1), with minimal solutions of scalar integral equations as estimators. Also strict solutions of (1) in the parabolic case and mild solutions of

$$x'(t) + Ax(t) = f(t, x_t), \tag{3}$$

where A is the infinitesimal generator of a strongly continuous semigroup in a Banach space, are studied.

All the solutions mentioned above satisfy an integral equation of the form

$$x(t) = y(t) + \int_0^{t^+} U(t^+, s)\, f(s, x_s)\, ds\,, \quad -\tau \leq t \leq T, \tag{4}$$

where $t^+ = \max\{t, 0\}$, whence most of the results obtained for these solutions will be first derived for solutions of (4).

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2. Notations

Let X be a Banach space with norm $\|\cdot\|$ and B (X) be the Banach space of all bounded linear maps on X, with the norm of uniform operator topology. Denote by C (Z) and C_+ (Z) the sets of all continuous mappings from a topological space Z into X and into the nonnegative reals $\mathbf{R}_+$, respectively. In case $Z = [a, b]$, $a \leq b$, these sets are endowed with the uniform topology. The uniform norm of $\mathscr{C} = C[-\tau, 0]$, $\tau \geq 0$, is denoted by $\|\cdot\|_0$. Given $J = [0, T]$, $T > 0$, and $u, v \in C_+(J)$, define

$$u \leq v \quad \text{iff} \quad u(t) \leq v(t) \quad \text{for each} \quad t \in J,$$

and for $y \in C[-\tau, T]$ and $t \in J$ define $y_t \in \mathscr{C}$ by

$$y_t(s) = y(t+s), \qquad -\tau \leq s \leq 0.$$

Let $U : \{0 \leq s \leq t \leq T\} \to B(X)$ be given and assume that $U(t, s)$ is strongly continuous, i.e. $(t, s) \mapsto U(t, s)x$ is continuous for each $x \in X$. The Uniform Boundedness Theorem implies that the constant α given by

$$\alpha = \sup\{\|U(t, s)\| \mid 0 \leq s \leq t \leq T\} \tag{2.1}$$

is finite. Given $f \in C(J \times \mathscr{C})$ we can define a mapping F on $C[-\tau, T]$ by

$$Fx(t) = \int_0^{t^+} U(t^+, s) f(s, x_s)\, ds \quad , \quad -\tau \leq t \leq T. \tag{2.2}$$

In view of this definition (4) can be written in the form

$$x = y + Fx.$$

In (2.2) t^+ denotes $\max\{t, 0\}$, and analogous to that we shall denote $t^- = \min\{t, 0\}$.

3. An existence and uniqueness theorem

In this section U is assumed to be a mapping from $\{0 \leq s \leq t \leq T\}$ into B (X) such that $U(t, s)$ is strongly continuous. α denotes the constant defined by (2.1).

Theorem 3.1. *Assume that* $f \in C(J \times \mathscr{C})$ *and for* $(t, \varphi), (t, \bar{\varphi}) \in J \times \mathscr{C}$

$$\|f(t, \varphi) - f(t, \bar{\varphi})\| \leq g(t, \|\varphi - \bar{\varphi}\|_0) \tag{3.1}$$

where $g \in C_+(J \times \mathbf{R}_+)$ *satisfying*

(i) $g(t, r)$ *is nondecreasing in* $r \in \mathbf{R}_+$ *for each* $t \in J$;

(ii) *for each* $v \in C_+(J)$ *the integral equation*

$$u(t) = v(t) + \alpha \int_0^t g(s, u(s))\, ds \tag{3.2}$$

has a solution on J;

(iii) $u(t) = 0$ *is the only solution of* (3.2) *with* $v(t) = 0$.

Then for each $y \in C[-\tau, T]$ *the successive approximations*

$$x^{n+1} = y + Fx^n \quad ; \quad n \in \mathbf{N} = \{1, 2, \cdots\}, \tag{3.3}$$

with F *given by* (2.2) *and with any* $x^1 \in C[-\tau, T]$ *as the first approximation, converge on* $[-\tau, T]$ *uniformly to a unique solution of* (4).

Proof. Let $x^1 \in C[-\tau, T]$ be given. From (3.1) one can easily deduce that

$$\| Fx_t - F\bar{x}_t \|_0 \leq \alpha \int_0^t g(s, \| x_s - \bar{x}_s \|_0)\, ds, \qquad t \in J, \tag{3.4}$$

whenever $x, \bar{x} \in C[-\tau, T]$, whence for each $n \in \mathbf{N}$ and $t \in J$

$$\| x_t^{n+1} - x_t^1 \|_0 \leq v(t) + \alpha \int_0^t g(s, \| x_s^n - x_s^1 \|_0)\, ds \tag{3.5}$$

where

$$v(t) = \| y_t - x_t^1 \|_0 + \alpha \int_0^t \| f(s, x_s^1) \|\, ds. \tag{3.6}$$

Applying (i) one can obtain from (3.5) by induction that

$$\| x_t^{n+1} - x_t^1 \|_0 \leq u(t), \qquad t \in J,$$

where u is any solution of (3.2) with v given by (3.6).

The uniform convergence of the sequence (x^n) to a solution of (4) and the uniqueness of this solution can then be proved, by the method of Walter, as in [2] in case of the equation (2).

Remarks 3.1. A local existence and uniqueness theorem is obtained without the hypothesis (ii). On the other hand, (ii) and the continuity of g can be replaced by the Caratheodory conditions (cfr. [1], p. 665). Similarly, instead of f being continuous it suffices that $f(t, \varphi)$ is continuous in $\varphi \in \mathscr{C}$ for almost every $t \in J$, $f(t, y_t)$ is strongly measurable in $t \in J$ for each $y \in C[-\tau, T]$, and that for each $M > 0$ there is a Lebesgue integrable function $h : J \to \mathbf{R}_+$ such that

$$\| f(t, \varphi) \| \leq h(t)$$

whenever $(t, \varphi) \in J \times \mathscr{C}, \|\varphi\|_0 \leq M$. These hypotheses and the Dominated Convergence Theorem for Bochner integrals imply that the integral in (2.2) exists in the Bochner sense whenever $x \in C[-\tau, T]$, and that the uniform limit of the successive approximations x^n is a solution of (4).

If the equation (3.2) has for $v(t) = 0$ the zero function as the only solution when $\alpha = 1$, the same holds trivially also when $0 \leq \alpha < 1$. But this is no longer true when $\alpha > 1$ as we see from the following example [(1)].

Given $\bar{\alpha} > 1$, define $g \in C_+(J \times R_+)$ by

$$g(t, r) = \frac{2t}{\bar{\alpha}} \quad \text{for} \quad r \geq t^2, t \in J ;$$

$$g(t, r) = \frac{2r}{\bar{\alpha}t} \quad \text{for} \quad 0 \leq r < t^2, 0 < t \leq T .$$

It is easy to show that $u(t) = 0$ is the only solution of (3.2) when $\alpha = 1$ and $v(t) = 0$, whereas

$$u(t) = Ct^2, \quad t \in J ,$$

is for each $C \in [0, 1]$ a solution of (3.2) when $\alpha = \bar{\alpha}$ and $v(t) = 0$ (cfr. [1], p. 676).

4. Inequalities

Let the mapping U be as in the preceding section, and let x denote the solution of (4), with the given $y \in C[-\tau, T]$.

Lemma 4.1. *Let $g \in C_+(J \times R_+)$ satisfy the hypotheses (i) and (ii) of Theorem 3.1, and let $v \in C_+(J)$ be given. Then the equation (3.2) has the minimal solution u. If (y^n) is a sequence in $C[-\tau, T]$ such that $y^n \to y$ uniformly on $[-\tau, T]$ and that for each $t \in J$*

$$(4.1) \qquad \|y_t^1\|_0 \leq v(t) \quad and \quad \|y_t^{n+1}\|_0 \leq v(t) + \alpha \int_0^t g(s, \|y_s^n\|_0)\, ds , n \in N ,$$

then

$$(4.2) \qquad \|y_t\|_0 \leq u(t) , \qquad t \in J .$$

Proof. The existence of the minimal solution u of (3.2) is trivially verified by the method of successive approximations (cfr. [2], Lemma 2.1). By the monotonicity of g, the sequence (y^n), satisfying (4.1), is bounded above by u, as we see by induction. This implies (4.2).

(1) This example is suggested by Dr. J. Tienari.

As a consequence of Theorem 3.1 and Lemma 4.1 we have

THEOREM 4.1. *With the hypotheses of Theorem 3.1, and for* $z \in \mathrm{C}[-\tau, \mathrm{T}]$ *and* $v \in \mathrm{C}_+(\mathrm{J})$ *satisfying*

$$\| z_t - y_t - \mathrm{F} z_t \|_0 \le v(t), \quad t \in \mathrm{J}, \tag{4.3}$$

we have

$$\| z_t - x_t \|_0 \le u(t), \quad t \in \mathrm{J}, \tag{4.4}$$

where u *is the minimal solution of* (3.2).

Proof. Let (x^n) be the sequence of the successive approximations, with $x^1 = y + \mathrm{F} z$ as the first approximation, converging to x. Then it is easy to verify that the conditions (4.1) are satisfied by $y^n = z - x^n$, whence (4.4) follows from Lemma 4.1.

COROLLARY 4.1. *If the hypotheses of Theorem 3.1 hold and if* $\bar{x}$ *and* x *are the solutions of* (4) *with given* y *and* $\bar{y}$ *from* $\mathrm{C}[-\tau, \mathrm{T}]$, *respectively, then*

$$\| \bar{x}_t - x_t \|_0 \le u(t), \quad t \in \mathrm{J}, \tag{4.5}$$

where u *is the minimal solution of* (3.2) *with* $v(t) = \| \bar{y}_t - y_t \|_0$.

Proof. Choose $z = \bar{x}$ in Theorem 4.1.

THEOREM 4.2. *Assume that the hypotheses of Theorem 3.1 hold, and that for some* $z \in \mathrm{C}[-\tau, \mathrm{T}]$

$$\| f(t, \varphi) \| \le q(t, \| \varphi - z_t \|_0), \quad (t, \varphi) \in \mathrm{J} \times \mathscr{C}, \tag{4.6}$$

where $q \in \mathrm{C}(\mathrm{J} \times \mathbf{R}_+)$ *satisfying the hypotheses* (i) *and* (ii) *given for* g *in Theorem 3.1. Then*

$$\| x_t - z_t \|_0 \le w(t), \quad t \in \mathrm{J}, \tag{4.7}$$

where w *is the minimal solution of*

$$w(t) = \| y_t - z_t \|_0 + \alpha \int_0^t q(s, w(s))\, ds. \tag{4.8}$$

Proof. (4.1) holds for $y^n = x^n - z$, $x^1 = y$, $g = q$ and $v(t) = \| x_t - z_t \|_0$, $t \in \mathrm{J}$.

An example of evaluation functions q in Theorem 4.2 is

$$q(t, r) = \| f(t, z_t) \| + \alpha g(t, r), \quad (t, r) \in \mathrm{J} \times \mathbf{R}_+,$$

where g is the evaluation function in Theorem 3.1. With the choices $z = 0$ and $z = y$, we get estimates on the norm of x and on its distance from y, respectively.

THEOREM 4.3. *Assume that f depends on a parameter belonging to a non-empty set* Y, *and satisfies the hypotheses of Theorem 3.1 for given $\mu, \bar{\mu} \in$ Y. If x and $\bar{x}$ denote the corresponding solutions of* (4) *with the same $y \in C[-\tau, T]$, then*

$$\| x_t - \bar{x}_t \|_h \leq u(t), \qquad t \in J, \tag{4.9}$$

where u is the minimal solution of (3.2) *with*

$$v(t) = \alpha \int_0^t \| f(s, \bar{x}_s, \bar{\mu}) - f(s, \bar{x}_s, \mu) \| \, ds. \tag{4.10}$$

Proof. Let (x^n) denote the sequence of the successive approximations, with $x^1 = y + F\bar{x}$ as the first approximation, converging to x. Then (4.1) holds for $y^n = x^n - \bar{x}$.

Remarks 4.1. The results derived in this section hold also under the weaker hypotheses given in Remarks 3.1 for f and g. For q the hypotheses of g, without (iii), are sufficient.

The closed interval $J = [0, T]$ can be replaced in this and in preceding section by $[0, T)$, $0 < T \leq \infty$, provided that U is assumed to be bounded in $\{0 \leq s \leq t < T\}$. The convergence of the successive approximations x^n is then uniform on each compact subset of $[-\tau, T)$. In particular, if the hypotheses of Theorem 4.2 hold with $J = [0, T)$, and if the minimal solution w of (4.8) is bounded for bounded $y \in C[-\tau, T]$, then also the solution x of (4) is bounded. If further y and z have limits from the left as t tends to T, the same holds also for x.

To see that the boundedness of the minimal solution of (4.8) is a weaker condition than the boundedness of the corresponding maximal solution, define

$$q(t, r) = \frac{1}{\alpha} e^{-t} \quad \text{for} \quad 0 \leq r < 2(1 - e^{-t}), \qquad t \geq 0;$$

$$q(t, r) = \frac{1}{\alpha} \left(e^{-t} + \sqrt{r - 2(1 - e^{-t})} \right) \quad \text{for} \quad r \geq 2(1 - e^{-t}), \quad t \geq 0.$$

This q has the properties required in Theorem 4.2, and the integral equation

$$w(t) = 1 - e^{-t} + \alpha \int_0^t q(s, w(s)) \, ds \tag{4.8'}$$

has a bounded minimal solution

$$w_*(t) = 2(1 - e^{-t}),$$

but an unbounded maximal solution

$$w^*(t) = 2(1 - e^{-t}) + \frac{t^2}{4}.$$

5. On mild solutions

Let $\{A(t) \mid t \in J\}$ be a family of closed linear operators from a dense subset of X into X. Assume that

1) there exist $\theta \in \left(\frac{\pi}{2}, \pi\right)$ and $C > 0$ such that $[\lambda I + A(t)]^{-1}$ exists, belongs to $B(X)$, and $\|[\lambda I + A(t)]^{-1}\| \leq \frac{C}{1 + |\lambda|}$ whenever $t \in J$ and λ is a complex number for which $|\arg \lambda| \leq \theta$ (I denotes the identity operator of X).

The operators $-A(t)$ form then a parabolic family in the sense that $-A(t)$ is for each $t \in J$ the infinitesimal generator of an analytic semigroup (cfr. [3], Remark 3.1.1).

Assume further that

2) $t \mapsto A(t)(A(s))^{-1}$ is for each $s \in J$ a uniformly Hölder continuous mapping from J into $B(X)$; the Hölder constant and exponent being independent of $s \in J$.

The hypotheses above ensure (see [3], Theorem 3.2.1) that the evolution equation

$$y'(t) + A(t)\, y(t) = 0 \tag{1'}$$

has a unique fundamental solution $U : \{0 \leq s \leq t \leq T\} \to B(X)$, and that for each $x_0 \in X$ the mapping $t \mapsto U(t, 0)\, x_0$ is the unique solution of (1') with the value x_0 at $t = 0$. This motivates us to call the mapping $y \in C[-\tau, T]$, defined by

$$y(t) = U(t^+, 0)\, \varphi_0(t^-) \qquad \text{for} \quad -\tau \leq t \leq T, \tag{5.1}$$

the solution of (1') with the given $\varphi_0 \in \mathscr{C}$ as the initial mapping.

By a *mild solution* of the functional differential equation

$$x'(t) + A(t)\, x(t) = f(t, x_t) \tag{1}$$

with the given $\varphi_0 \in \mathscr{C}$ as the initial mapping, we mean a solution of the integral equation

$$x(t) = U(t^+, 0)\, \varphi_0(t^-) + \int_0^{t^+} U(t^+, s)\, f(x, s_s)\, ds, \qquad -\tau \leq t \leq T. \tag{5.2}$$

From (5.1) and (5.2) it follows that the study of mild solutions of (1) is reduced to the study of (4). Furthermore, the fundamental solution $U(t, s)$ of (1') is strongly continuous (cfr. [3], Lemma 3.4.3), whence

all the results derived in sections 3 and 4 for solutions of (4) *hold also for mild solutions of* (1), *when y in* (4) *is considered as the solution of* (1') *with a given initial mapping, and* U *as the fundamental solution of* (1').

Particularly we have

THEOREM 5.1. *Assume $f \in C(J \times \mathscr{C})$ and for $(t, \varphi), (t, \overline{\varphi}) \in J \times \mathscr{C}$*

(5.1) $$\| f(t, \varphi) - f(t, \overline{\varphi}) \| \leq g(t, \| \varphi - \overline{\varphi} \|_0)$$

where $g \in C(J \times R_+)$, $g(t, r)$ is nondecreasing in $r \in R_+$ for each $t \in J$, the initial value problem

(5.3) $$u'(t) = \alpha g(t, u(t)) \quad , \quad u(0) = r_0 , \quad \text{with } \alpha \text{ given by (2.1),}$$

has for each $r_0 \geq 0$ a solution on J, and $u(t) = 0$ is the only solution of (5.3) with $r_0 = 0$. Then for each $\varphi_0 \in \mathscr{C}$ the functional differential equation (1) has a unique mild solution x on $[-\tau, T]$ with φ_0 as the initial mapping. Moreover, x depends continuously on φ_0.

Proof. Let $\varphi_0 \in \mathscr{C}$ be given. It is easy to see that the hypotheses of Theorem 3.1 hold for f (cfr. [2], Lemma 2.1). This proves the existence and uniqueness of the mild solution x of (1) with φ_0 as the initial mapping.

Let $\overline{\varphi}_0$ be another initial mapping from $\mathscr{C}$ and $\overline{x}$ be the corresponding mild solution of (1). If y and $\overline{y}$ denote the solutions of (1') with φ_0 and $\overline{\varphi}_0$ as the initial mappings, respectively, then it follows from (5.1) that for each $t \in J$

$$\| y_t - \overline{y}_t \|_0 \leq \| \varphi_0 - \overline{\varphi}_0 \|_0 + \alpha \| \varphi_0(0) - \overline{\varphi}_0(0) \| .$$

This, together with Corollary 4.1, implies that

(5.4) $$\| x_t - \overline{x}_t \|_0 \leq u(t) \leq u(T) , \quad t \in J ,$$

where u is the minimal solution of (5.3) with

(5.5) $$r_0 = \| \varphi_0 - \overline{\varphi} \|_0 + \alpha \| \varphi_0(0) - \overline{\varphi}(0) \| .$$

If now $\overline{\varphi}_0 \to \varphi_0$ in $\mathscr{C}$, then $r_0 \to 0$ and, since the zero function is the only solution of (5.3) with $r_0 = 0$, then

$$u(T) \to 0 \quad \text{as} \quad \overline{\varphi}_0 \to \varphi_0$$

(cfr. [7], Theorem II.8.VIII). Hence, by (5.4), $\overline{x}(t) \to x(t)$ uniformly on $[-\tau, T]$ as $\overline{\varphi}_0 \to \varphi_0$ in $\mathscr{C}$.

From Theorem 4.3 we obtain

THEOREM 5.2. *Let Y be a metric space and assume that $f \in C(J \times \mathscr{C} \times Y)$ and f satisfies for each fixed $\mu \in Y$ the hypotheses of Theorem 5.1. Then the mild solution of (1), with the given initial mapping, depends continuously on the parameter μ.*

Proof. Let x and $\overline{x}$ denote the mild solutions of (1) corresponding to given $\varphi_0 \in \mathscr{C}$ and $\mu, \overline{\mu} \in Y$. Since the set $\{(s, \overline{x}_s, \mu) \mid 0 \leq s \leq T\}$ is compact in the product space $J \times \mathscr{C} \times Y$, where f is continuous, one can show by elementary analysis (cfr. [5], p. 34) that v, given by (4.10), tends to zero uniformly on J as $\overline{\mu}$ tends to μ in Y. The conclusion follows the from (4.9).

As a consequence of Theorem *5.1* we get

Corollary 5.1. *The conclusions of Theorem* 5.1 *hold if* $f \in C(J \times \mathscr{C})$ *and if for all* $(t, \varphi), (t, \overline{\varphi}) \in J \times \mathscr{C}$

$$(5.6) \qquad \| f(t, \varphi) - f(t, \overline{\varphi}) \| \leq p(t) \psi(\| \varphi - \overline{\varphi} \|_0)$$

where $p \in C_+(J)$, $\psi \in C_+(R_+)$, ψ *is nondecreasing, and the integrals* $\int_0^1 \frac{dr}{\psi(r)}$ *and* $\int_1^\infty \frac{dr}{\psi(r)}$ *diverge.*

Proof. The hypotheses of Theorem 5.1 hold with

$$g(t, r) = p(t) \psi(r) \quad \text{for} \quad (t, r) \in J \times R_+ .$$

The most important special case of Corollary 5.1 is obtained when ψ is the identity mapping on R_+, in which case the Osgood condition (5.6) is reduced to the Lipschitz condition.

By Remarks 3.1 it suffices that p in (5.6) is a Lebesgue integrable function from J into R_+.

6. On strict solutions

By a *strict solution* of (1) with the given intial mapping $\varphi_0 \in \mathscr{C}$ we mean a mapping $x \in C[-\tau, T]$ which equals to φ_0 on $[-\tau, 0]$ and which is strongly continuously differentiable and satisfies (1) on $(0, T]$.

Theorem 6.1. *Let* $\{A(t) \mid t \in J\}$ *be a family of operators in* B(X) *satisfying the hypotheses* 1) *and* 2) (*p.* 7) *and being uniformly bounded over* $t \in J$. *Assume further that* f *satisfies the hypotheses of Theorem 5.1. Then for each* $\varphi_0 \in \mathscr{C}$ *the equation* (1) *has on* $[-\tau, T]$ *a unique strict solution with* φ_0 *as the initial mapping.*

Proof. By Theorem 5.1, (1) has a unique mild solution x with the given initial mapping $\varphi_0 \in \mathscr{C}$. In view of Lemmas 3.5.1 and 3.5.2 of [3], with $u_0 = \varphi_0(0)$ and $f(t) = f(t, x_t)$, one can deduce that x is also a strict solution of (1). The uniqueness follows from Theorem 5.1, because for any strict solution x of (1) the mapping $t \mapsto f(t, x_t)$ is continuous, whence x is also a mild solution of (1) (cfr. [4], Vol. II, p. 250).

If the operators $A(t)$ are not assumed to be bounded we have

Theorem 6.2. *Let* $A(t)$, $t \in J$ *be closed linear operators from a dense subset* D *of* X *into* X *satisfying the hypotheses* 1) *and* 2) *and*

3) $\quad \limsup_{t \to t_0} \| A(t) z(t) \| < \infty \quad$ *for* $z \in C(J)$, $\quad \operatorname{Im} z \subset D$.

Assume further that f *is a mapping from* $J \times \mathscr{C}$ *into* X *and for all* (s, φ), $(t, \bar{\varphi}) \in J \times \mathscr{C}$

$$\|f(s, \varphi) - f(t, \bar{\varphi})\| \leq \psi(|s-t| + \|\varphi - \bar{\varphi}\|_0) \tag{6.1}$$

where $\psi \in C_+(\mathbf{R}_+)$, ψ *is nondecreasing, the integrals* $\int_0^1 \frac{\mathrm{d}r}{\psi(r)}$ *and* $\int_1^\infty \frac{\mathrm{d}r}{\psi(r)}$ *diverge, and the integral* $\int_0^1 \frac{\psi(r)}{r}\,\mathrm{d}r$ *converges. Then to each Lipschitz continuous initial mapping* $\varphi_0 : [-\tau, 0] \to X$ *the equations* (1) *has on* $[-\tau, T]$ *a unique and Lipschitz continuous strict solution.*

Proof. Let $\varphi_0 : [-\tau, 0] \to X$ be a Lipschitz continuous mapping. From (6.1) it follows that f is continuous and satisfies the Osgood condition (5.6) with $p(t) = 1$. Thus by Corollary 5.1 the equation (1) has a unique mild solution x with φ_0 as the initial mapping.

The additional hypothesis 3), the continuous differentiability of the solution y of (1') on $(0, T]$, the Lipschitz continuity of φ_0, the boundedness of U and the continuity of $t \to f(t, x_t)$ imply that x is Lipschitz continuous on $[-\tau, T]$. In particular, there is $k > 0$ such that

$$\|x_s - x_t\|_0 \leq k|s-t|, \quad 0 \leq s \leq t \leq T. \tag{6.2}$$

From the proof of Theorem 3.2.2 in [3] one can easily deduce that x is a strict solution of (1) if the integral $\int_0^t \frac{\|f(s, x_s) - f(t, x_t)\|}{|s-t|}\,\mathrm{d}s$ converges for $t \in J$. This condition holds, since by (6.1) and (6.2)

$$\frac{\|f(s, x_s) - f(t, x_t)\|}{|s-t|} \leq \frac{\psi((k+1)|s-t|)}{|s-t|}, \; 0 \leq s < t \leq T,$$

and since the integral $\int_0^1 \frac{\psi(r)}{r}\,\mathrm{d}r$ converges. Thus the mild solution x of (1) is also a strict solution. The uniqueness can be verified as in Theorem 6.1.

Remark 6.1. The hypotheses of Theorem 6.2 for f hold particularly if f is Lipschitz continuous in both its arguments, i.e. if (6.1) holds with

$$\psi(r) = \mathrm{M}r \qquad (\mathrm{M} > 0).$$

The same hypotheses are valid also when ψ in (6.1) is given by

$$\psi(0) = 0; \qquad \psi(r) = \mathrm{M}r, \; r > 1/e;$$

$$\psi(r) = \mathrm{M}r \prod_{i=1}^{n} \log_i\left(\frac{1}{r}\right), \; 1/\exp_n(e) < r \leq 1/\exp_n(1), \qquad n \in \mathbf{N},$$

where $\log_n$ and $\exp_n$ denote n-fold iterated logarithm and exponential functions, respectively. This can be verified by elementary analysis.

A counter-example. The hypotheses given for ψ in Corollary 5.1 do not imply the convergence of the integral $\int_0^1 \frac{\psi(r)}{r}\,\mathrm{d}r$. To see this, define the sequences (a_n) and (b_n) by

$$a_1 = b_1 = 1 \ ; \ a_{n+1} = a_n \exp(-b_n) \quad \text{and} \quad b_{n+1} = 1/(a_{n+1} b_n), \quad n \in \mathbf{N}.$$

It is easy to see that $\lim_{n\to\infty} a_n = 0$ and that

$$\int_{a_{n+1}}^{a_n} \frac{\mathrm{d}r}{b_n r} = 1 \qquad \text{for each} \quad n \in \mathbf{N}.$$

Thus the equations

$$\begin{aligned}
&\psi(0) = 0 && ; \quad \psi(r) = r, r > 1 ; \\
&\psi(r) = b_{2i-1} r && , \quad a_{2i} < r \le a_{2i-1}, \qquad i \in \mathbf{N} ; \\
&\psi(r) = 1/b_{2i} && , \quad a_{2i+1} < r \le a_{2i}, \qquad i \in \mathbf{N},
\end{aligned}$$

define a continuous and nondecreasing (and also subadditive) function $\psi : \mathbf{R}_+ \to \mathbf{R}_+$ for which

$$\int_0^1 \frac{\mathrm{d}r}{\psi(r)} = \int_1^\infty \frac{\mathrm{d}r}{\psi(r)} = \infty, \qquad \textit{and also} \int_0^1 \frac{\psi(r)}{r}\,\mathrm{d}r = \infty.$$

Thus the proof of Theorem 6.2 fails if f is only Osgood continuous in $\mathrm{J} \times \mathscr{C}$.

7. Further consequences

Let $\mathrm{A}(t), t \in \mathrm{J}$ be closed linear operators from a dense subset of X into X. Assume that $\{-\mathrm{A}(t) \mid t \in \mathrm{J}\}$ is a hyperbolic family in the sense that each of its members is the infinitesimal generator of a contraction semigroup, and that $t \mapsto \mathrm{A}(t)(\mathrm{A}(0))^{-1}$ is continuously differentiable mapping from J into B(X). The evolution equation (1′) has also in this case a unique fundamental solution $\mathrm{U} : \{0 \le s \le t \le \mathrm{T}\} \to \mathrm{B}(\mathrm{X})$ (cfr. [4], 12.5). Hence

all the results derived above for the mild solution of (1) *in the parabolic case hold in the hyperbolic case in question.*

Let now $\{\mathrm{E}(t) \mid t \in \mathrm{J}\}$ be a strongly continuous semigroup in X, and let $-\mathrm{A}$ be its infinitesimal generator. If $f \in \mathrm{C}(\mathrm{J} \times \mathscr{C})$ and if the equation

$$x'(t) + \mathrm{A}x(t) = f(t, x_t) \tag{3}$$

has a strict solution x with the given initial mapping $\varphi_0 \in \mathscr{C}$, then x satisfies also the equation

$$(7.1) \qquad x(t) = \mathrm{E}(t^+)\,\varphi_0(t^-) + \int_0^{t^+} \mathrm{E}(t^+ - s)\, f(s, x_s)\,\mathrm{d}s\,, \qquad -\tau \leq t \leq \mathrm{T}$$

(cfr. [6]). A solution of (7.1) is therefore called a *mild* (or generalized) *solution* of (3).

Comparing (7.1) with (4) and (5.2) we deduce that

all the results derived above for solutions of (4) *and for mild solutions of* (1) *can be restated for mild solutions of* (3) *by the choices*

$$(7.2) \qquad y(t) = \mathrm{E}(t^+)\,\varphi_0(t^-)\,, \qquad -\tau \leq t \leq \mathrm{T}\,,$$

and

$$(7.3) \qquad \mathrm{U}(t, s) = \mathrm{E}(t - s)\,, \qquad 0 \leq s \leq t \leq \mathrm{T}\,.$$

From the defining properties of $\{\mathrm{E}(t)\}$ it namely follows that $y \in \mathrm{C}[-\tau, \mathrm{T}]$ and that $\mathrm{U}(t, s)$ is strongly continuous. In particular, Theorem 2.2 of [6] follows as a consequence of Corollary 5.1. By Remarks 4.1 the interval $\mathrm{J} = [0, \mathrm{T}]$ can be replaced by $[0, \infty)$ if $t \mapsto \mathrm{E}(t)$ is bounded, for ex. if $\{\mathrm{E}(t)\}, t \geq 0$, is a contraction semigroup.

In the special case $\mathrm{E}(t) = \mathrm{I} =$ the identity operator of X the considerations above are reduced to the considerations of solutions

$$(7.4) \qquad x(t) = \varphi_0(t^-) + \int_0^{t^+} f(s, x_s)\,\mathrm{d}s$$

of the functional differential equation

$$(2) \qquad x'(t) = f(t, x_t)\,.$$

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