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# On Subspaces of Subspaces of a Finsler Space (**)

Summary: B. Y. Chen and K. Yano ([4]) have investigated the properties of a submanifold of a submanifold of a Riemannian manifold. In this paper, using the induced covariant differentiation process given by Rund ([1, 2]), we shall study the properties of a subspace $F_l$ of a subspace $F_m$ of a Finsler space $F_n$. The conditions under which $F_l$ is minimal in $F_m$ or it is minimal in $F_n$ will be discussed.

## 1. Introduction

Let $F_n$ be an n-dimensional Finsler space of class $\overset{\infty}{C}$ associated with a coordinate system $x^i$ (i = 1,......, n) and $g_{ij}(x, \dot{x})$ be its metric tensor.

We denote $F_m$, the m-dimensional differentiable subspace of class $\overset{\infty}{C}$ of $F_n$ represented parametrically by the equations

(1.1) $$x^i = x^i(u^\alpha) , \quad (i = 1 , \ldots\ldots , n ; \alpha = 1 , \ldots\ldots , m) ; \quad m \leq n .$$

The matrix with intities

(1.2) $$B^i_\alpha = \partial_\alpha x^i \ (^1)$$

has rank m.

The metric tensors of $F_n$ and $F_m$ are such that

(1.3) $$g_{\alpha\beta}(u , \dot{u}) = g_{ij}(x , \dot{x}) B^i_\alpha B^j_\beta .$$

There exist (n — m) vectors $N^i_{(\mu)}(x , \dot{x})$ ($\mu = m + 1, \ldots, n$), called the normal vectors to $F_m$, satisfying the conditions

(1.4) (a) $g_{ij}(x , \dot{x}) N^i_{(\mu)} B^j_\alpha = 0$ , (b) $g_{ij}(x , \dot{x}) N^i_{(\mu)} N^j_{(\nu)} = \delta_{(\mu\nu)}$ .

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(1) $\partial_\alpha = \dfrac{\partial}{\partial u^\alpha}$ .

Further, we have the relation [1]

$$g^{ij}(x, y) = g^{\alpha\beta}(u, v)\, B^j_\beta B^i_\alpha + \sum_{\mu=m+1}^{n} N^i_{(\mu)} N^j_{(\mu)} \tag{1.5}$$

where $g^{ij}(x, y)$ and $g^{\alpha\beta}(u, v)$ are the contravariant components of the metric tensors of $F_n$ and $F_m$ respectively.

2. The induced covariant derivative [1,2] of $B^i_\alpha$, being denoted by $I^i_{\alpha\beta}$, is given by

$$I^i_{\alpha\beta} = \delta^0_\beta B^i_\alpha = B^i_{\alpha\beta} - B^i_\epsilon \Gamma^{*\epsilon}_{\alpha\beta} + \Gamma^{*i}_{hk} B^h_\alpha B^k_\beta \tag{2.1}$$

where $\Gamma^{*i}_{hk}$ are connection coefficients of the embedding space and $\Gamma^{*\epsilon}_{\alpha\beta}$ are induced connection coefficients of $F_m$ defined by

$$\Gamma^{*\epsilon}_{\alpha\beta} = B^\epsilon_i (B^i_{\alpha\beta} + \Gamma^{*i}_{hk} B^h_\alpha B^k_\beta) \tag{2.2}$$

where $B^i_{\alpha\beta} = \dfrac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}$ and $B^\epsilon_i = g^{\epsilon\gamma} B^j_\gamma g_{ij}$.

Since the derivative (2.1) is normal to $F_m$, we may write

$$I^i_{\alpha\beta} = \sum_{\mu=m+1}^{n} \Omega_{(\mu)\alpha\beta}(u, \acute{u})\, N^i_{(\mu)} \tag{2.3}$$

where $\Omega_{(\mu)\alpha\beta}(u, \acute{u})$ are components of second fundamental form.

The vector field

$$K^i_{(x, \acute{x})}\{F_m, F_n\} \overset{\text{def}}{=} \frac{1}{m} g^{\alpha\beta} I^i_{\alpha\beta} \tag{2.4}$$

of $F_n$ is normal to $F_m$. This is called the mean curvature vector of $F_m$ immersed in $F_n$. If $K^i_{(x, \acute{x})}\{F_m, F_n\} = 0$, $F_m$ is called a minimal subspace of $F_n$.

Furthermore, we consider an l-dimensional differentiable subspace $F_l$ of class $\overset{\infty}{C}$ of $F_m$ represented parametically by the equations

$$u^\alpha = u^\alpha(z^\theta), \quad \begin{pmatrix} \alpha, \beta, \gamma, \ldots = 1, \ldots, m \\ \theta, \Phi, \psi, \ldots = 1, \ldots, l \end{pmatrix} : l < m \leq n. \tag{2.5}$$

The projection factors $B^\alpha_\theta$ are defined by

$$B^\alpha_\theta = \partial_\theta u^\alpha \,(^1). \tag{2.6}$$

(1) $\partial_\theta = \dfrac{\partial}{\partial z^\theta}$

The metric tensors of $F_l$ and $F_m$ are related by

(2.7) $$g_{\theta\Phi}(z\,,\,\acute{z}) = g_{\alpha\beta}(u\,,\,\acute{u})\, B^{\alpha}_{\theta} B^{\beta}_{\Phi}.$$

Since the rank of the matrix $\| B^{\alpha}_{\theta} \|$ is l, there exist (m — l) normal vectors $N^{\alpha}_{(p)}(z\,,\,\acute{z})$ , $(p = l + 1 \,,\, \ldots\ldots , m)$ which satisfy the following conditions

(2.8) $$g_{\alpha\beta}(u\,,\,\acute{u})\, B^{\alpha}_{\theta} N^{\beta}_{(p)} = 0 \quad ; \quad g_{\alpha\beta}(u\,,\,\acute{u})\, N^{\alpha}_{(p)} N^{\beta}_{(q)} = \delta_{(p\,q)}\,, (p\,,\,q = l+1\,,\,\ldots\ldots , m).$$

Thus we have

(2.9) $$g^{\alpha\beta}(u\,,\,v) = g^{\theta\Phi} B^{\alpha}_{\theta} B^{\beta}_{\Phi} + \sum_{p=l+1}^{m} N^{\alpha}_{(p)} N^{\beta}_{(p)}$$

where $g^{\theta\Phi}$ are the contravariant components of the metric tensor of $F_l$.

The covariant derivative of $B^{\alpha}_{\theta}$ of the type (2.1) will be given by

(2.10) $$I^{\alpha}_{\theta\Phi} = \delta^{o}_{\Phi} B^{\alpha}_{\theta} = B^{\alpha}_{\theta\Phi} - B^{\alpha}_{\psi} \Gamma^{*\psi}_{\theta\Phi} + \Gamma^{*\alpha}_{\beta\gamma} B^{\beta}_{\theta} B^{\gamma}_{\Phi}$$

and it is normal to $F_l$, where $\Gamma^{*\psi}_{\theta\Phi}$ are the induced connection coefficients of $F_l$ defined by

(2.11) $$\Gamma^{*\psi}_{\theta\Phi} = B^{\psi}_{\alpha}\,(B^{\alpha}_{\theta\Phi} + \Gamma^{*\alpha}_{\beta\gamma} B^{\beta}_{\theta} B^{\gamma}_{\Phi})$$

where $$B^{\alpha}_{\theta\Phi} = \frac{\partial^2 u^{\alpha}}{\partial z^{\theta}\,\partial z^{\Phi}} \qquad \text{and} \qquad B^{\psi}_{\alpha} = g^{\psi\Phi} B^{\beta}_{\Phi}\, g_{\alpha\beta}.$$

The vector field

(2.12) $$K^{\alpha}_{(u\,,\,\acute{u})}\{F_l\,,\,F_m\} \overset{\text{def}}{=} \frac{1}{l}\, g^{\theta\Phi}\, I^{\alpha}_{\theta\Phi}$$

of $F_m$ is normal to $F_l$ and this is called the mean curvature vector of $F_l$ in $F_m$.

If $K^{\alpha}_{(u\,,\,\acute{u})}\{F_l\,,\,F_m\} = 0$ , $F_l$ is a minimal subspace of $F_m$.

The subspace $F_l$ of $F_m$ can be regarded as a subspace of the Finsler space $F_n$ and it can be expressed parametically by the equations

(2.13) $$x^i = x^i\,(u^{\alpha}\,(z^{\theta}))\;,\;(i = 1\,,\,\ldots\ldots , n\;;\;\theta = 1\,,\,\ldots\ldots , l\;;\;l < m \leq n)$$

and consequently

(2.14) $$B^{i}_{\theta} = B^{i}_{\alpha}\, B^{\alpha}_{\theta}.$$

The fundamental tensors of $F_l$, $F_m$ and $F_n$ respectively are related by

(2.15) $$g_{\theta\Phi}(z\ ,\ \acute{z}) = g_{\alpha\beta}(u\ ,\ \acute{u})\, B^\alpha_\theta\, B^\beta_\Phi = g_{ij}(x\ ,\ \acute{x})\, B^i_\theta B^j_\Phi .$$

The mutually orthogonal unit normals of $F_l$ in $F_n$ are

(2.16) $$N^i_{(p)} = N^\alpha_{(p)}\, B^i_\alpha \quad \text{and} \quad N^i_{(\mu)} \quad , \quad \begin{pmatrix} p = l+1\ ,\ \ldots\ldots\ ,\ m \\ \mu = m+1\ ,\ \ldots\ldots\ ,\ n \end{pmatrix}$$

where $N^i_{(p)}$ is tangent to $F_m$ and $N^i_{(\mu)}$ is normal to $F_m$. The induced covariant derivative of $B^i_\theta$ will be

(2.17) $$I^i_{\theta\Phi} = \delta^o_\Phi\, B^i_\theta = B^i_{\theta\Phi} - B^i_\psi\, \Gamma^{*\psi}_{\theta\Phi} + \Gamma^{*i}_{hk}\, B^h_\theta\, B^k_\Phi$$

and is normal to $F_l$, where $\Gamma^{*\psi}_{\theta\Phi}(z\ ,\ \acute{z})$ are defined by equation (2.11).

The mean curvature vector of $F_l$ in $F_n$ is given by

(2.18) $$K^i_{(x\ ,\ \acute{x})}\{F_l\ ,\ F_n\} \overset{\text{def}}{=} \frac{1}{l}\, g^{\theta\Phi}\, I^i_{\theta\Phi}$$

and is normal to $F_l$. If $K^i_{(x\ ,\ \acute{x})}\{F_l\ ,\ F_n\} = 0$ , $F_l$ is minimal in the Finsler space $F_n$.

### 3. Relation between mean curvature vectors

The covariant derivative of the type (2.1) of the equation (2.14) gives us

(3.1) $$I^i_{\theta\Phi} = I^\alpha_{\theta\Phi}\, B^i_\alpha + I^i_{\alpha\beta}\, B^\alpha_\theta\, B^\beta_\Phi .$$

On multiplying (3.1) by $\frac{1}{l}\, g^{\theta\Phi}$, we get

(3.2) $$\frac{1}{l}\, g^{\theta\Phi}\, I^i_{\theta\Phi} = \left(\frac{1}{l}\, g^{\theta\Phi}\, I^\alpha_{\theta\Phi}\right) B^i_\alpha + \left(\frac{1}{l}\, g^{\theta\Phi}\, B^\alpha_\theta\, B^\beta_\Phi\right) I^i_{\alpha\beta} .$$

Defining

(3.3) $$K^i_{(x\ ,\ \acute{x})}\{F_l\ ,\ F_m\ ,\ F_n\} \overset{\text{def}}{=} \left(\frac{1}{l}\, g^{\theta\Phi}\, B^\alpha_\theta\, B^\beta_\Phi\right) I^i_{\alpha\beta}$$

as the relative mean curvature vector of $F_l$ with respect to $F_m$ and $F_n$, we obtain from (2.12), (2.18) and (3.2)

(3.4) $$K^i_{(x\ ,\ \acute{x})}\{F_l\ ,\ F_n\} = K^\alpha_{(u\ ,\ \acute{u})}\{F_l\ ,\ F_m\}\, B^i_\alpha + K^i_{(x\ ,\ \acute{x})}\{F_l\ ,\ F_m\ ,\ F_n\} .$$

The vector field defined by (3.3) is normal to $F_m$.

With the help of (2.4), (2.9) and (3.3) we find

$$(3.5)\quad K^i_{(x\,,\,\acute{x})}\{F_l\,,\,F_m\,,\,F_n\} = \frac{m}{l}K^i_{(x\,,\,\acute{x})}\{F_m\,,\,F_n\} - \frac{1}{l}\sum_{p=l+1}^{m} N^\alpha_{(p)}\,N^\beta_{(p)}\,\Gamma^i_{\alpha\beta}\,.$$

The equation (3.4) yields the following theorems:

Theorem 3.1 - *The mean curvature vector of a Finsler subspace $F_l$ in Finsler space $F_n$ is the sum of the mean curvature vector of $F_l$ in $F_m$ and the relative mean curvature vector of $F_l$ with respect to $F_m$ and $F_n$.*

Corollary 3.2 - *In order that $F_l$ is minimal in $F_m$, it is necessàry and sufficient that the mean curvature vector of $F_l$ in $F_n$ be normal to $F_m$.*

Theorem 3.3 - *In order that the subspace $F_l$ be minimal in Finsler space $F_n$, it is necessary and sufficient that $F_l$ is minimal in $F_m$ and the relative mean curvature vector of $F_l$ with respect to $F_m$ and $F_n$ vanishes.*

## 4. Concurrent vector field

Let $V^i$ be a vector field of Finsler space $F_n$ and concurrent along $F_l$, that is, we have

$$(4.1)\quad B^i_\theta + \delta^o_\theta\,V^i = 0\,.$$

This equation may be written as

$$(4.2)\quad \delta^o_\Phi\,B^i_\theta + \delta^o_\Phi\,\delta^o_\theta\,V^i = 0$$

which in view of (2.18) reduces to

$$(4.3)\quad K^i_{(x\,,\,\acute{x})}\{F_l\,,\,F_n\} + \frac{1}{l}\,g^{\theta\Phi}\,\delta^o_\Phi\,\delta^o_\theta\,V^i = 0\,.$$

The equations (3.4) and (4.3) yield

$$(4.4)\quad K^\alpha_{(u\,,\,\acute{u})}\{F_l\,,\,F_m\}\,B^i_\alpha + K^i_{(x\,,\,\acute{x})}\{F_l\,,\,F_m\,,\,F_n\} + \frac{1}{l}\,g^{\theta\Phi}\,\delta^o_\Phi\,\delta^o_\theta\,V^i = 0\,.$$

Since $K^i_{(x\,,\,\acute{x})}\{F_l\,,\,F_m\,,\,F_n\}$ is normal to $F_m$,

$$(4.5)\quad K^\alpha_{(u\,,\,\acute{u})}\{F_l\,,\,F_m\} = 0$$

iff $\frac{1}{l}\,g^{\theta\Phi}\,\delta^o_\Phi\,\delta^o_\theta\,V^i$ is normal to the space $F_m$. Hence we get the following theorems:

THEOREM 4.1. - *Suppose that there exists a vector field $V^i$ in a Finsler space $F_n$ and concurrent along $F_l$. In order that $F_l$ be minimal in $F_m$, it is necessary and sufficient that $\frac{1}{l}\, g^{\theta\Phi}\, \delta^{\alpha}_{\Phi}\, \delta^{\alpha}_{\theta}\, V^i$ is normal to $F_m$.*

In particular, let $F_n = F_m$. In this case the equation (4·3) yields

THEOREM 4.2 - *Suppose that there exists a vector field $V^{\alpha}$ of $F_m$ and concurrent along $F_l$. In order that $F_l$ be minimal in $F_m$, it is necessary and sufficient that $g^{\theta\Phi}\, \delta^{\alpha}_{\Phi}\, \delta^{\alpha}_{\theta}\, V^{\alpha} = 0$.*

5. SUBSPACES UMBILICAL TO A NORMAL

Let us consider a unit vector field $\lambda^i$ of Finsler space $F_n$ and normal to $F_m$. Also, let $F_l$ be umbilical with mean curvature B with respect to this unit vector field in $F_m$. Now we choose $\lambda^i$ as the first normal $N^i_{(m+1)}$ to $F_m$, then we have equations of $F_l$ in $F_n$:

$$(5.1)\qquad I^i_{\theta\Phi} = \sum_{p=l+1}^{m} \bar{\Omega}_{(p)\,\theta\Phi}\, N^i_{(p)} + B\, g_{\theta\Phi}\, N^i_{(m+1)} + \bar{\Omega}_{(m+2)}\,\theta\Phi\, N^i_{(m+2)} + $$
$$+ \bar{\Omega}_{(m+3)}\,\theta\Phi\, N^i_{(m+3)} + \cdots\cdots + \bar{\Omega}_{(n)}\,\theta\Phi\, N^i_{(n)}$$

where $\bar{\Omega}_{(p)}\,\theta\Phi$, $B\, g_{\theta\Phi}$, $\bar{\Omega}_{(m+2)}\,\theta\Phi$, $\bar{\Omega}_{(m+3)}\,\theta\Phi$, ..... $\bar{\Omega}_{(n)}\,\theta\Phi$ are second fundamental forms with respect to $N^i_{(p)}$, $N^i_{(m+1)}$, $N^i_{(m+2)}$, $N^i_{(m+3)}$, ....., $N^i_{(n)}$ respectively.

Multiplying (5.1) by $\frac{1}{l}\, g^{\theta\Phi}$ and using (2.18), we obtain

$$(5.2)\qquad K^i_{(x,\,\dot{x})}\{F_l, F_n\} = \frac{1}{l}\, g^{\theta\Phi} \sum_{p=l+1}^{m} \bar{\Omega}_{(p)}\,\theta\Phi\, N^i_{(p)} + B\, N^i_{(m+1)} + $$
$$+ \frac{1}{l}\, g^{\theta\Phi}\, \bar{\Omega}_{(m+2)}\,\theta\Phi\, N^i_{(m+2)} + \cdots\cdots + \frac{1}{l}\, g^{\theta\Phi}\, \bar{\Omega}_{(n)}\,\theta\Phi\, N^i_{(n)}.$$

Let A be the mean curvature of $F_l$ in the Finsler space $F_n$ and $A^2 \leqslant B^2$ then we find $A^2 = B^2$ and

$$(5.3)\qquad g^{\theta\Phi}\, \bar{\Omega}_{(p)}\,\theta\Phi = 0\,,\ g^{\theta\Phi}\, \bar{\Omega}_{(m+2)}\,\theta\Phi = g^{\theta\Phi}\, \bar{\Omega}_{(m+3)\,\Phi\theta} =$$
$$= \cdots\cdots = g^{\theta\Phi}\, \bar{\Omega}_{(n)}\,\theta\Phi = 0\,.$$

Thus the subspace $F_l$ is minimal in $F_m$ and $F_l$ is minimal in $F_n$ iff $B = 0$. This gives the following theorem:

THEOREM 5.1. - *Suppose that $\lambda^i$ is a unit vector field of the Finsler space $F_n$ and normal to $F_m$ and also $F_l$ is umbilical with mean curvature $B$ with respect to the unit vector field $\lambda^i$. If the mean curvature $A$ of $F_l$ in $F_n$ is such that $A^2 \leq B^2$, then $F_l$ is minimal in $F_m$ and is minimal in $F_n$ iff $B = 0$.*

Again, let $F_m$ be umbilical in the Finsler space $F_n$ and $A_{(\mu)}$ be a vector field in normal $N^i_{(\nu)}$ of $F_m$ in $F_n$. Then we have [3]

$$\Omega_{(\mu)\,\alpha\beta} = A_{(\mu)}\, g_{\alpha\beta} \tag{5.4}$$

which in view of (2.3) gives

$$I^i_{\alpha\beta} = \sum_{\mu=m+1}^{n} A_{(\mu)}\, g_{\alpha\beta}\, N^i_{(\mu)}\,. \tag{5.5}$$

From equations (2.4), (2.15), (3.3) and (5.5) we get

$$K^i_{(x\,,\,\dot{x})}\{F_m\,,\,F_n\} = \sum_{\mu=m+1}^{n} A_{(\mu)}\, N^i_{(\mu)}\,, \tag{5.6}$$

$$K^i_{(x\,,\,\dot{x})}\{F_m\,,\,F_n\} = K^i_{(x\,,\,\dot{x})}\{F_l\,,\,F_m\,,\,F_n\} = \sum_{\mu=m+1}^{n} A_{(\mu)}\, N^i_{(\mu)}\,. \tag{5.7}$$

Hence we have

THEOREM 5.2. - *Suppose that $F_m$ is umbilical in the Finsler space $F_n$. Then the mean curvature vector of $F_m$ in $F_n$ coincides with relative mean curvature vector of $F_l$ with respect to $F_m$ and $F_n$.*

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