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On a submanifold of a manifold with areal metric (**)

Summary. Manifolds with areal metric have been defined and studied by various authors namely Davies [3], Kawaguchi [6], Martin [7] and Rund [9], [10] etc. Also the submanifolds of different type of manifolds have been studied by Bompiani [1], Davies [2], Douglas [4], Eliopoulos [5], Rund [8] and others, but no attempt has been made to study the submanifolds of the manifolds with areal metric, defined by Rund [10]. The purpose of the present paper is to study the submanifolds of the manifolds cited above. In this paper I have studied properties of the normal vectors, induced connection parameters, induced covariant derivative, normal curvatures of the submanifold etc. Finally Gauss-Codazzi equations in such manifolds have been obtained.

1. INTRODUCTION. — Let X_n be a differentiable manifold referred to local coordinates x^i $(i = 1, \ldots, n)$ such that

$$x^i = x^i(t^\alpha), \quad (\alpha = 1, \ldots, l), \tag{1.1}$$

where t^α denotes a system of independent parameters of l-dimensional subspace C_l of X_n $(l < n)$, such that Rund [10]:

$$\dot{x}^i_\alpha = \partial x^i / \partial t^\alpha . \tag{1.2}$$

Let us suppose that we are given a function $L(x^i, \dot{x}^i_\alpha)$ of the $n + nl$ variables $x^i, \dot{x}^i_\alpha$ which satisfies:

I) the Lagrangian L is of class C^4 in all its arguments and it is a scalar with respect to the transforms of the local coordinates x^i of X_n,

II) L is positive for all independent set of arguments $\dot{x}^i_\alpha$,

III) the integral

$$\int_{R_l} L(x^i, \dot{x}^i_\alpha)\, dt^1 \ldots dt^l,$$

where R_l is a finite simply connected region, is independent of the choice of parameters t^α of the subspace,

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IV the $n\,l \times n\,l$ determinant

$$D = \det\left[\frac{1}{2}\,\frac{\partial^2 L^{2/l}}{\partial \dot{x}^i_\alpha \partial \dot{x}^j_\beta}\right],$$

is non-vanishing for linearly independent $\dot{x}^i_\beta$.

The third condition of the above can equivalently be expressed as

(1.3) $$(\partial L/\partial \dot{x}^i_\alpha)\,\dot{x}^i_\beta = \delta^\alpha_\beta L\,.$$

For such a manifold the metric tensor is defined by

(1.4) $$g^{\alpha\beta}_{ij}(x^h, \dot{x}^h_\epsilon) = \frac{1}{2}\,\frac{\partial^2 \{L(x^h, \dot{x}^h_\epsilon)\}^{2/l}}{\partial \dot{x}^i_\alpha \partial \dot{x}^j_\beta}$$

which satisfies :

(1.5) a $$C^{\alpha\beta\gamma}_{ijk} = \frac{1}{2}\left(\frac{\partial g^{\alpha\beta}_{ij}}{\partial \dot{x}^k_\gamma}\right)$$

and

(1.5) b $$g^{ij}_{\alpha\beta}\, g^{\alpha\gamma}_{ik} = \delta^\gamma_\beta\, \delta^j_k\,,$$

where

$$C^{\alpha\beta\gamma}_{ijk}\,\dot{x}^i_\alpha = 0 \qquad \text{and} \qquad C^{\alpha\beta\gamma}_{ijk}\,\dot{x}^i_\beta\,\dot{x}^j_\alpha = 0\,.$$

The coefficients of connection for the manifold are defined by Rund [10]:

(1.6) $$\Gamma^i_{kj} = l^{-1}\left[P^{i\beta}_{kj\beta} + \frac{\partial P^{i\alpha}_{hj\beta}}{\partial \dot{x}^k_\beta}\,\dot{x}^h_\alpha\right],$$

where

$$P^{h\alpha}_{ij\gamma} = G^{kh}_{\beta\gamma}\,(V^{(\alpha\beta)}_{ij,k} - C^{\alpha\beta\delta\lambda}_{ijkl}\, f^l_{\delta\lambda})\,,$$

$$V^{(\alpha\beta)}_{ij,k} = \frac{1}{2}\left(\frac{\partial g^{\alpha\beta}_{kj}}{\partial x^i} + \frac{\partial g^{\alpha\beta}_{ik}}{\partial x^j} - \frac{\partial g^{\alpha\beta}_{ij}}{\partial x^k}\right),$$

$$C^{\delta\beta\lambda\alpha}_{kjli} = \frac{\partial C^{\delta\beta\lambda}_{kjl}}{\partial \dot{x}^i_\alpha}\,,\; G^{\alpha\beta}_{kj} = \frac{1}{2}\,(g^{\alpha\beta}_{kj} + g^{\alpha\beta}_{jk})\,.$$

Also these connection coefficients have the following properties :

(1.7) a $$\Gamma^i_{kj}\,\dot{x}^k_\gamma = P^{i\alpha}_{kj\gamma}\,\dot{x}^k_\alpha$$

and

(1.7) b $$\Gamma^i_{kj}\,\dot{x}^k_\gamma\,\dot{x}^j_\epsilon = P^{i\alpha}_{hj\gamma}\,\dot{x}^h_\alpha\,\dot{x}^j_\epsilon\,.$$

If X^i_ϵ is a set of linearly independent differentiable vector fields tangent to C_l in X_n, then we have

(1.8) $$X^i_{\epsilon|j} = \frac{\partial X^i_\epsilon}{\partial x^j} - \frac{\partial X^i_\epsilon}{\partial \dot{x}^l_\lambda}\,\Gamma^l_{pj}\,\dot{x}^p_\lambda + \Gamma^i_{jl}\,X^l_\epsilon\,,$$

where $X^i_{\epsilon|j}$ denotes the covariant derivative of X_ϵ with respect to x^j.

Also we have

$$X^i_{\in|kh} - X^i_{\in|hk} = X^j_\in K^{*i}_{jkh} - \frac{\partial X^i_\in}{\partial \dot{x}^l_\alpha} K^l_{pkh} \dot{x}^p_\alpha + X^i_{\in|l} T^l_{kh} , \tag{1.9}$$

where

$$K^{*i}_{jkh} = \left(\frac{\partial \Gamma^i_{kj}}{\partial x^h} - \frac{\partial \Gamma^i_{kj}}{\partial \dot{x}^l_\alpha} \Gamma^l_{ph} \dot{x}^p_\alpha\right) + \Gamma^i_{hp} \Gamma^p_{kj} \tag{1.10}$$

$$- \left(\frac{\partial \Gamma^i_{hj}}{\partial x^k} - \frac{\partial \Gamma^i_{hj}}{\partial \dot{x}^l_\alpha} \Gamma^l_{pk} \dot{x}^p_\alpha\right) - \Gamma^i_{kp} \Gamma^p_{hj} ,$$

$$T^i_{kh} = \Gamma^i_{kh} - \Gamma^i_{hk} \tag{1.11}$$

and

$$K^{*i}_{jkh} = K^i_{jkh} + T^i_{kj|h} - T^i_{hj|k} + T^i_{lj} T^l_{hk} + (T^i_{kl} T^l_{hj} - T^i_{hl} T^l_{kj}) . \tag{1.12}$$

2. Submanifold of X_n. — An m-dimensional submanifold X_m of X_n $(1 < m < n)$ may be represented parametrically by the equations

$$x^i = x^i(u^A) , \ (A = 1, \ldots, m) , \tag{2.1}$$

where we suppose that the variables u^A form a coordinate system of X_m.

Now we shall parametrize the submanifold X_m by the vectors of C_l such that

$$\dot{u}^A_\alpha = \partial u^A / \partial t^\alpha , \ (\alpha = 1, \ldots, l) . \tag{2.2}$$

Along any coordinate curve of parameter u^A in X_m, the vector whose n components are $B^i_A = \partial x^i / \partial u^A$, where the matrix $\| B^i_A \|$ is of rank m, is tangential to the curve and corresponding to m independent variables u^A, there are m such linearly independent vector fields tangential to X_m, in terms of which any vector tangential to X_m is linearly expressibile.

In particular, if dx^i is a small displacement tangential to X_m, it follows that

$$dx^i = B^i_A \, du^A , \tag{2.3}$$

where du^A denotes the same displacement in terms of the coordinates of X_m. Thus if we denote the components of a vector $X^i_\in$ tangent to X_m by $X^A_\in$ in terms of u^A system, we have

$$X^i_\in = B^i_A X^A_\in . \tag{2.4}$$

By virtue of equation (2.4) we can also obtain

$$\partial \dot{x}^i_\alpha / \partial \dot{u}^A_\beta = B^{i\beta}_{A\alpha} . \tag{2.5}$$

In case $\alpha = \beta$ in (2.5) we shall express $B^{i\beta}_{A\alpha}$ by B^i_A. Now in analogy with the definition of the metric tensor for the manifold X_n we can also define the metric tensor for the manifold X_m as follows:

$$g^{\alpha\beta}_{AB}(u^c, \dot{u}^c_\in) = \frac{1}{2} \left[\frac{\partial^2 \{ \bar{L}(u^c, \dot{u}^c_\in) \}^{2}}{\partial \dot{u}^A_\alpha \, \partial \dot{u}^B_\beta}\right] , \tag{2.6}$$

where $\overline{L}(u^c, \dot{u}^c_\in)$ satisfies similar properties as $L(x^h, \dot{x}^h_\in)$. Also for $\overline{L}$ we have

$$(\partial \overline{L} / \partial \dot{u}^c_\alpha)\, \dot{u}^c_\beta = \delta^\alpha_\beta \overline{L} . \tag{2.7}$$

Since the metric of the manifold X_n induces a similar metric on the manifold X_m we can write

$$\overline{L}(u^c, \dot{u}^c_\alpha) = L(x^i(u^c), \dot{u}^c_\alpha B^i_c) . \tag{2.8}$$

If we differentiate (2.8) successively with respect to $\dot{u}^A_\alpha$ and $\dot{u}^D_\beta$ and use (1.4) and (2.6) we can establish a relationship between the two metric tensors in the following form

$$g^{\alpha\beta}_{AD}(u^c, \dot{u}^c_\in) = g^{\alpha\beta}_{ij}(x^h, \dot{x}^h_\in)\, B^i_A B^j_D . \tag{2.9}$$

As we can define the inverse of the metric tensor $g^{\alpha\beta}_{ij}(x^h, \dot{x}^h_\in)$ by $g^{ij}_{\alpha\beta}(x^h, y^h_\in)$ such that Rund [10]:

$$g^{ij}_{\alpha\beta}(x^h, y^h_\in) = \frac{1}{2} \frac{\partial^2 \{ H(x^h, y^h_\in) \}^2}{\partial y^\alpha_i\, \partial y^\beta_j} , \tag{2.10}$$

where

$$H(x^i, y^\beta_j) = L(x^j, \dot{x}^j_\alpha) ,$$

we can also define inverse of the metric tensor of the submanifold X_m such that it satisfies:

$$g^{AB}_{\alpha\beta}(u^c, z^c_\gamma) = \frac{1}{2} \frac{\partial^2 \{ \overline{H}(u^c, z^c_\gamma) \}^2}{\partial z^\alpha_A\, \partial z^\beta_B} , \tag{2.11}$$

where

$$z^\alpha_A = g^{\alpha\beta}_{AB}\, \dot{u}^B_\beta = \partial \overline{L}^2 / \partial \dot{u}^A_\alpha .$$

From these relations we can obviously have

$$B^A_i = g^{AD}_{\alpha\beta}\, g^{\alpha\beta}_{il}\, B^l_D \tag{2.12 a}$$

and

$$B^A_i\, B^i_c = \delta^A_c . \tag{2.12 b}$$

3. Normal vectors. — A covariant vector Y^α_i is said to be normal to X_m at a point P, if it satisfies

$$Y^\alpha_i\, B^i_A = 0 . \tag{3.1}$$

Since the rank of the matrix $\| B^i_A \|$ is assumed to be m, it follows that there exist l (n-m) linearly independent vectors $\underset{\mu}{N}{}^\alpha_i$, ($\mu = m + 1, \ldots, n$) normal to X_m. These normals may be choosen in the multiply infinite number of ways:

$$\underset{\mu}{N}{}^\alpha_i\, B^i_A = 0 . \tag{3.2}$$

With respect to a given direction $\dot{x}^i_\alpha$ in the tangent space of X_n we may choose a set of normals satisfying the relations :

(3.3) a $$\underset{\mu}{N}{}^i_\alpha = g^{ij}_{\alpha\beta} (x^h, y^h_\epsilon) \underset{\mu}{N}{}^\alpha_j ,$$

and

(3.3) b $$g^{\alpha\beta}_{ij} (x^h, \dot{x}^h_\epsilon) \underset{\mu}{N}{}^i_\alpha \underset{\upsilon}{N}{}^j_\beta = \delta^\mu_\upsilon .$$

As a consequence of (3.3) it can be noted that

(3.4) $$B^A_i \underset{\mu}{N}{}^i_\alpha (x^h, y^h_\epsilon) = 0 .$$

Now we shall define a relation

(3.5) $$\psi^{\;i}_j = B^i_A B^A_j - \delta^i_j ,$$

which by virtue of zquation (2.12) b leads to

$$\psi^{\;i}_j B^j_c = 0 ,$$

which implies that ψ_j is of the form $\sum_\mu \underset{\mu}{N}{}^\alpha_j \underset{\mu}{\lambda}{}^i_\alpha$, where the factors $\underset{\mu}{\lambda}{}^i_\alpha$ are given by

$$\psi^{\;i}_j \underset{\upsilon}{N}{}^j_\alpha = \underset{\upsilon}{\lambda}{}^i_\alpha \quad , \quad (\upsilon = m + 1 , \ldots , n) .$$

But from equations (3.4) and (3.5) we have

(3.6) $$\psi_j \underset{\upsilon}{N}{}^j_\alpha = - \underset{\upsilon}{N}{}^i_\alpha ,$$

which implies

(3.7) $$B^i_A B^A_j = \delta^i_j - N^i_j (x^h, \dot{x}^h_\epsilon) ,$$

where

(3.8) $$N^i_j (x^h, \dot{x}^h_\epsilon) \overset{\text{def.}}{=\!=} \sum_{\mu = m+1}^{n} \underset{\mu}{N}{}^i_\alpha \underset{\mu}{N}{}^\alpha_j (x^h, \dot{x}^h_\epsilon) .$$

The immediate consequence of the above and the preceeding formulae is the relation

(3.9) $$g^{\alpha\beta}_{ij} (x^h, \dot{x}^h_\epsilon) = g^{\alpha\beta}_{AC} (u^D, \dot{u}^D_\epsilon) B^A_i B^C_j + N^{\alpha\beta}_{ij} (x^h, \dot{x}^h_\epsilon) ,$$

and

(3.10) $$g^{ij}_{\alpha\beta} = g^{AC}_{\alpha\beta} B^i_A B^j_C + N^{ij}_{\alpha\beta} .$$

Furthermore we can also obtain

(3.11) $$\frac{1}{2} \frac{\partial g^{\alpha\beta}_{AE} (u^D, \dot{u}^D_\epsilon)}{\partial \dot{u}^F_\gamma} = C^{\alpha\beta\gamma}_{AEF} (u^D, \dot{u}^D_\epsilon)$$
$$= C^{\alpha\beta\gamma}_{ijk} (x^h, \dot{x}^h_\epsilon) B^i_A B_E B^k_F ,$$

which by virtue of (2.12) a gives rise to

$$\frac{\partial B_i^A}{\partial \dot{u}_\alpha^C} = 2\, g_{\delta\gamma}^{AD} \left\{ C_{ijh}^{\alpha\delta\gamma}\, B_C^j\, B_D^h - C_{kjh}^{\alpha\delta\gamma}\, B_D^k\, B_C^j\, B_E^h\, B_i^E \right\}. \tag{3.12}$$

Applying equation (3.7) to the last term of (3.12) we obtain after simplification

$$\frac{\partial B_i^A}{\partial \dot{u}_\alpha^C} = 2\, C_{jh\delta}^{\alpha\delta k}\, N_i^j\, B_k^A\, B_C^h \tag{3.13}$$

or alternatively

$$\frac{\partial B_i^A}{\partial \dot{u}_\alpha^C} = 2\, N_i^j\, B_C^h\, B_D^k\, g_{\delta\gamma}^{AD}\, C_{jkh}^{\alpha\delta\gamma}. \tag{3.13 a}$$

From equation (3.7) and (3.13) a we deduce that

$$\frac{\partial N_j^i}{\partial \dot{u}_\alpha^C} = -\, 2\, g_{\delta\gamma}^{AD}\, B_A^i\, B_C^h\, B_D^k\, C_{rkh}^{\alpha\delta\gamma}\, N_j^r\,, \tag{3.14}$$

which togeather with (3.13) a implies

$$\frac{\partial B_i^A}{\partial \dot{u}_\alpha^C}\, \dot{u}_\alpha^C = \frac{\partial N_j^i}{\partial \dot{u}_\alpha^C}\, \dot{u}_\alpha^C = 0\,. \tag{3.15}$$

So far we have discussed the general properties of normals, but now we shall see that in general we have two sets of normals to X_m at a point P of X_m. The first set of normals n_α^i, which are independent of the directions $\dot{x}_\in^h$, satisfy

$$n_i^\alpha\, B_A^i \equiv g_{ij}^{\alpha\beta}\, n_\beta^j\, B_A^i = 0\,. \tag{3.16}$$

The solutions of the equations (3.16) are normalized by means of the relation

$$L\,(x\,,n) = 1\ r\ g_{ij}^{\alpha\beta}\,(x^h\,, n_\in^h)\, n_\alpha^i\, n_\beta^j = 1\,. \tag{3.17}$$

The second set of normals can be defined by the solutions $n_\alpha^{*i}\,(x^h, \dot{x}_\in^h)$ of the equations

$$g_{ij}^{\alpha\beta}\,(x^h\,, \dot{x}_\in^h)\, B_A^i\, n_\beta^{*j}\,(x^h\,, \dot{x}_\in^h) = 0 \tag{3.18}$$

and its solutions are normalized by

$$g_{ij}^{\alpha\beta}\,(x^h\,, n_\in^{*h}\,(x^j\,, \dot{x}_\gamma^j))\, n_\alpha^{*i}\, n_\beta^{*j} = 1\,. \tag{3.19}$$

The normals n_α^{*i} will be called secondary normals. From equations (3.17) and (3.19) it follows that $n_i^{*\alpha}$ is proportional to $n_i^\alpha\,(x^h, n_\in^h)$ and therefore we can define tensors $\gamma_{(\mu)AD}^{\alpha\beta}\,(u^C)$, $(\mu = m+1\,, \ldots, n)$ which are independent of the directions as follows :

$$\gamma_{(\mu)AD}^{\alpha\beta} = g_{ij}^{\alpha\beta}\,(x^h\,, \underset{(\mu)}{n_\in^h})\, B_A^i\, B_D^j\,. \tag{3.20}$$

We also define the following set of inverse parameters corresponding to B^i_A:

(3.21) $$y^A_{(\mu)i}(x^h) = g^{\alpha\beta}_{ij}(x^h, \underset{(\mu)}{n}{}^h_\gamma)\, \gamma^{AD}_{(\mu)\alpha\beta}\, B^j_D ,$$

so that we have

(3.22) $$\underset{(\mu)}{n}{}^{*i}_\alpha\, B^A_i = 0 \quad , \quad y^A_{(\mu)i}\, \underset{(\mu)}{n}{}^i_\alpha = 0 ,$$

and

(3.23) $$y^A_{(\mu)i}\, B^i_C = \delta^A_C .$$

4. Induced connection parameters. — Let $x^i = x^i(s)$ be a curve C of X_m which is continuous and continuously differentiable, then if we change the parameter from t to s and the dot by dash we can say that x'^i_α is tangential to X_m. Let us consider a continuous and continuously differentiable vector field tangent to X_m; which satisfies (2.4), then the induced covariant derivative of the vector field along C in the space X_m is defined by

(4.1) $$X^A_{\in|B} = \frac{\partial X^A_\in}{\partial u^B} - \frac{\partial X^A_\in}{\partial \acute{u}^C_\alpha}\, \Gamma^C_{DB}\, x'^D_\alpha + \Gamma^A_{BC}\, X^C_\in ,$$

where Γ^A_{BC} are induced connection parameters.

The tensor $X^A_{\in|B}$ defined by (4.1) is given by the projection onto X_m of the covariant derivative $X^i_{\in||j}$ of $X^i_\in$ with respect to X_n. Hence

(4.2) $$g^{\alpha\beta}_{ij}\, B^j_A\, B^k_C\, X^i_{\in|k} = g^{\alpha\beta}_{AD}\, X^D_{\in|C} .$$

If $T^{i\gamma}_{A\alpha}$ is some mixed tensor, its intrinsic derivative will be given by

(4.3) $$\frac{D\, T^{i\gamma}_{A\alpha}}{D\, s^\beta} = \left(\frac{\partial T^{i\gamma}_{A\alpha}}{\partial u^C} - \frac{\partial T^{i\gamma}_{A\alpha}}{\partial \dot{u}^D_\delta}\, \Gamma^D_{EC}\, u'^E_\delta + \Gamma^i_{kC}\, T^{k\gamma}_{A\alpha} - \Gamma^B_{AC}\, T^{i\gamma}_{B\alpha} \right) \frac{d\, u^C}{d\, s^\beta} ,$$

where Γ^i_{kC} is mixed connection.

Now applying equation (4.3) to the well known relation $x'^h_\in = B^h_A\, u'^A_\in$ we obtain on simplification

(4.4) $$\frac{D\, x'^i_\alpha}{D\, s^\beta} = B^i_{AC}\, \acute{u}^A_\alpha\, \acute{u}^C_\beta + B^i_A\, \frac{D\, \acute{u}^A_\alpha}{D\, s^\beta} ,$$

where

(4.5) $$B^i_{AC} = \frac{\partial^2 x^i}{\partial u^A\, \partial u^C} - \frac{\partial^2 x'^i_\in}{\partial \acute{u}^A_\in\, \partial \acute{u}^B_\beta}\, \Gamma^B_{DC}\, \acute{u}^D_\beta + \Gamma^i_{hk}\, B^h_A\, B^k_C - B^i_D\, \Gamma^D_{AC} .$$

The expression B^i_{AC} which is a tensor may be considered as the generalised covariant derivative of B^i_A with respect to u^C. Because of the second term on the right hand side it can be seen that the tensor B^i_{AC} is not symmetric in its lower indices.

Now multiplying equation (4.2) by u'^{C}_{γ}, we obtain

$$g^{\alpha\beta}_{ij}\, B^{j}_{C} \frac{D\, \acute{x}^{i}_{\in}}{D\, \overset{\theta}{s}} = g^{\alpha\beta}_{AC} \frac{D\, \acute{u}^{A}_{\in}}{D\, \overset{\theta}{s}} ,$$

which is satisfied by the tangent vector u'^{A}_{α} to any curve C in X_m.

Now in case of a submanifold X_m we define a curve C to be a geodesic G, if it satisfies

$$D\, \acute{u}^{A}_{\in} / D\, s^{\theta} = 0 ,$$

hence obviously from the last relation we can get

$$g^{\alpha\beta}_{ij}\, (x^{h}, \acute{x}^{h}_{\gamma})\, B^{j}_{C} \left(\frac{D\, \acute{x}^{i}_{\in}}{D\, \overset{\theta}{s}}\right)_{G} = 0 .$$

Since the vector $D\, x'^{i}_{\in} / D\, \overset{\theta}{s}$, which defines a principal normal to a geodesic G, satisfies equation (3.18), it belongs to the space spanned by $\underset{(\mu)}{n}^{*i}_{\beta}$, therefore we have:

Theorem (4.1). — *The principal normal of a geodesic G of* X_m *lies in the space spanned by the secondary normals* $\underset{(\mu)}{n}^{*i}_{\beta}$.

Since we know that

$$g^{\alpha\beta}_{ij}\, (x^{h}, \acute{x}^{h}_{\in})\, B^{j}_{A}\, B^{i}_{C}\, \Gamma^{C}_{DE}$$

$$= g^{\alpha\beta}_{ij}\, B^{j}_{A} \left(\frac{\partial^{2} x^{i}}{\partial u^{D} \partial u^{E}} - \frac{\partial^{2} \acute{x}^{i}_{\in}}{\partial \acute{u}^{D}_{\in} \partial \acute{u}^{C}_{\gamma}}\, \Gamma^{C}_{BE}\, \acute{u}^{B}_{\gamma} + \Gamma^{i}_{hk}\, B^{h}_{D}\, B^{k}_{E}\right),$$

therefore by virtue of (4.5) we obtain

$$g^{\alpha\beta}_{ij}\, (x^{h}, \acute{x}^{h}_{\in})\, B^{j}_{A}\, B^{i}_{CD} = 0 , \tag{4.7}$$

which implies:

Theorem (4.2). — *The tensor* B_{AC} *lies in the space spanned by the secondary normals* $\underset{(\mu)}{n}^{*j}_{\beta}$.

5. Normal curvatures. — Since B^{i}_{AC} lies in the space spanned by secondary normals it can be expressed as a linear combination of $\underset{(\mu)}{n}^{*i}_{x}$, thus

$$B^{i}_{AC} = \sum_{(\mu)} \Omega^{*x}_{(\mu)AC}\, (u^{D}, \acute{u}^{D}_{\in})\, \underset{(\mu)}{n}^{*i}_{x} , \tag{5.1}$$

where $\Omega^{*x}_{(\mu)AC}$ is called secondary second fundamental tensor of the submanifold X_m. Also this tensor is not symmetric in A and C.

Multiplying equation (5.1) by $\underset{(\nu)}{n}^{j}_{\beta}$ and putting

$$\sum_{\mu} \Omega^{*x}_{(\mu)AC} \cos\, (\underset{(\nu)}{n}, \underset{(\mu)}{n^{*}}) = \Omega^{x}_{(\nu)AC} , \tag{5.2}$$

we find

$$B^i_{AC} \underset{(\nu)}{n}{}^{\beta}_{j} = \Omega^{\beta}_{(\nu)AC} \cdot \tag{5.3}$$

The tensor $\Omega^{\alpha}_{(\mu)AC}$ is also called the second fundamental tensor and in (5.2) $\cos(\underset{(\nu)}{n}, \underset{(\mu)}{n^*})$ is the cosine of the angle between the two types of normals and is expressed as

$$\cos(\underset{(\mu)}{n}, \underset{(\mu)}{n^*}) = \frac{g^{\alpha\beta}_{ij}(x, \underset{(\mu)}{n}) \underset{(\mu)}{n}{}^{*i}_{\alpha} \underset{(\nu)}{n}{}_{\beta}}{\left[g^{\alpha\beta}_{ij}(x, \underset{(\nu)}{n}) \underset{(\nu)}{n}{}^{i}_{\alpha} \underset{(\nu)}{n}{}^{j}_{\beta} \cdot g^{\gamma\delta}_{kh}(x, \underset{(\mu)}{n^*}) \underset{(\mu)}{n}{}^{*k}_{\gamma} \underset{(\mu)}{n}{}^{*h}_{\delta} \right]^{1/2}} \cdot \tag{5.4}$$

Multiplying equation (5.3) by $u'^A_\alpha \, u'^C_\gamma$ and using

$$\underset{(\nu)}{n}{}^{\beta}_{j} \frac{d \, x'^j_\alpha}{d \, s^\gamma} = \underset{(\nu)}{n}{}^{\beta}_{j} \frac{\partial^2 x^j}{\partial u^A \partial u^C} \, u'^A_\alpha \, u'^C_\gamma \, ,$$

we get on simplification

$$\Omega^{\beta}_{(\nu)AC} \, u'^A_\alpha \, u'^C_\gamma = \underset{(\nu)}{n}{}^{\beta}_{j} \frac{D \, x'^j_\alpha}{D \, s^\gamma} \, . \tag{5.5}$$

Equation (5.5) holds for all curves of X_m with tangent vector x'^i_α, but depends on the choice of L. If we differentiate the relation

$$\underset{(\nu)}{n}{}^{\alpha}_{i} \, x'^i_\alpha = 0 \, ,$$

we find

$$\frac{D \underset{(\nu)}{n}{}^{\alpha}_{i}}{D \, s^\gamma} \, x'^i_\alpha = - \underset{(\nu)}{n}{}^{\alpha}_{i} \frac{D \, x'^i_\alpha}{D \, s^\gamma} \, ,$$

which implies that $\underset{(\nu)}{n}{}^{\alpha}_{i} \, D \, x'^i_\alpha / D \, s^\gamma$ depends on the choice of line element.

Using (5.5) in the identity

$$\underset{(\nu)}{n}{}^{\alpha}_{i} \frac{D \, x'^i_\alpha}{D \, s^\gamma} = \left| \frac{D \, x'^i_\alpha}{D \, s^\gamma} \right| \left| \underset{(\nu)}{n}{}^{\alpha}_{i} \right| \cos\left(\underset{(\nu)}{n}{}^{\alpha}_{i} , \frac{D \, x'^i_\alpha}{D \, s^\gamma} \right) \, ,$$

we obtain

$$\left| \frac{D \, x'^i_\alpha}{D \, s^\gamma} \right| = \frac{\Omega^{\beta}_{(\nu)AC} \, u'^A_\beta \, u'^C_\gamma}{\cos\left(\underset{(\nu)}{n}{}^{\alpha}_{i} , \frac{D \, x'^i_\alpha}{D \, s^\gamma} \right) \left| \underset{(\nu)}{n}{}^{\alpha}_{i} \right|} \, , \tag{5.6}$$

which gives an expression for the curvature of a curve of X_n.

Since $\Omega^{\alpha}_{(\nu)AC} \, u'^A_\alpha \, u'^C_\gamma$ is same for all curves of X_m, tangent to x'^i_α, equation (5.6) gives a generalisation of the Meuniers theorem of classical differential geometry.

Therefore we may regard

$$\frac{1}{R^{\gamma}_{(\nu)}(u^C, \acute{u}^C_{\epsilon})} \overset{\text{def.}}{=} \Omega^{\alpha}_{(\nu)AC}\, \acute{u}^A_{\alpha}\, \acute{u}^C_{\gamma}\,, \tag{5.7}$$

as normal curvature corresponding to the normals $\underset{(\nu)}{n}{}^{\alpha}_{i}$.

So far we have shown that to each direction at a point P of X_m correspond l (n-m) normal curvatures associated with the given direction u'^A_{α}, it can be easily proved in analogy to Finsler spaces Rund [8], that the principal directions will be given by the extreme values of $g^{\alpha\gamma}_{AC}(u^B, u'^B_{\epsilon})\, u'^A_{\alpha}\, u'^C_{\gamma}$ where u^A is kept fixed. In other words we can say that principal directions are those for which normal curvatures assume extreme values.

Now we shall define a secondary normal curvature associated to a line element (x^h, x'^h_{ϵ}) and depending on Ω^*. Let us consider

$$\frac{D\, x'^i_{\alpha}}{D\, s^{\beta}} = \sum_{\mu} \lambda_{(\mu)\beta}\, \underset{(\mu)}{n}{}^{*i}_{\alpha} + B^i_A \frac{D\, \acute{u}^A_{\alpha}}{D\, s^{\beta}}\,, \tag{5.8}$$

for an arbitrary curve of X_m. If we multiply (5.8) by $\underset{(\nu)}{n}{}^{\alpha}_{i}$, we get

$$\underset{(\nu)}{n}{}^{\alpha}_{i} \frac{D\, x'^i_{\alpha}}{D\, s^{\beta}} = \sum_{\mu} \lambda_{(\mu)\beta} \cos(\underset{(\nu)}{n}, \underset{(\mu)}{n^*})\,, \tag{5.9}$$

which by virtue of (5.6) implies

$$\Omega^{\alpha}_{(\nu)AC}\, \acute{u}^A_{\alpha}\, \acute{u}^C_{\beta} = \sum_{\mu} \lambda_{(\mu)\beta} \cos(\underset{(\nu)}{n}, \underset{(\mu)}{n^*})\,. \tag{5.10}$$

Thus we have

$$\lambda_{(\mu)\beta} = \Omega^{*\alpha}_{(\mu)AC}\, \acute{u}^A_{\alpha}\, \acute{u}^C_{\beta}$$

and hence for a geodesic (5.8) yields

$$\frac{D\, x'^i_{\alpha}}{D\, s^{\beta}} = \sum_{\mu} \Omega^{*\gamma}_{(\mu)AB}\, \acute{u}^A_{\alpha}\, \acute{u}^B_{\beta} \cdot \underset{(\mu)}{n}{}^{*i}_{\gamma}\,. \tag{5.11}$$

We now define secondary normal curvature as follows:

$$\frac{1}{R^{*\gamma}_{(\nu)}\, R^{*\delta}_{(\nu)}} \overset{\text{def.}}{=} g^{\alpha\delta}_{ij}(x^h, x'^h_{\epsilon})\, \frac{D\, x'^i_{\gamma}}{D\, s^{\alpha}}\, \frac{D\, x'^j_{\delta}}{D\, s^{\beta}}\,, \tag{5.12}$$

which on simplification yields:

$$\frac{1}{R^{*\gamma}_{(\nu)}\, R^{*\delta}_{(\nu)}} = \sum_{\nu} \Omega^{*\alpha}_{(\nu)AC}\, \Omega^{*\beta}_{(\nu)BD}\, \acute{u}^A_{\alpha}\, \acute{u}^C_{\gamma}\, \acute{u}^B_{\beta}\, \acute{u}^D_{\delta}\,, \tag{5.13}$$

from which we can say that the secondary normal curvature is independent of the choice of secondary normals.

6. Covariant derivative of normals. — We define the tensor $\underset{(\mu)}{n}{}^{*i}_{\alpha|A}$ as the covariant derivative of the vector $\underset{(\mu)}{n}{}^{*i}_{\alpha}$ and it is the projection of $\underset{(\mu)}{n}{}^{*i}_{\alpha|k}$ onto X_m. Thus

$$\underset{(\mu)}{n}{}^{*i}_{\alpha|A} = \underset{(\mu)}{n}{}^{*i}_{\alpha|k} B^k_A \,. \tag{6.1}$$

Since we know that

$$\underset{(\mu)}{n}{}^{*i}_{\alpha|A} = \frac{\partial \underset{(\mu)}{n}{}^{*i}_{\alpha}}{\partial u^A} - \frac{\partial \underset{(\mu)}{n}{}^{*i}_{\alpha}}{\partial \acute{u}^C_\beta} \Gamma^C_{DA} \acute{u}^D_\beta + \Gamma^i_{hk} \underset{(\mu)}{n}{}^{*h}_{\alpha} B^k_A \,, \tag{6.2}$$

therefore differentiating relation (3.18) with respect to u^C we get on simplification and rearrangement of terms the following relation

$$\begin{aligned} & g^{\alpha\beta}_{ij} B^i_A \underset{(\mu)}{n}{}^{*j}_{\beta|C} + \underset{(\mu)}{n}{}^{*j}_{\beta} B^i_A \left(\frac{\partial g^{\alpha\beta}_{ij}}{\partial u^C} - g^{\alpha\beta}_{hl} \Gamma^h_{jk} B^k_C \right) \\ & + \frac{\partial \underset{(\mu)}{n}{}^{*j}_{\beta}}{\partial \acute{u}^E_\gamma} \Gamma^E_{DC} \acute{u}^D_\gamma g^{\alpha\beta}_{ij} B^i_A + g^{\alpha\beta}_{ij} \frac{\partial^2 x^i}{\partial u^A \partial u^C} \underset{(\mu)}{n}{}^{*j}_{\beta} = 0 \,. \end{aligned} \tag{6.3}$$

In equation (6.3) if we add and subtsract

$$g^{\alpha\beta\delta}_{\beta h} \Gamma^h_{ik} B^k_C \underset{(\mu)}{n}{}^{*j}_{\beta} B^i_A \quad , \quad \frac{\partial g^{\alpha\beta}_{ij}}{\partial \acute{u}^B_\gamma} \Gamma^B_{DC} \acute{u}^D_\gamma \underset{(\mu)}{n}{}^{*j}_{\beta} \,,$$

on rearrangement of terms we get

$$\begin{aligned} & g^{\alpha\beta}_{ij} \underset{(\mu)}{n}{}^{*j}_{\beta|C} B^i_A + g^{\alpha\beta}_{ij} \left(\frac{\partial^2 x^i}{\partial u^A \partial u^C} + \Gamma^i_{hk} B^h_A B^k_C \right) \underset{(\mu)}{n}{}^{*j}_{\beta} \\ & + \frac{\partial g^{\alpha\beta}_{ij}}{\partial \acute{u}^B_\gamma} \Gamma^B_{DC} \acute{u}^D_\gamma \underset{(\mu)}{n}{}^{*j}_{\beta} B^i_A + \frac{\partial \underset{(\mu)}{n}{}^{*j}_{\beta}}{\partial \acute{u}^E_\gamma} \Gamma^E_{DC} g^{\alpha\beta}_{ij} B^i_A \acute{u}^D_\gamma \\ & + \left(\frac{\partial g^{\alpha\beta}_{ij}}{\partial u^C} - g^{\alpha\beta}_{hi} \Gamma^h_{jk} B^k_C - g^{\alpha\beta}_{jh} B^k_C \Gamma^h_{ik} - \frac{\partial g^{\alpha\beta}_{ij}}{\partial \acute{u}^B_\gamma} \Gamma^B_{DC} \acute{u}^D_\gamma \right) \cdot \underset{(\mu)}{n}{}^{*j}_{\beta} B^i_A = 0 \,. \end{aligned} \tag{6.4}$$

If we put

$$g^{\alpha\beta}_{hj|k} (x^i, \acute{x}^i_\epsilon) \overset{\text{def.}}{=} E^{*\alpha\beta}_{hjk} (x^i, \acute{x}^i_\epsilon) \,,$$

in (6.4) we get on simplification

$$\begin{aligned} & g^{\alpha\beta}_{ij} B^i_A \underset{(\mu)}{n}{}^{*j}_{\beta|C} + \frac{\partial g^{\alpha\beta}_{ij}}{\partial \acute{u}^E_\gamma} \Gamma^E_{DC} \acute{u}^D_\gamma B^i_A \underset{(\mu)}{n}{}^{*j}_{\beta} \\ & + \psi_{(\mu)} \Omega^{*\alpha}_{(\mu)AC} + E^{*\alpha\beta}_{ijk} B^k_C B^i_A \underset{(\mu)}{n}{}^{*j}_{\beta} = 0 \,. \end{aligned} \tag{6.5}$$

Since $\underset{(\mu)}{n}{}^{*i}_{\beta|C}$ is not tangential to X_m we can decompose it as follows:

$$\underset{(\mu)}{n}{}^{*i}_{\beta|C} = \underset{(\mu)}{D}{}^{E}_{\beta C}\, B^{i}_{E} + \sum_{\nu} N^{(\mu)}_{(\nu)C}\, \underset{(\nu)}{n}{}^{*i}_{\beta}\,, \tag{6.6}$$

where $\underset{(\mu)}{D}{}^{E}_{\beta C}$ and $N^{(\mu)}_{(\nu)C}$ are to be determined.

Multiplying equation (6.6) by $g^{\alpha\beta}_{ij}\, B^{j}_{A}$ and using (6.5) we obtain

$$\underset{(\mu)}{D}{}^{E}_{\beta C} = -\, g^{AE}_{\alpha\beta}\left[E^{*\alpha\gamma}_{ihk}\,(x^{l}, \acute{x}^{l}_{\epsilon})\, B^{k}_{C}\, B^{i}_{A}\, \underset{(\mu)}{n}{}^{*h}_{\gamma} + \frac{\partial g^{\alpha\beta}_{ij}}{\partial \acute{u}^{F}_{\gamma}}\, \Gamma^{F}_{DC}\, \acute{u}^{D}_{\gamma}\, B^{i}_{A}\, \underset{(\mu)}{n}{}^{*j}_{\beta} + \psi_{(\mu)}\, \Omega^{*\alpha}_{(\mu)AC}\right]. \tag{6.7}$$

If we multiply (6.6) by $\underset{(\nu)}{n}{}^{*\beta}_{j}$ we obtain

$$N^{(\mu)}_{(\nu)C}\, \psi_{(\nu)} = \underset{(\mu)}{n}{}^{*j}_{\beta|C}\, \underset{(\nu)}{n}{}^{*\beta}_{j}\,. \tag{6.8}$$

Now we shall consider the relation

$$g^{\alpha\beta}_{ij}\,(x^{h}, \acute{x}^{h}_{\epsilon})\, \underset{(\mu)}{n}{}^{*i}_{\alpha}\, \underset{(\nu)}{n}{}^{*j}_{\beta} = \psi_{(\mu)}\, \delta^{\nu}_{\mu}\,, \tag{6.9}$$

(no summation on μ), if we differentiate (6.9) with respect to u^{C} and eliminate $\partial \underset{(\mu)}{n}{}^{*i}_{\alpha} / \partial u^{C}$, from (6.2) and (6.9), we get by virtue of (6.8) on simplification

$$\psi_{(\nu)}\, N^{(\mu)}_{(\nu)C} + \psi_{(\mu)}\, N^{(\nu)}_{(\mu)C} \tag{6.10}$$

$$= \frac{\partial \psi_{(\mu)}}{\partial u^{C}}\, \delta^{\nu}_{\mu} - \frac{\partial \psi_{(\mu)}}{\partial \acute{u}^{B}_{\gamma}}\, \delta^{\nu}_{\mu}\, \Gamma^{B}_{DC}\, \acute{u}^{D}_{\gamma} - E^{*\alpha\beta}_{ijk}\, \underset{(\mu)}{n}{}^{*i}_{\alpha}\, \underset{(\nu)}{n}{}^{*j}_{\beta}\, B^{k}_{C}\,,$$

which implies the properties of $N^{(\mu)}_{(\nu)C}$.

As we have defined $\underset{(\mu)}{n}{}^{*i}_{\alpha|C}$, we can also define $\underset{(\mu)}{n}{}^{i}_{\alpha|C}$ by projecting $\underset{(\mu)}{n}{}_{\alpha|k}$ onto X_m, thus

$$\underset{(\mu)}{n}{}^{i}_{\alpha|C} = \frac{\partial \underset{(\mu)}{n}{}^{i}_{\alpha}}{\partial u^{C}} - \frac{\partial \underset{(\mu)}{n}{}^{i}_{\alpha}}{\partial \acute{u}^{B}_{\gamma}}\, \Gamma^{B}_{DC}\, \acute{u}^{D}_{\gamma} + \Gamma^{i}_{hk}\, \underset{(\mu)}{n}{}^{h}_{\alpha}\, B^{k}_{C}\,. \tag{6.11}$$

If we put

$$g^{\alpha\beta}_{ij|k}\,(x^{h}, \underset{(\mu)}{n}{}^{h}_{\epsilon}) = E^{\alpha\beta}_{(\mu)ijk}\,,$$

we can find

$$\Omega^{\alpha}_{(\mu)AC} = -\, g^{\alpha\beta}_{ij}\, B^{i}_{A}\, \underset{(\mu)}{n}{}^{j}_{\beta|C} - E^{\alpha\beta}_{(\mu)ijk}\, B^{k}_{C}\, B^{i}_{A}\, \underset{(\mu)}{n}{}^{j}_{\beta} - \frac{\partial g^{\alpha\beta}_{ij}}{\partial \acute{u}^{E}_{\gamma}}\, \Gamma^{E}_{DC}\, \acute{u}^{D}_{\gamma}\, B^{i}_{A}\, \underset{(\mu)}{n}{}^{j}_{\beta}\,. \tag{6.12}$$

Now we decompose the tensor $\underset{(\mu)}{n}{}^{j}_{\beta|C}$ as follows:

$$\underset{(\mu)}{n}{}^{j}_{\beta|C} = \underset{(\mu)}{A}{}^{D}_{\beta C}\, B^{j}_{D} + \sum_{\nu} V^{(\mu)}_{(\nu)C}\, \underset{(\nu)}{n}{}^{j}_{\beta}\,, \tag{6.13}$$

where $\underset{(\mu)}{A}{}^{D}_{\beta C}$ and $V^{(\mu)}_{(\nu)C}$ are to be determined.

Multiplying equation (6.13) by $g^{\alpha\beta}_{ij}(x^h, \underset{(\mu)}{n}{}^h_\in)\, B^i_A$ and $\underset{(\sigma)}{n}{}^\beta_j$ respectively we get on simplification

$$\underset{(\mu)}{A}{}^D_{\beta C} = -\gamma^{AD}_{(\mu)\alpha\beta} \left\{ \Omega^\alpha_{(\mu)AC} + E^{\alpha\gamma}_{(\mu)ijk} B^k_C B^i_A \underset{(\mu)}{n}{}^j_\gamma + \frac{\partial g^{\alpha\beta}_{ij}}{\partial \acute{u}^B_\gamma} \Gamma^B_{EC} \acute{u}^E_\gamma B^i_A \underset{(\mu)}{n}{}^j_\beta \right\}, \tag{6.14}$$

and

$$\underset{(\mu)}{n}{}^j_{\beta|C} \underset{(\sigma)}{n}{}^\beta_j = \sum_\nu V^{(\mu)}_{(\nu)C} \cos(\underset{(\nu)}{n}, \underset{(\sigma)}{n}), \tag{6.15}$$

where

$$\cos(\underset{(\nu)}{n}, \underset{(\sigma)}{n}) = \underset{(\nu)}{n}{}^j_\beta \underset{(\sigma)}{n}{}^\beta_j .$$

By virtue of equation (6.13) we can write

$$\frac{D \underset{(\mu)}{n}{}^j_\beta}{D s^\gamma} = -\gamma^{AB}_{(\mu)\alpha\beta} \left\{ \frac{\partial g^{\alpha\beta}_{ii}}{\partial \acute{u}^B_\in} \Gamma^B_{EC} B^i_A B^j_D \underset{(\mu)}{n}{}^i_\delta \acute{u}^E_\in \acute{u}^C_\gamma + \right. \tag{6.16}$$

$$\left. + \Omega^\alpha_{(\mu)AC} B^j_D \acute{u}^C_\gamma + E^{\alpha\delta}_{(\mu)ijk} B^i_A B^j_D \underset{(\mu)}{n}{}^l_\delta \acute{x}^k_\gamma \right\} + \sum_\nu V^{(\nu)}_{(\mu)C} \underset{(\nu)}{n}{}^j_\beta \acute{u}^C_\gamma .$$

Since we know that

$$\underset{(\mu)}{n}{}^\alpha_i = g^{\alpha\beta}_{ij}(x, \underset{(\mu)}{n}) \underset{(\mu)}{n}{}^j_\beta ,$$

it gives on differentiation

$$\frac{D \underset{(\mu)}{n}{}^\alpha_i}{D s^\gamma} = g^{\alpha\beta}_{ij}(x, \underset{(\mu)}{n}) \frac{D \underset{(\mu)}{n}{}^j_\beta}{D s^\gamma} + E^{\alpha\beta}_{(\mu)ijk} \underset{(\mu)}{n}{}^j_\delta \acute{x}^k_\gamma , \tag{6.17}$$

which by virtue of (6.16) implies :

$$\frac{D \underset{(\mu)}{n}{}^\alpha_i}{D s^\gamma} = -\gamma^{AD}_{(\mu)\beta\delta}\, g^{\alpha\beta}_{ij}(x, \underset{(\mu)}{n}) \left\{ \Omega^\delta_{(\mu)AC} \acute{u}^C_\gamma B^j_D + \frac{\partial g^{\delta\theta}_{ml}}{\partial \acute{u}^B_\in} \Gamma^B_{EC} \acute{u}^C_\gamma \acute{u}^E_\in g^{\alpha\beta}_{ij} \underset{(\mu)}{n}{}^l_\theta B^m_D B^j_A \right\} \tag{6.18}$$

$$+ \sum_\nu V^{(\mu)}_{(\nu)C} g^{\alpha\beta}_{ij}(x, \underset{(\mu)}{n}) \underset{(\nu)}{n}{}^j_\beta \acute{u}^C_\gamma .$$

Multiplying equation (6.18) by B^i_F we obtain

$$\frac{D \underset{(\mu)}{n}{}^\alpha_i}{D s^\gamma} B^i_F = -\Omega^\alpha_{(\mu)FC} \acute{u}^C_\gamma - \frac{\partial g^{\alpha\delta}_{ml}}{\partial \acute{u}^B_\in} \Gamma^B_{EC} \acute{u}^C_\gamma \acute{u}^E_\in \underset{(\mu)}{n}{}^l_\delta B^m_F + \sum_\nu V^{(\mu)}_{(\nu)C} g^{\alpha\beta}_{ij}(x, \underset{(\mu)}{n}) \acute{u}^C_\gamma \underset{(\nu)}{n}{}^j_\beta B^i_F , \tag{6.19}$$

which when combined with

$$g^{\alpha\beta}_{AB}(u^C, \acute{u}^C_\in) \acute{u}^B_\delta = \Omega^\alpha_{(\mu)AB} \acute{u}^B_\delta R^\delta_{(\mu)}(x^h, \acute{x}^h_\in) ,$$

yields

$$\frac{D \underset{(\mu)}{n}{}^i_\alpha}{D s^\gamma} B^i_F = -g^{\alpha\beta}_{FC}(u^D, \acute{u}^D_\in) (R^\gamma_{(\mu)})^{-1} \acute{u}^C_\delta - \frac{\partial g^{\alpha\delta}_{ml}}{\partial \acute{u}^B_\in} \Gamma^B_{EC} \acute{u}^C_\gamma \acute{u}^E_\in \underset{(\mu)}{n}{}^l_\delta B^m_F \tag{6.20}$$

$$+ \sum_\nu V^{(\mu)}_{(\nu)C} g^{\alpha\beta}_{ij}(x, \underset{(\mu)}{n}) \underset{(\nu)}{n}{}^j_\beta \acute{u}^C_\gamma B^i_F .$$

If we choose a particular set of normals $\underset{(\mu)}{n}$ in such a way that the last term of (6.20) vanisches, then (6.20) reduces to

$$(6.21) \qquad \frac{D \underset{(\mu)}{n}^{i}_{\alpha}}{D s^{\gamma}} B^{i}_{A} = - g^{\alpha\delta}_{AC} (u^{D}, \acute{u}^{D}_{\in}) (R^{\gamma}_{(\mu)})^{-1} \acute{u}^{C}_{\delta} - \frac{\partial g^{\alpha\delta}_{ml}}{\partial \acute{u}^{B}_{\in}} \Gamma^{B}_{EC} \acute{u}^{C}_{\gamma} \acute{u}^{E}_{\in} \underset{(\mu)}{n}^{l}_{\delta} B^{m}_{A} ,$$

which is a generalisation of the Rodrigue's formula of subspace of a Finsler space Eliopoulos [5].

7. Gauss-Codazzi equations. — To obtain the Gauss-Codazzi equations we consider the covariant derivative of B^{i}_{A} with respect to u^{E}, provided we consider the metric of the submanifold X_{m}, thus

$$(7.1) \qquad B^{i}_{A\,|\,E} = \frac{\partial B^{i}_{A}}{\partial u^{E}} - \frac{\partial B^{i}_{A}}{\partial \acute{u}^{C}_{\alpha}} \Gamma^{C}_{DE} \acute{u}^{D}_{\alpha} - \Gamma^{D}_{AE} B^{i}_{D} ,$$

which by virtue of equation (4.5) leads to

$$(7.2) \qquad B^{i}_{A\,|\,C} = B^{i}_{AC} - \Gamma^{i}_{hk} B^{h}_{A} B^{k}_{C} .$$

Using equation (5.1) we can write (7.2) as

$$(7.3) \qquad B^{i}_{A\,|\,C} = \sum_{\mu} \Omega^{*\alpha}_{(\mu)AC} (u^{D}, \acute{u}^{D}_{\in}) \underset{(\mu)}{n}^{*i}_{\alpha} - \Gamma^{i}_{hk} B^{h}_{A} B^{k}_{C} ,$$

which on further differentiation with respect to u^{D} and the application of

$$(7.4) \qquad B^{i}_{A\,|\,C\,|\,D} - B^{i}_{A\,|\,D\,|\,C} = K^{*E}_{ACD} B^{i}_{E} + B^{i}_{A\,|\,E} T^{E}_{CD} - \frac{\partial B^{i}_{A}}{\partial \acute{u}^{F}_{\alpha}} K^{F}_{GCD} \acute{u}^{G}_{\alpha} ,$$

implies by virtue of (6.6), (6.7) and (7.2), after a long calculation the following equation

$$(7.5) \qquad K^{*E}_{ACD} B^{i}_{E} - \frac{\partial B^{i}_{A}}{\partial \acute{u}^{F}_{\alpha}} K^{F}_{GCD} \acute{u}^{G}_{\alpha} + T^{E}_{CD} (B^{i}_{AE} - \Gamma^{i}_{hk} B^{h}_{A} B^{k}_{E})$$

$$= \left\{ K^{*i}_{hkl} B^{h}_{A} - \frac{\partial B^{i}_{A}}{\partial \acute{x}^{m}_{\alpha}} K^{m}_{pkl} \acute{x}^{p}_{\alpha} + T^{m}_{kl} B^{E}_{m} (B^{i}_{AE} - \Gamma^{i}_{hn} B^{h}_{A} B^{n}_{E}) \right\} B^{k}_{C} B^{l}_{D}$$

$$+ \sum_{\mu} \underset{(\mu)}{n}^{*i}_{\alpha} (\Omega^{*\alpha}_{(\mu)AC\,|\,D} - \Omega^{*\alpha}_{(\mu)AD\,|\,C}) + \sum_{\mu} (\Omega^{*\alpha}_{(\mu)AC} \Omega^{*\beta}_{(\mu)FD} - \Omega^{*\alpha}_{(\mu)AD} \Omega^{*\beta}_{(\mu)FC}) g^{GF}_{\alpha\beta} B^{i}_{G} \psi_{(\mu)}$$

$$- g^{GE}_{\beta\alpha} \frac{\partial g^{\beta\delta}_{lm}}{\partial \acute{u}^{F}_{\alpha}} (\Omega^{*\alpha}_{(\mu)AC} \Gamma^{F}_{LD} - \Omega^{*\alpha}_{(\mu)AD} \Gamma^{F}_{LC}) \acute{u}^{L}_{\gamma} B^{l}_{E} B^{i}_{G} \underset{(\mu)}{n}^{*m}_{\delta}$$

$$- E^{*\alpha\delta}_{ihk} g^{ij}_{\alpha\delta} \sum_{\mu} (\Omega^{*\gamma}_{(\mu)AC} B^{k}_{D} - \Omega^{*\gamma}_{(\mu)AD} B^{k}_{C}) \underset{(\mu)}{n}^{*h}_{\gamma} + \sum_{\mu} \sum_{\nu} (N^{(\mu)}_{(\nu)C} \Omega^{*\alpha}_{(\mu)AD} - N^{(\mu)}_{(\nu)D} \Omega^{*\alpha}_{(\mu)AC}) \underset{(\nu)}{n}^{*i}_{\alpha} ,$$

where K^{*E}_{ACD}, K^{F}_{GCD} and T^{E}_{CD} are corresponding terms for the submanifold X_{m} as already defined by Rund [10] for the manifold X_{n}.

Multiplying equation (7.5) by $g_{ij}^{\epsilon\theta}\, B_H^j$ and $g_{ij}^{\epsilon\theta}\, \underset{(\nu)}{n_\delta^{*j}}$ respectively we get on simplification

$$(7.6)\qquad g_{EH}^{\epsilon\theta}\, K_{ACD}^{*E} - g_{ij}^{\epsilon\theta}\, B_H^j\, K_{GCD}^{F} \frac{\partial B_A^i}{\partial \acute{u}_\alpha^F}\, \acute{u}_\alpha^G - T_{CD}^{E}\, \Gamma_{jhk}^{\epsilon\theta}\, B_H^j\, B_A^h\, B_E^k$$

$$+ E_{jhk}^{*\epsilon\theta} \sum_{\mu} (\Omega_{(\mu)AC}^{*\gamma}\, B_D^k - \Omega_{(\mu)AD}^{*\gamma}\, B_C^k)\, \underset{(\mu)}{n_\gamma^{*j}}\, B_H^h$$

$$+ \frac{\partial g_{lm}^{\theta\delta}}{\partial \acute{u}_\alpha^F} (\Omega_{(\mu)AC}^{*\epsilon}\, \Gamma_{LD}^{F} - \Omega_{(\mu)AD}^{*\epsilon}\, \Gamma_{LC}^{F})\, \underset{(\mu)}{n_\delta^{*m}}\, B_H^l\, \acute{u}_\gamma^L$$

$$= K_{jhkl}^{*\epsilon\theta}\, B_A^h\, B_C^k\, B_D^l\, B_H^j - g_{ij}^{\epsilon\theta} \frac{\partial B_A^i}{\partial \acute{x}_\alpha^m}\, K_{pkl}^{m}\, B_H^j\, B_C^k\, B_D^l\, \acute{x}_\alpha^p + T_{kl}^{m}\, B_C^k\, B_D^l\, \Gamma_{jhm}^{\epsilon\theta}\, B_H^j\, B_A^h$$

and

$$(7.7)\qquad K_{jhkl}^{*\alpha\delta}(x^h, \acute{x}_\epsilon^h)\, B_A^h\, B_C^k\, B_D^l\, \underset{(\nu)}{n_\delta^{*j}} + \psi_{(\nu)} (\Omega_{(\nu)AC|D}^{*\alpha} - \Omega_{(\nu)AD|C}^{*\alpha})$$

$$- \frac{\partial B_A^i}{\partial \acute{x}_\gamma^m}\, K_{pkl}^{m}\, B_C^k\, B_D^l\, \acute{x}_\gamma^p\, \underset{(\nu)}{n_i^{*\alpha}} + T_{kl}^{m}\, B_C^k\, B_D^l\, (\Omega_{(\nu)AE}^{*\alpha}\, \psi_{(\nu)}\, B_m^E - \Gamma_{jhm}^{\alpha\delta}\, B_A^h\, \underset{(\nu)}{n_\delta^{*j}})$$

$$- E_{jhk}^{*\epsilon\theta} \sum_{\mu} (\Omega_{(\mu)AC}^{*\alpha}\, B_D^k - \Omega_{(\mu)AD}^{*\alpha}\, B_C^k)\, \underset{(\mu)}{n_\epsilon^{*h}}\, \underset{(\nu)}{n_\theta^{*j}}$$

$$+ \sum_{\mu} \psi_{(\mu)} (N_{(\nu)C}^{(\mu)}\, \Omega_{(\mu)AD}^{*\alpha} - N_{(\nu)D}^{(\mu)}\, \Omega_{(\mu)AC}^{*\alpha}) - T_{CD}^{E} (\psi_{(\nu)}\, \Omega_{(\mu)AE}^{*\alpha} - \Gamma_{jhm}^{\alpha\delta}\, B_A^h\, B_E^m\, \underset{(\nu)}{n_\delta^{*j}})$$

$$+ \frac{\partial B_A^i}{\partial \acute{u}_\gamma^F}\, K_{GCD}^{F}\, \underset{(\nu)}{n_i^{*\alpha}}\, \acute{u}_\gamma^G = 0\ ,$$

where

$$\Gamma_{jhk}^{\epsilon\theta} = g_{ij}^{\epsilon\theta}\, \Gamma_{hk}^{i} \quad\text{and}\quad K_{jhkl}^{*\epsilon\theta} = g_{ij}^{\epsilon\theta}\, K_{hkl}^{*i}\ .$$

Equations (7.6) and (7.7) are the Gauss-Codazzi equations for the normals $\underset{(\lambda)}{n_\alpha^{*i}}$ in a submanifold X_m of a manifold X_n.

To obtain the Gauss-Codazzi equations for the normals $\underset{(\nu)}{n_\alpha^i}$, we decompose B_{AC}^i (considered as a vector with respect to the index i) into components along the tangent plane at the point considered and normals $\underset{(\lambda)}{n_\alpha^i}$. Thus we have

$$(7.8)\qquad B_{CD}^i = \sum_{\mu} A_{(\mu)CD}^{\alpha}\, \underset{(\mu)}{n_\alpha^i} + W_{CD}^i\ ,$$

where $A_{(\mu)CD}^{\alpha}$ and W_{CD}^i are to be determined and W_{CD}^i satisfies

$$W_{CD}^i\, \underset{(\mu)}{n_i^\alpha} = 0\ .$$

Multiplying equation (7.8) by $\underset{(\nu)}{n_i^\alpha}$ we obtain

$$(7.9)\qquad \Omega_{(\nu)CD}^{\alpha} = \underset{(\nu)}{n_i^\alpha}\, B_{CD}^i = \sum_{\mu} A_{(\mu)CD}^{\alpha} \cos(\underset{(\nu)}{n}, \underset{(\mu)}{n})\ ,$$

which implies

$$\text{(7.10)} \qquad W^i_{CD} = \sum_{\mu} \Omega^{*\alpha}_{(\mu)CD} \underset{(\mu)}{n}^{*i}_{\alpha} - \sum_{\mu} A^{\alpha}_{(\mu)CD} \underset{(\mu)}{n}^{i}_{\alpha} .$$

Using equation (7.2) in (7.10) we get

$$\text{(7.11)} \qquad B^i_{C|D} = \sum_{\mu} A^{\alpha}_{(\mu)CD} \underset{(\mu)}{n}^{i}_{\alpha} + W^i_{CD} - \Gamma^i_{hk} B^h_C B^k_D .$$

In analogy with the Finsler space Eliopoulos [5] we can also express W^i_{CD} as follows :

$$\text{(7.12)} \qquad W^i_{CD} = B^i_E \sum_{\mu} A^{\alpha}_{(\mu)CD} \underset{(\mu)}{M}^{E}_{\alpha} ,$$

where

$$\underset{(\mu)}{M}^{E}_{\alpha} = \underset{(\mu)}{n}^{j}_{\alpha} B^E_j .$$

From (7.12) we can easily obtain

$$\text{(7.13)} \qquad W^i_{CDF} = B^i_E \Big(\sum_{\mu} A^{\alpha}_{(\mu)CD|F} \underset{(\mu)}{M}^{E}_{\alpha} + \sum_{\mu} A^{\alpha}_{(\mu)CD} \underset{(\mu)}{M}^{E}_{\alpha|F}\Big) + \sum_{\mu} \big(A^{\alpha}_{(\mu)CD} \underset{(\mu)}{M}^{E}_{\alpha}\big) B^i_{EF} .$$

Now differentiating (7.11) with respect to the metric of the submanifold X_m, interchanging the last two indices, substracting the resulting equation from first and using equations (6.11), (7.4) and (7.13) we get on simplification the following relation :

$$\text{(7.14)} \qquad K^{*E}_{CDF} B^i_E - \frac{\partial B^i_C}{\partial \acute{u}^A_\alpha} K^A_{GDF} \acute{u}^G_\alpha + T^E_{DF} (B^i_{CE} - \Gamma^i_{hk} B^h_C B^k_E)$$

$$= \Big\{ K^{*i}_{hkl} B^h_C - \frac{\partial B^i_C}{\partial \acute{x}^m_\alpha} K^m_{pkl} \acute{x}^p_\alpha + T^m_{kl} B^E_M (B^i_{CE} - \Gamma^i_{hn} B^h_C B^n_E) \Big\} B^k_D B^l_F$$

$$+ B^i_E \Big[\Big\{ \sum_{\mu} \gamma^{LE}_{(\mu)\alpha\beta} (\Omega^{\alpha}_{(\mu)LD} A^{\beta}_{(\mu)CF} - \Omega^{\alpha}_{(\mu)LF} A^{\beta}_{(\mu)CD}) \Big\}$$

$$+ \Big\{ \sum_{\mu} \underset{(\mu)}{M}^{E}_{\alpha} (A^{\alpha}_{(\mu)CD|F} - A^{\alpha}_{(\mu)CF|D}) + \sum_{\mu} (\underset{(\mu)}{M}^{E}_{\alpha|F} A^{\alpha}_{(\mu)CD} - \underset{(\mu)}{M}^{E}_{\alpha|D} A^{\alpha}_{(\mu)CF}) \Big\} \Big]$$

$$- \sum_{\mu} g^{ri}_{\alpha\beta} E^{\beta\gamma}_{(\mu)rhk} \underset{(\mu)}{n}^{h}_{\gamma} (A^{\alpha}_{(\mu)CD} B^k_F - A^{\alpha}_{(\mu)CF} B^k_D) + \sum_{\mu} \underset{(\mu)}{n}^{i}_{\alpha} (A^{\alpha}_{(\mu)CD|F} - A^{\alpha}_{(\mu)CF|D})$$

$$+ \sum_{\mu} \sum_{\nu} (A^{\alpha}_{(\mu)CD} \Omega^{*\beta}_{(\nu)EF} - A^{\alpha}_{(\mu)CF} \Omega^{*\beta}_{(\nu)ED}) \underset{(\nu)}{n}^{*i}_{\beta} \underset{(\mu)}{M}^{E}_{\alpha} + \sum_{\mu} \sum_{\nu} \underset{(\nu)}{n}^{i}_{\alpha} (A^{\alpha}_{(\mu)CD} V^{(\mu)}_{(\nu)F} - A^{\alpha}_{(\mu)CF} V^{(\mu)}_{(\nu)D}) .$$

Multiplying equation (7.14) by $g^{\epsilon\theta}_{ij} B^j_H$ and solving we get

$$\text{(7.15)} \qquad \gamma^{\epsilon\theta}_{(\nu)EH} \Big[K^{*E}_{CDF} - \sum_{\mu} \gamma^{LE}_{(\mu)\alpha\beta} (\Omega^{\alpha}_{(\mu)LD} A^{\beta}_{(\mu)CF} - \Omega^{\alpha}_{(\mu)LF} A^{\beta}_{(\mu)CD})$$

$$- \sum_{\mu} \underset{(\mu)}{M}^{E}_{\alpha} (A^{\alpha}_{(\mu)CD|F} - A^{\alpha}_{(\mu)CF|D}) - \sum_{\mu} (\underset{(\mu)}{M}^{E}_{\alpha|F} A^{\alpha}_{(\mu)CD} - \underset{(\mu)}{M}^{E}_{\alpha|D} A^{\alpha}_{(\mu)CF}) \Big]$$

$$-\frac{\partial B^{l}_{C}}{\partial \acute{u}^{A}_{x}} K^{A}_{GDF}\, \acute{u}^{G}_{x}\, g^{\in 0}_{ij}\, B^{j}_{H} + T^{E}_{DF} \Big[\sum_{\mu} \underset{(\mu)}{n^{i}_{x}} \big(A^{x}_{(\mu)CE}\, g^{\in 0}_{ij}\, B^{j}_{H} + g^{\in 0}_{LH}\, B^{L}_{i}\, A^{x}_{(\mu)CE}\big)$$

$$-\Gamma^{\in 0}_{jhk}\, B^{h}_{C}\, B^{k}_{E}\, B^{j}_{H} \Big] + \sum_{\mu} g^{ri}_{x\beta}\, E^{s\beta}_{(\mu)rhk}\, g^{\in 0}_{ij}\, (x\,,\underset{(\nu)}{n})\, \underset{(\mu)}{n^{h}_{s}}\, B^{j}_{H}\, \big(A^{x}_{(\mu)CD}\, B^{k}_{F} - A^{x}_{(\mu)CF}\, B^{k}_{D}\big)$$

$$-\sum_{\mu} g^{\in 0}_{ij}\, (x\,,\underset{(\nu)}{n})\, \underset{(\mu)}{n^{i}_{x}}\, B^{j}_{H}\, \big(A^{x}_{(\mu)CD\,|\,F} - A^{x}_{(\mu)CF\,|\,D}\big)$$

$$-\sum_{\mu}\sum_{\sigma} \big(A^{x}_{(\mu)CD}\, V^{(\mu)}_{(\sigma)F} - A^{x}_{(\mu)CF}\, V^{(\mu)}_{(\sigma)D}\big)\, g^{\in 0}_{ij}\, \underset{(\sigma)}{n^{i}_{x}}\, B^{j}_{H}$$

$$= \Big[K^{*\in 0}_{jhkl}\, B^{j}_{H}\, B^{h}_{C} - \frac{\partial B^{l}_{C}}{\partial x'^{m}_{x}} K^{m}_{pkl}\, x'^{p}_{x}\, g^{\in 0}_{ij}\, B^{j}_{H} + T^{m}_{kl}\, B^{E}_{m} \Big\{ \sum_{\mu} A^{x}_{(\mu)CE}\, \underset{(\mu)}{n^{i}_{x}}\, \big(g^{\in 0}_{ij}\, B^{j}_{H} + g^{\in 0}_{LH}\, B^{L}_{i}\big)$$

$$-\Gamma^{\in 0}_{jhn}\, B^{h}_{C}\, B^{n}_{E}\, B^{j}_{H} \Big\} \Big]\, B^{k}_{D}\, B^{l}_{F}\,.$$

Similarly multiplying equation (7.14) by $g^{\in 0}_{ij}\, (x\,,\underset{(\nu)}{n})\, \underset{(\nu)}{n^{j}_{s}}$ we get

$$\text{(7.16)}\quad \Big\{ K^{*\in 0}_{jhkl}\, B^{h}_{C} - \frac{\partial B^{l}_{C}}{\partial x'^{m}_{x}} K^{m}_{pkl}\, g^{\in 0}_{ij}\, x'^{p}_{x} \Big\}\, \underset{(\nu)}{n^{j}_{s}}\, B^{k}_{D}\, B^{l}_{F} + \frac{\partial B^{l}_{C}}{\partial \acute{u}^{A}_{x}} K^{A}_{GDF}\, \acute{u}^{G}_{x}\, g^{\in 0}_{ij}\, (x\,,\underset{(\nu)}{n})\, \underset{(\nu)}{n^{j}_{s}}$$

$$- T^{E}_{DF}\, \Big(\sum_{\mu} \Omega^{*x}_{(\mu)CE}\, g^{\in 0}_{ij}\, \underset{(\mu)}{n^{*i}_{x}}\, \underset{(\nu)}{n^{j}_{s}} - \Gamma^{\in 0}_{jhk}\, B^{h}_{C}\, B^{k}_{E}\, \underset{(\nu)}{n^{j}_{s}}\Big)$$

$$+ T^{m}_{kl}\, g^{\in 0}_{ij}\, (x\,,\underset{(\nu)}{n})\, B^{E}_{M}\, \underset{(\nu)}{n^{j}_{s}}\, B^{k}_{D}\, B^{l}_{F}\, \Big(\sum_{\mu} \Omega^{*x}_{(\mu)CE}\, \underset{(\mu)}{n^{*i}_{x}} - \Gamma^{i}_{hn}\, B^{h}_{C}\, B^{n}_{E}\Big)$$

$$- \sum_{\mu} g^{ri}_{x\beta}\, E^{s\gamma}_{(\mu)rhk}\, g^{\in 0}_{ij}\, (x\,,\underset{(\nu)}{n})\, \big(A^{x}_{(\mu)CD}\, B^{k}_{F} - A^{x}_{(\mu)CF}\, B^{k}_{D}\big)\, \underset{(\mu)}{n^{h}_{\gamma}}\, \underset{(\nu)}{n^{j}_{s}}$$

$$+ \sum_{\mu} g^{\in 0}_{ij}\, (x\,,\underset{(\nu)}{n})\, \underset{(\mu)}{n^{i}_{x}}\, \underset{(\nu)}{n^{j}_{s}}\, \big(A^{x}_{(\mu)CD\,|\,F} - A^{x}_{(\mu)CF\,|\,D}\big)$$

$$+ \sum_{\mu}\sum_{\sigma} \underset{(\mu)}{M^{E}_{x}}\, g^{\in 0}_{ij}\, (x\,,\underset{(\nu)}{n})\, \underset{(\nu)}{n^{j}_{s}}\, \underset{(\sigma)}{n^{*i}_{\beta}}\, \big(A^{x}_{(\mu)CD}\, \Omega^{*\beta}_{(\sigma)EF} - A^{x}_{(\mu)CF}\, \Omega^{*\beta}_{(\sigma)ED}\big) +$$

$$+ \sum_{\mu}\sum_{\sigma} g^{\in 0}_{ij}\, (x\,,\underset{(\nu)}{n})\, \underset{(\sigma)}{n^{i}_{x}}\, \underset{(\nu)}{n^{j}_{s}}\, \big(A^{x}_{(\mu)CD}\, V^{(\mu)}_{(\sigma)F} - A^{x}_{(\mu)CF}\, V^{(\mu)}_{(\sigma)D}\big) = 0\,.$$

Equations (7.15) and (7.16) are Gauss-Codazzi equations for the normals $\underset{(\mu)}{n^{i}_{x}}$ in the submanifold X_m of the manifold X_n.

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